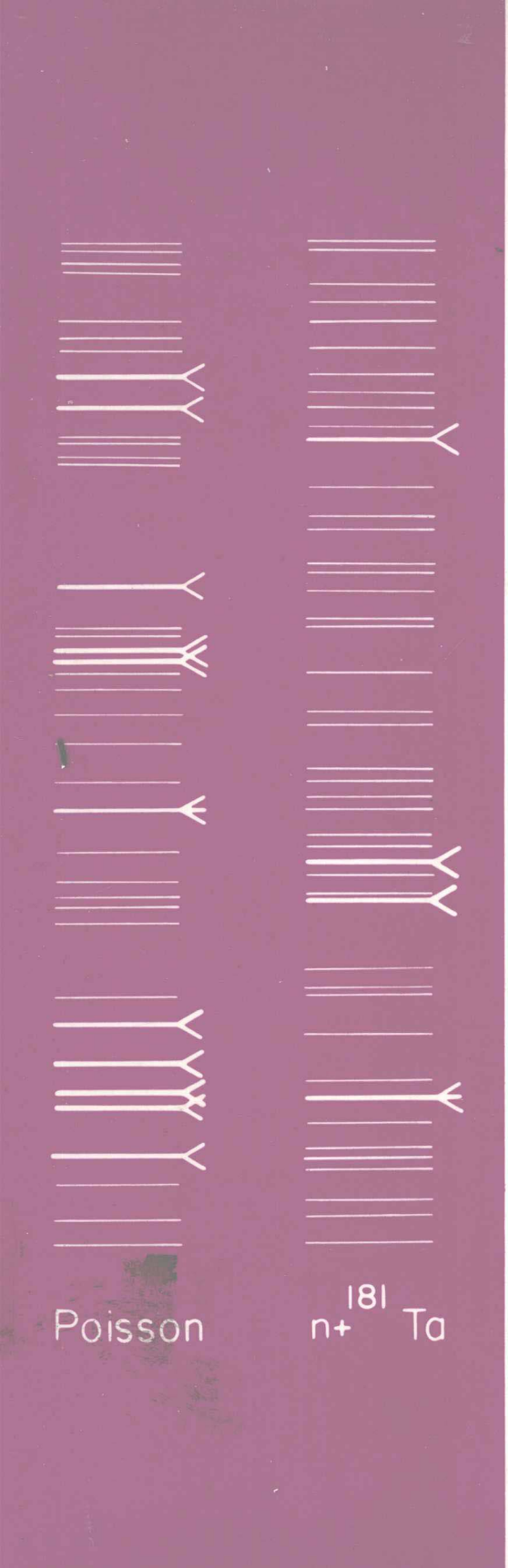

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Moshe Carmeli

STATISTICAL THEORY AND RANDOM MATRICES

Moshe Carmeli

Center for Theoretical Physics
Ben Gurion University of the Negev
Beer Sheva
Israel

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PREFACE

The question whether the highly excited states of a physical system may be understood by assuming no structure to the system, and that no quantum number other than the spin and the parity remains good, leads to the *statistical theory of energy levels* and its relation to *random matrices*. Such a statistical theory is designed to describe the general appearance and the degree of irregularity of the level structure that occurs in a complex physical system, which is otherwise too complicated to be understood in detail, rather than to predict the detailed sequence of the energy levels in any particular nucleus or atom.

The standard type of statistical mechanics is clearly inadequate for the discussion of energy levels since statements on the *fine detail* of the energy level structure cannot be made in terms of an ensemble of states. What is required is a *different* kind of statistical mechanics in which one renounces the exact knowledge *not* of the state of the system but of the *nature* of the system itself.

The problem then is to define in a mathematically precise way an ensemble of systems in which all possible laws of interactions are equally probable. The idea of a statistical mechanics of nuclei, which is based on an ensemble of systems, is due to Wigner and to von Neumann. This book summarizes the fundamentals of this theory.

After introducing the basic concepts of the statistical theory of energy levels and their relations to random matrices in Chapters 1

and 2, we discuss the symmetry properties of physical systems in Chapter 3. Different kinds of ensembles are subsequently introduced and discussed. This includes the Gaussian and the orthogonal ensembles which are discussed in Chapter 4, followed by the unitary ensemble discussed in Chapter 5.

In Chapter 6 the problem of eigenvalue-eigenvector distributions of the Gaussian ensemble is discussed, while Chapter 7 deals with the distribution of the widths. We then discuss the symplectic group and its relation to quaternions in Chapter 8. A detailed discussion on the Gaussian ensemble is subsequently given in Chapter 9. Chapter 10 is devoted to the summary, whereas Appendices A and B include a detailed discussion on the statistical aspects of multivariate distributions, and Appendix C on the ergodic properties of random matrices.

I am indebted to Professor P. R. Krishnaiah for stimulating my interest in the statistical theory of multivariate distributions of random matrices. I am also most thankful to Professor E. P. Wigner for emphasizing the uniqueness of the statistical theory of energy levels and for his comments on the use of complex Wishartian ensemble (see Chapter 10). Also, I am indebted to Mrs. Sara Corrogosky for her technical assistance in preparing and typing the manuscript of the book, and to Professor J. B. French and the American Physical Society for their permission to reproduce Figures 5.1-5.5, 6.1-6.3, and 7.1.

Part of this book was written while the author was a Visiting Professor and Member of the Institute for Theoretical Physics, State University of New York at Stony Brook. I am indebted to Professor Chen Ning Yang for his kind hospitality at Stony Brook in 1977-78 and 1981. Finally, I am most grateful to my wife Elisheva for her moral encouragement and technical help without which the book would probably have never been written.

Moshe Carmeli

CONTENTS

CHAPTER 1
INTRODUCTION

1.1 COMPLEX SYSTEMS

Within the theory of quantum mechanics [see, for instance, Schiff (*1*)] the behavior of a physical system is determined by a state function Ψ. The state function is then a solution of the familiar Schrödinger equation,

$$H\Psi = E\Psi, \tag{1.1}$$

where H is the *Hamiltonian operator*, which is a *Hermitian operator*, and E is a constant which denotes the energy levels of the system.

Thus the energy levels are *characteristic values* (*eigenvalues*, or *roots*) of Hermitian operators. The *stationary states* of the system are the corresponding *characteristic vectors* (or *eigenfunctions*).

Although theoretical analyses have had impressive success, as was pointed out by Kisslinger and Sorensen (*2*), and by Baranger (*3*), in interpreting the detailed structure of the *low-lying* excited states of complex systems [here the word system is used for a physical quantum system that can be described by the Schrödinger equation; a system could be, for example, a complex nucleus, as is discussed in (*2*) and (*3*), or an atomic system], still, there must come a point beyond which such analyses of individual levels cannot usefully go [Dyson (*4*)]. For example, observations of levels of heavy nuclei in the neutron-capture region [Rosen, Desjardins, Rainwater, and Havens (*5*)] give exact information on the energy levels from number N to number $N + n$, where n is an integer of the order of 100 whereas N is an integer of the order of one million. It appears improbable that energy level assignments, based on various models, can ever be pushed as far as the millionth level.

1.2 STATISTICAL THEORY OF LEVELS

One is then led to ask whether the highly excited states may be understood from the opposite point of view, by assuming no structure for the system and that no quantum numbers other than spin and parity remain good. Such an inquiry leads to a *statistical theory of energy levels*.

Such a statistical theory is not supposed to predict the detailed sequence of energy levels in any one nucleus or atom, but is expected to describe the *general appearance* and the *degree of irregularity* of the level structure that is to occur in a complex system which is otherwise too complicated to be understood in detail.

As Dyson (*4*) has pointed out, in ordinary statistical mechanics a comparable renunciation of exact knowledge about the system is also made. By assuming that all states of a very large ensemble are equally probable, one obtains useful information about the overall behavior of a complex system when the observation of the state of the system in all its detail is impossible. This standard type of statistical mechanics is clearly inadequate for the discussion of energy

levels. What one wishes is to make statements on the fine detail of the energy level structure, and such statements cannot be made in terms of an ensemble of states. What is required is a *different* kind of statistical mechanics in which one renounces exact knowledge *not* of the state of a system but of the *nature* of the system itself. One might picture a complex nucleus as a "black box" in which a large number of particles are interacting according to unknown laws. The problem then is to define in a mathematically precise way an ensemble of systems in which all possible laws of interaction are equally probable. The idea of a statistical mechanics of nuclei based on an ensemble of systems is due to Wigner.

1.3 COMPARISON WITH STATISTICAL MECHANICS

The difference between the usual statistical mechanics and the statistical theory of energy levels can also be seen, according to Wigner (*6*), as follows.

A system in quantum mechanics can be characterized by the Hamiltonian H, which is a self-adjoint linear operator in the infinite-dimension Hilbert space of functions Ψ. If one introduces a coordinate system in the Hilbert space, the Hamiltonian operator may then be looked at as a Hermitian matrix of infinitely many dimensions. Therefore, an ensemble of systems can be considered as an ensemble of Hermitian matrices. At this stage one might consider matrices of very high dimensionality rather than infinite matrices. However, the question arises as to what ensemble of such matrices one has to consider. Herein lies the difference between the ensembles of statistical mechanics and the ensemble of the statistical theory of energy levels.

In statistical mechanics one considers a system of particles with definite masses interacting among themselves by a given law. The state of such a system can be specified, in classical mechanics, by the generalized coordinates q_i and the generalized momenta p_i of the particles, where both q_i and p_i are functions of time. The physical quantities one is then interested in are the time averages of continuous functions f of the coordinates and momenta,

$$\lim_{T\to\infty}\left(\frac{1}{T}\int_t^{t+T} f(q_1(\tau), q_2(\tau), \ldots, p_1(\tau), p_2(\tau), \ldots)\, d\tau\right). \qquad (1.2)$$

Using Newton's law of motion, one can, in principle, determine the coordinates and momenta as functions of time and their initial values [see, e.g., Goldstein (*7*)]. Hence the averaging process is an entirely *definite* one, and the result is a function only of the constants of motion, such as energy, but independent of other initial conditions. This result, except for rare cases, has long been proved and known by von Neumann and others [Koopman (*8*); Birkhoff (*9, 10*); von Neumann (*11*); Birkhoff and Koopman (*12*)].

The averaging process in the theory of random processes, on the other hand, is *not* defined. One again deals with a specific system with its proper, though in many cases unknown, Hamiltonian and pretends that one deals with a multitude of systems, all with their own Hamiltonians, and averages over the properties of these systems. Such a procedure can be meaningful only if it turns out that the properties in which one is interested are the same for the vast majority of the admissible Hamiltonians.

1.4 EXAMPLES

What are the admissible Hamiltonians, and what is the proper measure in the ensemble of these Hamiltonians? And suppose the ensemble of admissible Hamiltonians with a proper measure is given. Are the properties in which we are interested common for the vast majority of them?

Figures 1.1-1.3 illustrate the situation which leads to the idea of the statistical properties of the spectrum in the higher-energy region, as compared to low-energy region, where one desires to have a rather complete description of the stationary states and as complete a listing as possible of the exact values of the energy levels.

Figure 1.1 gives the energy levels of the nuclei beryllium, boron, and carbon (^{10}Be, ^{10}B, and ^{10}C). The diagram shows the eight lowest energy levels of ^{10}B and the lowest two energy levels of ^{10}Be and ^{10}C. It gives the position of these energy levels, their total angular momenta J, and parities T [Ajzenberg and Lauritsen (*13*)].

Of much interest, but not shown in the diagram, are the transition probabilities between these levels. Such transition probabilities can be calculated if the characteristic functions associated with the characteristic values are known. Conversely, agreement between the

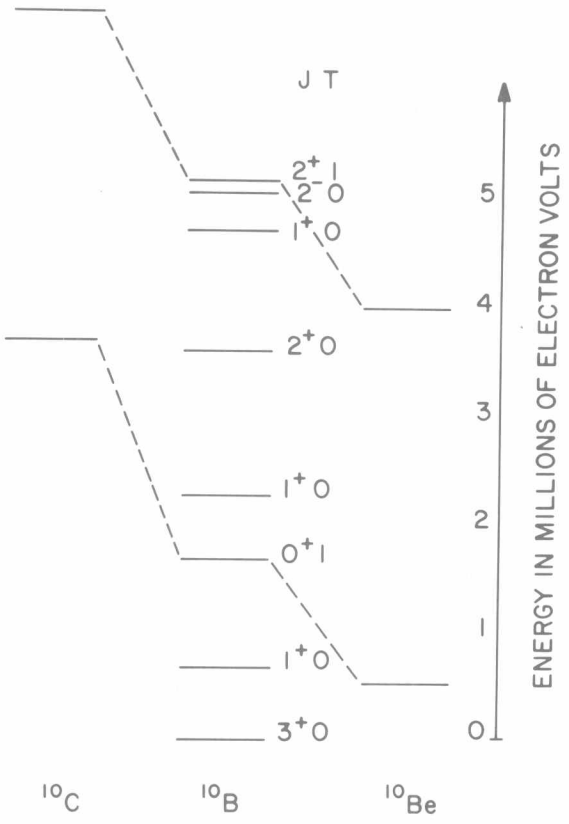

Figure 1.1 Energy levels of the nuclei ^{10}Be, ^{10}B, and ^{10}C [Ajzenberg and Lauritsen (*13*)].

observed transition probabilities and the calculated values of these quantities gives an indication of the accuracy of the calculated characteristic function [Wigner (*14*)].

Figure 1.2 gives the energy levels of ^{180}Hf. This nucleus has a rotational band [Mihelich, Scharff-Goldhaber, and McKeown (*15*)]. The angular momenta of the states shown are $J = 0, 2, 4, 6, 8$ in units of $h/2\pi$, where h is Planck's constant. The energy levels of these states are proportional to $J(J + 1)$, where J is the angular momentum quantum number.