COMBINATORIAL GROUP THEORY AND TOPOLOGY

组合群论和拓扑学〔英〕

EDITED BY

S. M. GERSTEN

AND

JOHN R. STALLINGS

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PREFACE

Combinatorial Group Theory lives in the fertile region between pure group theory and pure topology. The interplay between the logical precision of algebra and the intuitive depths of geometry gives it charm and strength. Lyndon suggests in his article that perhaps there is no exact definition of the subject. Certainly it includes the study of equations over groups (which tends to involve the study of geometric diagrams), much of 3-manifold theory (the Poincaré Conjecture is equivalent to a group-theoretic question), group actions on geometric-combinatorial objects such as trees, the theory of surface automorphisms, and many other developments. As the Walrus said, the time has come to talk of many things.

At Alta Lodge in the spectacular Wasatch Mountains of Utah, on July 15-18, 1984, sixty of us gathered for an intense conference. Roger Lyndon's opening address, in the tradition of Hilbert, offered a score of open problems to guide the field in the future. In the style of the Séminaire Bourbaki, six speakers were assigned topics for expository talks; these were the origins of the papers in this book by Alperin and Bass, Bleiler, Eckmann, Hempel, and Howie. In addition, there were about twenty-five shorter talks on current research, from which we selected the remaining articles here.

Our goal was to produce a book full of ideas, understandable to ourselves and to students, that will open up the future development of this field. We consciously tried to reach a large class of readers, including good graduate students with backgrounds in group theory and topology. We are grateful for the cooperation of the authors who have helped us in pursuing this goal.

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It is a pleasure to acknowledge the assistance given us by Roger Lyndon, with whom we frequently consulted in the planning of the conference. In tribute to his work in the field and his insight into directions for future research, his article has a prominent place in this volume.

We are greatly indebted to James Howie and Geoffrey Mess for frequent consultations. In addition, we wish to thank some thirty referees, who by tradition must remain anonymous, for their valuable aid.

Special thanks are due to Marty Jones of Alta Lodge and to Ann Reed for helping make the conference run smoothly. Finally we acknowledge with thanks and appreciation the financial support of the University of Utah and the National Science Foundation.

STEVE GERSTEN
JOHN STALLINGS

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PROBLEMS IN COMBINATORIAL GROUP THEORY

Roger Lyndon

0. *Introduction*

Steve Gersten asked me to give a talk like Hilbert gave in Paris in 1900. I said I'd be happy to, but pointed out that I'm no Hilbert. Merzlyakov [117] came to my rescue with the following suggestion: "Rather than waiting [for a new Hilbert] group theorists have come to the more prosaic idea of a Present-day collective Hilbert." So I have appealed to some of you for amendments to a provisional outline, for which I thank you, and I hope this exonerates me of any charge of arrogance in presenting this biased survey.

What is Combinatorial Group Theory? This term acquired official status as the title of the book of Magnus-Karrass-Solitar. The first sentence of the History of Combinatorial Group Theory by Chandler-Magnus [28] says: "Combinatorial Group Theory may be characterized as the theory of groups which are given by generators and relations...." (But compare Magnus-Karrass-Solitar with Coxeter-Moser, Generators and Relations for Discrete Groups.) This hardly does justice to the goals or methods of the subject. On other occasions Chandler-Magnus speak of "group theory with the exception of Lie groups and of group representations and linear groups," but this seems to both include and exclude too much.

Maybe we could define Combinatorial Group Theory to be Very Low Dimensional Topology. In fact, group theorists, like others, will attack whatever problems interest them with whatever tools they have at hand, so perhaps Combinatorial Group Theory is just a state of mind. Maybe it is distinguished by a reluctance to make the great supposedly simplifying assumptions of Commutativity, Linearity, and either Finiteness or Continuity, or, put more positively, a relish for the combinatorial core of a problem. Altogether, it seems fruitless to try to define an elephant we have barely touched.

I intended to talk about the history of the subject, but Chandler-Magnus have let me off that hook. I do not need to tell anyone that the major source and strength of Combinatorial Group Theory has been Topology, wherein we tentatively include Discontinuous Groups, from Poincaré on, with an assist from the more or less abstract or axiomatic side, beginning with Cayley, through the influence of Finite Groups and of problems from Logic. These influences will be evident, and, if my discussion appears biased against topology, it is because as a non-topologist I am reluctant or unable to tell you what you know better than I. Altogether, though, I was agreeably surprised by the homogeneity of the subject, if somewhat inconvenienced by the many interconnections among the various approaches and problems in the subject.

My account is also biased for the most part toward rather recent work, and to pathways to the future more than monuments to the past. The space devoted to a subject should not be taken as a measure of the importance attached to it. Likewise, the mention of a name or citation of a paper is often more to draw attention to it than to bestow an honor. References are illustrative or suggestive, and should usually be followed by 'and others.' I have not cited papers that are well known or easily accessible (with a few exceptions), in particular papers more than about five years old, or listed in the bibliography of Lyndon-Schupp. Incomplete references indicate lack of knowledge.

Despite the unity of the subject, as a first step toward linearization I have divided it into seven unequal and gerrymandered sections. It took some cutting and pasting to bring my list of problems to twenty, a modest three fewer than Hilbert. Even so, many are really 'problem areas,' of the classical form: 'What can be said about ....?.'

1. *Properties of free groups, equations in and over groups*

The most 'abstract' and 'axiomatic' of our problems is a folklore problem of Alfred Tarski.

PROBLEM 1. *Do all nonabelian free groups have the same elementary theory?*

The elementary theory of a class of groups is the set of all sentences in first order logic (with symbols for equality and group composition but, emphatically, excluding set theory) that are true in all groups of the class. By contrast, among free *abelian* groups those of rank at most 2 are distinguished by the property that there exist elements a and b such that, for all x, one of x, xa, xb, xab is the square y^2 of some element y. See V. Dyson [47].

In this connection, R. Vaught posed the following test problem: In a free group, does $x^2y^2 = z^2$ always imply $xy = yx$? This was proved by Lyndon and generalized extensively. The method was close to that used by H. Zieschang in studying automorphisms of surface groups and Fuchsian groups.

A much earlier theorem of Frobenius (1895) deals with the number of solutions of the equation $x^n = 1$ in a finite group; for a survey of this problem see H. Finkelstein [57]. See also Finkelstein-Mandelberg [58].

Equations *over* groups entered Combinatorial Group Theory in a paper of B. H. Neumann (1943), where he showed that, for a positive integer n and an element g of a group G, the equation $x^n = g$ has as many solutions as desired in some group H containing G. Higman-Neumann-Neumann extensions are a generalization of the fact that the equation $t^{-1}g_1t = g_2$, for g_1 and g_2 in G, has a solution t in a group H containing G if and only if g_1 and g_2 have the same order.

The central problem in this area is the Kervaire-Lauderbach problem, which we state as follows.

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PROBLEM 2. *If* G *has a presentation* $G = (X:R)$ *and* $H = (X \cup t : R \cup w)$ *is obtained by adding one new generator and one defining relation, when does the inclusion* $X \to X \cup t$ *induce an injection of* G *into* H ?

In simpler language, for $g_1, \cdots, g_n$ in G, when does an equation $w(g_1, \cdots, g_n, t) = 1$ have a solution t in some group H containing G ?

We have seen that $t^{-1} g_1 t g_2^{-1} = 1$ has no solution if g_1 and g_2 have different orders. Gerstenhaber-Rothaus showed that if the sum of the exponents on t in w is not 0, and if G can be embedded in a compact connected Lie group, then a solution always exists. Rothaus [144] later improved the condition on G to local residual finiteness. The sufficient condition of local indicability has been studied by J. Howie [94]. The group $G = (a_1, \cdots, a_4 : a_{i+1}^{-1} a_i a_{i+1} = a_i^2,\ i \text{ modulo } 4)$ of G. Higman satisfies none of the known sufficient conditions for the solvability of an equation (with non-zero exponent sum) over G. Without the exponent condition, probing by Lyndon [113] suggests that the problem is difficult even for G a finite cyclic group. See also Brodskii [19, 20] and Short [152].

The result of Gerstenhaber-Rothaus raises the following question.

PROBLEM 2a. *If the sum of the exponents on* t *in* w *is not* 0, *does the equation* $w = 1$ *always have a solution?*

So much for equations *over* groups. Equations *in* groups are exemplified by the Vaught problem. This prompted Lyndon to ask about solutions of an equation $w(a_1, \cdots, a_n, t_1, \cdots, t_m) = 1$ *in* a free group G with basis $a_1, \cdots, a_n$. In the case $m = 1$ of one unknown he obtained a set of words containing parameters as exponents which, subject to conditions on the parameters, give precisely the set of solutions. For example, $t^{-1} a_1 t a_1^{-1} = 1$ has exactly the solutions $t = a_1^n$ for all integers n. This was substantially improved and extended, but without definitive result. Recently Makanin [115] has given an algorithm that associates with an equation $w = 1$, as above, an integer N such that, if any solution exists, there exists one with total length of the t_i at most N. This settles the question of existence of solutions and provides an algorithm for finding one if it exists, but leaves open the following problem.

PROBLEM 3. *Given an equation over a free group, find an algebraic description of the set of all solutions.*

See Howie [93, 95, 97], Ozhigov [130].

A special case of this problem, for free groups and other groups, is the Substitution Problem (for free groups, the Endomorphism Problem): Does an equation $w(t_1, \cdots, t_m) = g$ have a solution? Wicks [169] showed that an element g of a free group is a commutator if and only if, relative to any basis, g can be written in the form $g = abca^{-1}b^{-1}c^{-1}$ *without cancellation.* This result has been extended substantially by C. C. Edmunds [51, 52] and L. P. Comerford, Jr. [34, 35] and, using topological methods, by M. Culler [37] and by Goldstein-Turner [73, 74]. Related methods have been used by M. Scharlemann [146] and P. E. Schupp [148]; see also E. Rips [140], and P. Hill and S. J. Pride [90].

PROBLEM 4. *Let* $w(a_1, \cdots, a_n)$ *be a word in the free group* F *with basis* $a_1, \cdots, a_n$. *Is there an algorithm which, given* g *in* F, *decides if there exist* $t_1, \cdots, t_n$ *in* F *such that* $w(t_1, \cdots, t_n) = g$?

The Substitution Problem has been studied extensively for finite simple groups and for various classical infinite groups. See Finkelstein [57], Finkelstein-Mandelberg [58], Lyndon [112], Mycielski [122], and also Ehrenfeucht-Fajtlowicz-Malitz-Mycielski [53].

2. *Automorphisms of groups*

The general linear group $GL(n, R)$ is the group of automorphisms of the free R-module of rank n. The automorphism group $GL(2, Z)$ of the free Z module, or free abelian group, A_2 of rank 2, and also, in particular, the modular group $PSL(2, \mathbb{Z})$, have been much studied. But it is a giant stride to the study of the automorphism group $\mathrm{Aut}\, F_n$ of the free nonabelian group F_n of rank n.

Nielsen used automorphisms of free groups to great advantage, and obtained a finite presentation for Aut F. J. McCool recovered this presentation by different methods, which enabled him to obtain finite presentations for the stabilizers of finite sets of elements. His method is based on that used by Whitehead to decide whether two finite sequences of elements of a free group are equivalent under some automorphism. A related problem of Whitehead, to decide whether two finitely generated subgroups are equivalent under an automorphism, has been solved recently by Gersten [68], using new methods. By the same methods he has proved a conjecture of P. Scott that the subgroup of elements fixed by an automorphism of a finitely generated free group is finitely generated. J. L. Dye and Scott had shown earlier that the set of fixed points of a finite subgroup of Aut F is a free factor of F. There has been some study of the structure of a single automorphism, inducing the dream, or nightmare, of a 'Jordan structure theorem.'

Considerable work on the structure of Aut F, related mainly to its action on various naturally arising characteristic subgroups and their quotients, has been done by S. Andreadakis and by S. Bachmuth-H. Mochizuki

PROBLEM 5. *Determine the structure of* Aut F, *of its subgroups, especially its finite subgroups, and its quotient groups, as well as the structure of individual automorphisms.*

If N is a normal subgroup of a free group F and $\mathrm{Aut}_N F$ is its stabilizer, then $\mathrm{Aut}_N F$ induces a group of automorphisms of $G = F/N$. Nielsen showed that for the usual presentation of a surface group G the group $\mathrm{Aut}_N F$ maps onto Aut G. G. Rosenberger [143] and H. Zieschang have studied this 'lifting problem,' especially for Fuchsian groups. See also S. J. Pride and A. D. Vella [137].

PROBLEM 6. *If* G = F/N, F *free, what subgroups of* Aut G *are images of subgroups of* Aut F ?

The mapping class group M of a surface can be identified essentially with the group of outer automorphisms (automorphisms modulo inner automorphisms) of its fundamental group G. The results of J. McCool give M (or, directly, Aut G) as the fundamental group of a finite complex, which, unfortunately, exhibits too many natural symmetries to make it accessible to present day computation. W. Thurston (see A. Hatcher-W. Thurston [88]) has given a quite different method for obtaining a finite presentation of the mapping class groups, and, for the orientable case, B. Wajnryb [168] has used this method to obtain presentations that are reasonably concise but not entirely perspicuous.

PROBLEM 7. *Obtain finite presentations for the mapping class groups that are at once usably concise and yet in which both the generators and the relations have fairly obvious geometrical meanings.*

For example, Hatcher-Thurston say: "... all relations follow from relations supported in certain subsurfaces, finite in number, of genus at most 2."

The following is an obvious addendum.

PROBLEM 7a. *The same for all Fuchsian groups.*

3. *Morphisms of trees*

The prefix 'auto' is omitted as a salute to the recent work of Gersten [62, 64-72] and Stallings [158, 159] on morphisms in the category of graphs.

A basic paper of J. Tits [164] initiates the study of the group Aut T of automorphisms of a tree T. He shows that, if Aut T leaves invariant no proper subtree and no end of T, then the subgroup G generated by all stabilizers of branch points is a simple group, while the quotient (Aut T)/G is a free product of groups of order 2 and infinite cyclic groups. This frequently cited work deserves to be extended.

A great deal is coming to be known about certain very special but very remarkable groups of automorphisms of trees, introduced by N. Gupta and S. Sidki [84, 85, 86, 153, 154]. If A is an 'alphabet' with a prime number p of letters, then the monoid $T = A^*$ of all finite words, ordered by left divisibility, is a tree T. Generalizing a construction of R I. Grigorchuk [79], Gupta and Sidki show that certain easily described 2-generator subgroups G of Aut T have remarkable properties: they are 'Burnside groups,' that is, infinite 2-generator p-groups (with elements of unbounded order); they contain isomorphically all finite p-groups; they are residually finite, and all their proper quotient groups are finite. These groups are remarkable not only for these properties, but also because they are quite 'concrete' and accessible to detailed study.

PROBLEM 8. *Study the structure of the automorphism groups of trees and of their subgroups.*

For example, do these groups have Sylow subgroups?

Certain naturally arising instances of groups acting on trees encountered by J-P. Serre appear to have led to the Bass-Serre theory of graphs of groups, with their associated groups acting on trees. This method has become a standard tool in the study of infinite groups, especially as obtained by amalgamated product and HNN-extension, and with regard to subgroup theorems. Earlier, Lyndon, in seeking to unify cancellation arguments based on Nielsen transformations in the proofs of the Nielsen-Schreier and Kurosh Subgroup Theorems and of the Grushko-Neumann Theorem, introduced axiomatically characterized length functions on groups. I. M. Chiswell showed that this theory, for integer valued functions, is essentially equivalent to the Bass-Serre theory of groups acting on trees. J. W. Morgan and P. B. Shalen [120], following work of R.C. Alperin and K.N. Moss [4] on real valued length functions, have played off these two theories to obtain new proofs of two theorems of Thurston.

In connection with the Subgroup Theorems, we note that Rosenberger, Zieschang, and Karrass-Solitar all obtained refinements of the basic Subgroup Theorem of Hanna Neumann for amalgamated products. In particular, Karrass and Solitar introduced tree products, which agree with a special case of the Bass-Serre graph products. They also introduced polygonal products: a group G is generated by vertex groups, with the edge groups amalgamated. (This group G differs in a small but significant way from the corresponding Bass-Serre group; the Karrass-Solitar definition is natural in the context of presentations of certain geometrically constructed groups, while that of Bass-Serre is natural for a graph of complexes.) Polygonal products were motivated by the recognition that the Picard group PSL(2, Z [i]) can be obtained from four very small groups at the vertices of a square by amalgamating subgroups associated with the sides of the square. See, for example, B. Fine [55, 56] and A. Brunner, M. L. Frame, Y. W. Lee, N. J. Wielenberg [24]. Square products are studied also in a paper of D. Ž. Djokovič [42], which, although failing of its main objective, contains an extensive study of groups acting on cubic trees. See also Djokovič-G. L. Miller [43]. A. Brunner, Y. W. Lee, and N. J. Wielenberg [25] have used polygonal products to obtain elegant descriptions of various 3-dimensional Euclidean groups. M. W. Davis [40] has used a similar construction in a more abstract context to obtain generalized Coxeter groups by sewing together infinitely many copies of an abstract polytope by identifications at vertices according to specified orthogonal groups; in this way he obtains aspherical manifolds of dimension $n \geq 4$ not covered by Euclidean space. There are also various abstractly defined generalizations of symmetric groups, braid groups, and Coxeter groups; K. I. Appel and P. E. Schupp [5, 6] have used small cancellation theory to study certain such groups.

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