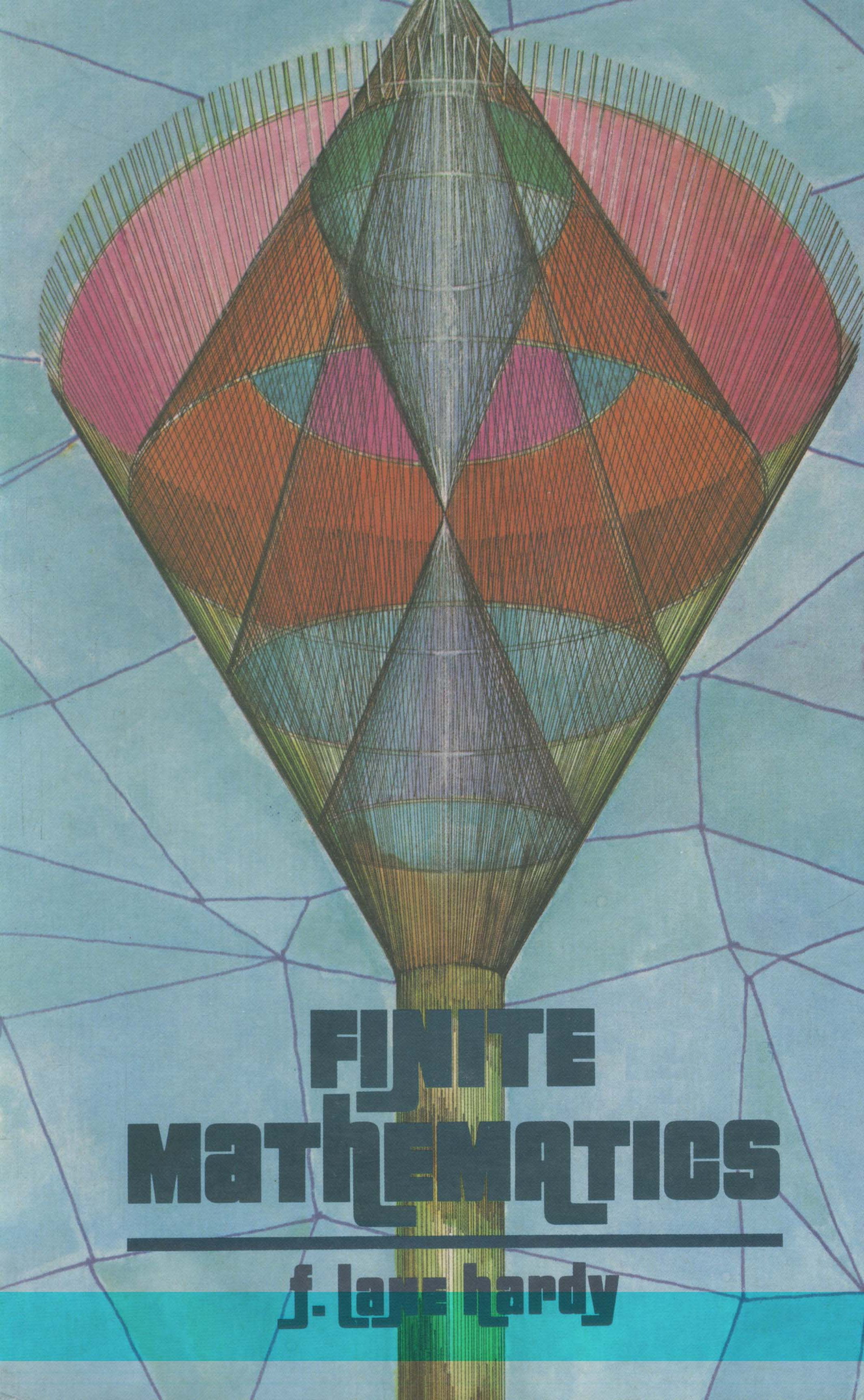

FINITE MATHEMATICS

f. Lane hardy

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DeKalb Community College

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PREFACE

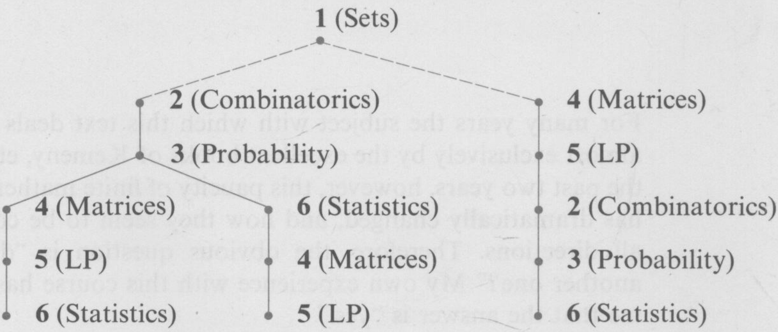
For many years the subject with which this text deals was served almost exclusively by the excellent books of Kemeny, et. al. Within the past two years, however, this paucity of finite mathematics texts has dramatically changed, and now they seem to be coming from all directions. Therefore, the obvious question is "do we need another one?" My own experience with this course has convinced me that the answer is "yes."

This text has been written for college freshmen whose mathematical background is very limited. Although there are many fine books available, it has been impossible for me to find one that includes all of the features that I find important:

(a) exposition at a very low level
(b) a large number of routine exercises*
(c) a student manual
(d) an instructor's manual
(e) a battery of tests
(f) computer programs
(g) material appropriate for both liberal arts majors and business majors

The material on sets in Chapter 1 is limited because most students are already familiar with it. For many readers this chapter will likely be a review, and instructors can skip it if so desired. Chapter 2 includes the standard topics on permutations, combinations, and the binomial and multinomial theorems. Chapter 3 contains a detailed introduction to finite probability including stochastic processes and Bayes' theorem. Topics covered in Chapter 2 are used in this chapter as well. Chapter 4 is a brief introduction to linear systems and matrices. Also included in this chapter is an introduction to finite Markov chains. Chapter 5 discusses linear programming and includes the simplex method and dual problems. The final chapter, on statistics, includes a brief discussion of descriptive statistics as well as normal distributions, estimation, confidence intervals, and confidence coefficients. Some sections have been starred to indicate that the material contained within is not needed for the sections that follow. These may be omitted at the instructor's discretion. There is enough material for a one semester course and more than enough for one quarter.

Some instructors begin their course with the topics related to probability, while others prefer to begin with matrices and linear programming. This text is designed so that either approach is possible. The following diagram suggests some possible arrangements.



* A few problems have been included that are either not routine or that extend the text material. In these cases, the problems have been starred.

For ease of reference, the most frequently used formulas are listed on pages xiii and xiv immediately following.

It is a pleasure to acknowledge the help that I have received from so many individuals during the writing of this book. The material was first taught in mimeograph form to Mathematics 121 students at DeKalb Community College, and I owe much to them for their comments and questions. My appreciation goes to Dean Edwin Davidson of DeKalb, who was kind enough to agree to the class testing. Special thanks are extended to my chairman and friend H. E. Hall, who taught from a preliminary version and made helpful comments. My colleague Linda Boyd wrote the computer programs that appear in Appendix A and to her I express sincere thanks. My friend and former colleague B. K. Youse of Emory University read the entire manuscript and made his usual helpful suggestions. I greatfully acknowledge the contribution of Ray Ankner, President of NBF Corporation, who allowed me access to his fine collection of problems in probability and statistics.

I am especially pleased to be able to express my thanks to my friends at "The Great House of Harper": to Jim Nye and George Telecki, who were first interested in my writing, and to Charlie Dresser, Mathematics Editor. To Lois Lombardo, who has been responsible for so much excellent work in the production stage of this book, I can only say that it has been a singular experience to be associated with such an efficient professional.

F. L. H.

formulas

CHAPTER 1

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ — The number of elements in $A \cup B$.

CHAPTER 2

(1) $P(n, r) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)$ — The number of permutations of n things taken r at a time.

(2) $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ — The number of combinations of n things taken r at a time. Also the number of r-subsets of an n-element set.

(3) $\dfrac{n!}{r_1!r_2!r_3!\cdots r_k!}$ — The number of permutations of n things of which r_1 are alike, r_2 are alike, r_3 are alike, ..., r_k are alike.

(4) $(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n} a^0 b^n$ — The Binomial theorem.

CHAPTER 3

(1) $P(A) = \dfrac{n(A)}{n(S)}$ — The probability the event A occurs in an experiment in which outcomes are *equally likely* and S is the sample space.

(2) $P(A) = P(a_1) + P(a_2) + P(a_3) + \cdots + P(a_k)$ — The probability that A occurs, where $A = \{a_1, a_2, a_3, \ldots, a_k\}$.

(3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ — The probability that A or B occurs.

(4) $P(A \cap B) = P(A) \cdot P(B)$ — The probability that A and B occurs in case (and only in the case) that A and B are independent.

(5) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ — The probability that A occurs, given that B occurs.

(6) $b(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$ — The probability of k successes in n trials of a binomial experiment in which p is the probability of success on any single trial.

CHAPTER 6

(1) $\overline{x} = \dfrac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$ — The sample mean.

(2) $s^2 = \dfrac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2}{n}$ — The sample variance.

(3) $s = \sqrt{\dfrac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2}{n}}$ — The sample standard deviation.

(4) $E = p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n)$ — Mathematical expectation.

(5) $z = \dfrac{x - \mu}{\sigma}$ — Formula for converting x-numbers of a normal distribution having parameters μ and σ to z-numbers.

conTenTs

sets

1.1 THE LANGUAGE OF SETS

The student is likely to be familiar with the language of sets and for this reason the beginning ideas will be discussed only briefly.

The words “collection,” “class,” and “family,” will be used as synonyms for “set.” The objects of a set will be called its *elements* or *members*. For example, the set which we describe as “the U.S. Senate” has 100 senators as members.

When we speak of a set we may refer to it in one of two ways:

(a) we list or tabulate its members
(b) we give a description or common property which characterizes the set.

To illustrate, we may say "the set whose members are 1, 2, 3" or "the set whose members are the first three natural numbers."[1] In each of these two cases it is convenient to use the *braces* or *bracket* notation. This simply means that the set is described by enclosing the members of the set between braces (or brackets). The above set is indicated in this way:

$\{1, 2, 3\}$

or

$\{x|x$ is one of the first three natural numbers$\}$

Both of these descriptions describe the same set. The expression "$\{x|x$ is one of the first three natural numbers$\}$" should be read as "the set of all x such that x is one of the first three natural numbers." This notation always contains the parts indicated below:

{①|②}

The set of all . . . such that . . .

The first bracket and the vertical line above should be read as indicated. The space ① is filled with some convenient letter of the alphabet, such as x, and space ② gives the appropriate description. As an example, we may wish to speak of the set consisting of Borman, Anders, and Lovell. In the notation just described this may be done as follows:

$\{x|x$ is a human who orbited the moon in 1968$\}$

Consider the sets below:

$\{6, 8, 10\}$, $\{4, 5, 6, 7, 8, 9, 10\}$

It is evident that each number of the first of these two sets is a member of the second set. For this reason we call the set $\{6, 8, 10\}$ a *subset* of the set $\{4, 5, 6, 7, 8, 9, 10\}$.

Definition. Let A and B represent sets. Then *A is a subset of set B* if and only if either of the following is true:
(a) each number of A is a member of B;
(b) no member of A fails to be a member of B.

[1] The *natural numbers* are 1, 2, 3, 4, . . . while the *whole numbers* are 0, 1, 2, 3, 4, ...

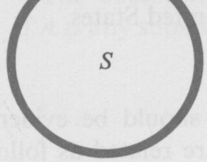
It should be clear that (a) and (b) of the definition above mean the same thing. A common notation for subset is "$\subseteq$," that is, we write "$A \subseteq B$" as a shorthand for the statement "A is a subset of B." For example, the statements below are true:

$$\{4, 6, 8\} \subseteq \{4, 5, 6, 7, 8, 9, 10\}$$

$$\{9, 1, 8, 2\} \subseteq \{y \mid y \text{ is a whole number}\}$$

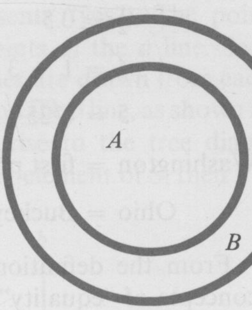
To indicate that a set A is not a subset of a set B, we write $A \nsubseteq B$.

A convenient way to represent a set S is to choose a closed figure such as the one illustrated in Figure 1.1 and let the points in the interior of the figure represent the elements of S. Using this device we may indicate that $A \subseteq B$ by enclosing the figure for A within the figure for B (see Figure 1.2).



Set S

FIGURE 1.1



$A \subseteq B$

FIGURE 1.2

If an object a is a member of a set A, we use the symbol "$\in$" to indicate this fact and write: $a \in A$. We read the statement "$a \in A$" as "a is a member of A." Using this notation we may correctly make the following statements:

$$4 \in \{3, 4, 5\}$$

$$8 \in \{z \mid z \text{ is a whole number}\}$$

With the set membership notation "$\in$" the definition of "subset" may be stated as follows:

$A \subseteq B$ if and only if for all x if $x \in A$, then $x \in B$.

It is convenient to employ the concept of a set which has no elements or members. This set is denoted by the symbols "$\varnothing$" or "{ }" and is called the *empty set* or *null set*. If X is any set, then no element of $\varnothing$ (since $\varnothing$ has no elements) fails to be a member of X.

Therefore, it follows that $\varnothing$ is a subset of X. Hence, we have for all sets X,

$$\varnothing \subseteq X$$

In mathematics different names are used for the objects under discussion. Usually it is convenient to use letters of the alphabet to name sets, numbers, geometric objects, and so on. In talking about sets, for example, we use upper case letters, $A, B, C \ldots$, for sets and the corresponding lower case letters, $a, b, c \ldots$, for elements of sets. To indicate that an object has more than one name the notion of *equality* is used.

Definition. $x = y$ means that x and y are names for the same object.

To illustrate the above definition, we have the following:

$$\tfrac{1}{2} = 0.5$$

$$4 = 1 + 3$$

$$5 = \sqrt{25}$$

Washington = first president of the United States

Ohio = Buckeye State

From the definitions just given it should be evident that the concepts of "equality" and "subset" are related as follows:

If A, B are sets, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

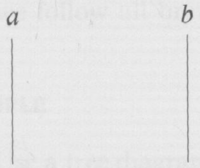
An immediate consequence of the observation just made is that if A is any set, then $A \subseteq A$; that is, *every set is a subset of itself.*

A problem that occurs frequently in finite mathematics is the determination of all the subsets of some given set S. Let us consider a few simple cases. If $A = \varnothing$, then S has only one subset, namely itself. Now suppose S contains exactly one element: $S = \{a\}$. Then S has two subsets: $\varnothing$ and $\{a\}$. If $S = \{a, b\}$, then S has the subsets $\varnothing$, $\{a\}$, $\{b\}$, $\{a, b\}$. These three cases have been tabulated in Table 1.1.

TABLE 1.1

Set S	*Subsets of S*	*Number of subsets*
$\varnothing$	$\varnothing$	1
$\{a\}$	$\varnothing, \{a\}$	2
$\{a, b\}$	$\varnothing, \{a\}, \{b\}, \{a, b\}$	4

As the number of elements in the set S increases it becomes evident that some routine procedure is needed to tabulate all the subsets of S. A *tree diagram* may be used for this and we illustrate for $S = \{a, b\}$. The first step in constructing the tree diagram is to draw a vertical line for each element of the set; in this case there is a vertical line for a and one for b:



To this we add a starting point P which is located to the left of the a line. We also choose two points on the a line and label them 0, 1 (0 represents "no" and 1 represents "yes"). The point P is now connected to the two chosen points of the a line. See Figure 1.3. To complete the diagram two lines are drawn from each of the two points on the a line to two points on the b line, as shown in Figure 1.4.

The observation which gives rise to the tree diagram is this: If A is any subset of S and x is any element of S, then x is a member

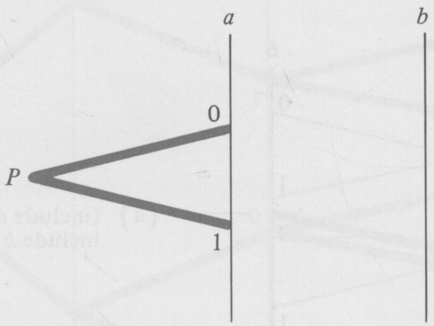


FIGURE 1.3

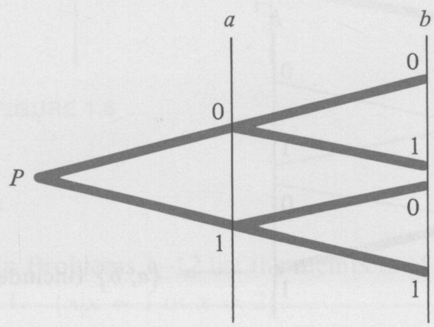


FIGURE 1.4

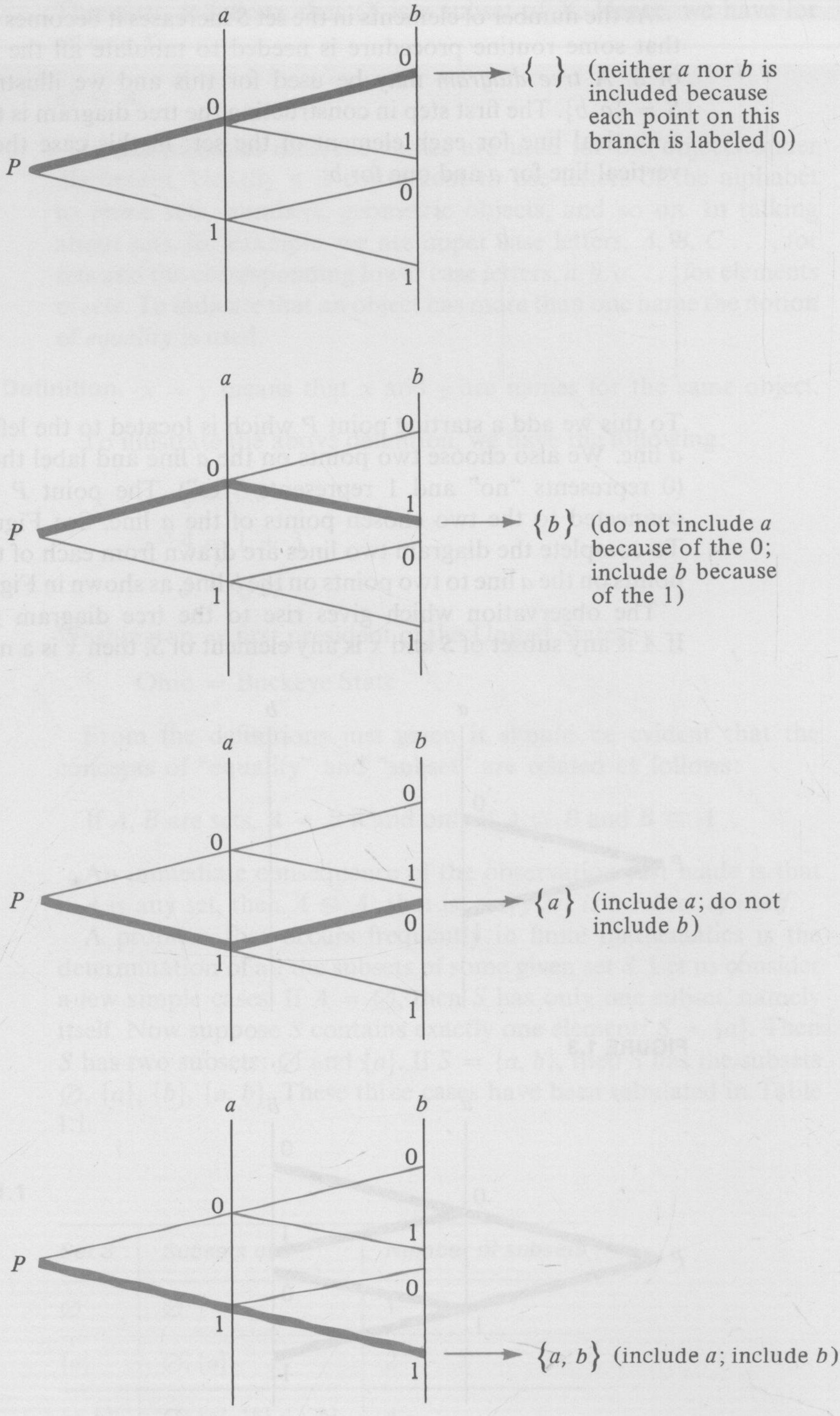

a
b
0
0
1
P
0
1
1
{ } (neither a nor b is included because each point on this branch is labeled 0)
a
b
0
0
1
P
0
1
1
{b} (do not include a because of the 0; include b because of the 1)
a
b
0
0
1
P
{a} (include a; do not include b)
0
1
1
a
b
0
0
1
P
0
1
1
{a, b} (include a; include b)

FIGURE 1.5