Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Subseries: Department of Mathematics
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24

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TABLE OF CONTENTS

INTRODUCTION

During the spring semester of 1979 we presented a program in Euclidean harmonic analysis at the University of Maryland. The six lecture series comprising this volume were a major part of our program.

Euclidean harmonic analysis has a rich basic theory and maintains a vital relationship with several other areas which, in fact, have molded the subject and enlivened it with significant applications for over 150 years. Wiener's Tauberian theorem provides a neat example of this fundamental and, to some extent, mysterious interplay. Wiener's theorem not only characterizes the prime number theorem but is used to define spectra properly for phenomena such as white light; this spectral theory provides perspective for the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, lead naturally to H^p spaces.

In the first lecture series of this volume L. CARLESON addressed the two main problems of classical statistical mechanics: a. the verification of expected equilibrium thermodynamic properties and b. the validity of the Gibbs theory for dynamical systems. The results of part a include proofs of the basic properties of the free energy function, as well as a rigorous verification of the existence of phase transition for certain classical models. In part b Carleson first discusses a Boltzmann equation and the approach to equilibrium that it describes. He then considers dynamical properties of harmonic oscillator systems and shows how one can verify the Gibbs theory for an ensemble of such systems. Classical harmonic analysis is pervasive in his approach; and the point of his lectures is to introduce some analytic results and problems which may eventually lead to further progress in applications.

The remaining lecture series contained in this volume, as well as the lectures by our other visitors, fell into one or the other of two categories of problems.

The first category of problems deals with the synthesis of prescribed harmonics to describe a given phenomenon.

The fundamental synthesis problem is to determine whether or not the Fourier series of a function f converges in some designated way to the function. The most famous question in this area treats the case in which f is an element of $L^2[0,2\pi)$ and convergence is pointwise almost everywhere. Carleson answered this question in 1966 by proving that every such function is the pointwise almost everywhere sum of its Fourier series. C. FEFFERMAN gave a conceptually different proof of

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Carleson's theorem in 1973, and an explanation of this proof as well as a comparison between it and Carleson's was the subject of his lecture series. Since Fefferman's paper has already appeared (Ann. Math., 98 (1973) 551-571) we have not included his lectures in this volume, and because of this omission we mention a few of his comments. We begin by recalling that in 1968 Hunt proved Carleson's theorem for $L^p[0,2\pi)$, $p > 1$, and that Carleson's method of proof can even be used for the space $L \log L(\log\log L)$. On the other hand, Fefferman's method is L^2 in nature and depends on an orthogonality property of linear operators first formulated by Cotlar. To begin with, Carleson's theorem is an easy consequence of the maximal function estimate,

$$\text{(1)} \qquad \forall f \in L^2[0,2\pi), \quad \|\sup_N |S_N f(\cdot)|\|_1 \leq C\|f\|_2,$$

where $S_N f$ is the N^{th} partial sum of the Fourier series of f. The classical formula, $S_N f = D_N * f$, where D_N is the Dirichlet kernel, expresses $S_N f$ as a Hilbert transform H, and the fundamental nature of the operator H in Euclidean harmonic analysis, including its boundedness on L^2, provides the basic direction for Fefferman's approach. Instead of substituting the Hilbert transform representation of $S_N f$ into (1), he begins by noting that (1) is equivalent to the estimate, $\|S_{N(\cdot)} f(\cdot)\|_1 \leq C\|f\|_2$, where N is a function depending on f and x. Then he observes that (1) follows from the inequalities, $\|T_N f\|_1 \leq C\|f\|_2$, for arbitrary functions $N(x)$, where $T_N f(x)$ is essentially $H(e^{iN(x)y} f(y))$. For each $N(x)$, he verifies the corresponding inequality by making a proper dyadic decomposition of T and applying Cotlar's result to the relatively independent and orthogonal pieces of the decomposition. In his lectures, Fefferman illustrated the method for the case of $N(x) = \lambda x$, which in fact contains the germ of the whole argument; and then, for arbitrary $N(x)$, he explained his combinatorial procedure and decomposition of T into sums of local operators which contain both space and frequency data on small intervals. Regardless of the simplicity or complexity of f or N, Carleson's method analyzes the given function f and is oblivious to the corresponding function N, and Fefferman's method does the opposite.

Synthesis was also the subject matter of the lecture series by both Y. DOMAR and L. HEDBERG. The problems they discussed fall into the category of spectral synthesis and have the following formulation: let X be a class of distributions with support contained in a fixed subset E of $\mathbb{R}^n$; determine whether or not a given element $\mu \in X$

is the limit in some designated topology of bounded measures contained in X. In Domar's case the Fourier transform of X is a subset of $L^\infty(\mathbb{R}^n)$ and the topology is weak * convergence. This is the setting of Beurling's classical spectral synthesis problem based ultimately on Wiener's Tauberian theorem. Domar considers the case in which E is a curve in $\mathbb{R}^2$ and he characterizes spectral synthesis in terms of the curvature of E. He also solves some analogous problems for manifolds E in $\mathbb{R}^n$, $n \geq 3$, and obtains spectral synthesis results in terms of the geometric properties of E. In Hedberg's case, X can be any one of a large collection of Sobolev spaces and the topology is the Sobolev space norm topology. This is the setting in which the spectral synthesis property for all elements of X is equivalent to the stability, in the sense of potential theory, of closed sets essentially complementary to E. Hedberg verifies this equivalence in various Sobolev spaces, and analyzes and generalizes Wiener's criterion for regular points in order to characterize Sobolev space spectral synthesis.

The second category of problems deals with the harmonic analysis of operators of L^p spaces. These problems have emerged from the research of Zygmund, Calderón, and Stein, as well as several of our guests. The omnipresent Hilbert transform H and its generalizations are an essential feature of the area, and multipliers, maximal functions, H^p theory, and interpolation are some of its major topics.

In order to verify various L^p estimates for H and related operators, R. COIFMAN and Y. MEYER presented a range of real and complex methods, from Boole's symbolic calculus of over a century ago to the latest $\bar{\partial}$ proofs of Calderón's theorem. Boole's theory systematically uses measure preserving maps and has long been a staple for ergodic theorists; in harmonic analysis it provides a means to calculate the distribution function of H. Large parts of Coifman's and Meyer's lectures were given in the context of commutators and bilinear maps. Commutators of H are used in the study of boundary value problems for elliptic equations, and they arise naturally when one wishes to extend the classical L^2 estimate for H to curves. Next, G. WEISS, in joint work with several others, set forth a theory of interpolation which includes the Riesz-Thorin theorem and Stein's theorem for analytic families of operators. He dealt with a continuum of Banach spaces associated with the boundary points of a domain $D \subseteq \mathbb{C}^n$ and constructed intermediate spaces for each point of D. The basic interpolation result is stated in terms of subharmonic functions. An interesting corollary of the theory is an extension of the celebrated Wiener-Masani theorem which, in turn, provides important factorization criteria for certain filtering and prediction problems. Finally, A. CÓRDOBA solved several specific problems involving a thorough mix of many of the real methods in this second category of problems and concepts. The first result settles a basic real variable question on the differentiation of integrals and depends on a covering theorem and estimates on the appropriate maximal function. The remaining results include a rather complete theory for multipliers arising from classical summability methods.

We wish to thank Berta Casanova, Cindy Edwards, Pat Pasternack, Becky Schauer, and June Slack, of our technical typing staff for their expert work; and to express our appreciation to Alice Chang, Robert Dorfman, Ward Evans, Raymond Johnson, and C. Robert Warner for their editorial assistance.

John J. Benedetto
College Park, Maryland

Besides many of the analysts at the University of Maryland, the participants in our program included:

M. Ash	L. Ehrenpreis	A. Picardello
A. Baernstein	E. Fabes	H. Pollard
M. Benedicks	C. Fefferman	E. Prestini
G. Benke	R. Fefferman	F. Ricci
R. Blei	A. Figà-Talamanca	L. Rubel
G. Bohnké	L. Hedberg	C. Sadosky
P. A. Boo	H. Heinig	D. Sarason
L. Carleson	R. Hunt	P. Sarnak
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SOME ANALYTIC PROBLEMS RELATED TO STATISTICAL MECHANICS

Lennart Carleson
Institut Mittag-Leffler

Apology. In the following lectures, I shall give some analytic results which derive from my interest in statistical mechanics. I do not claim any new results for applications, and any serious student of statistical mechanics should consult other sources. It is my hope that analysts will find, as I have, that interesting and difficult analytic problems are suggested by this material; and that they will eventually make contributions of real significance in applications.

I. Classical Statistical Mechanics. Background

1. We consider a system of N particles moving according to a Hamiltonian function

$$H(p,q) = H(p_1,\ldots,p_{3N},q_1,\ldots,q_{3N}).$$

The classical equations for the motion are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{1}$$

p_i are the momenta and q_i the position coordinates for the particles. It follows that $H(p,q)$ is constant during the motion and H is interpreted as the energy of the system. A typical situation is

$$H(p,q) = \frac{1}{2}\sum p_i^2 + \sum_{i\neq j} \Phi(\tilde{q}_i-\tilde{q}_j), \qquad \tilde{q}_i = (q_{3i+1}, q_{3i+2}, q_{3i+3}).$$

We now assume that the motion takes place inside a box Λ_N of volume $\sim\rho^{-1}N$, where ρ is the density of the particles. The total energy $H = E_N \sim \lambda N$, so that λ is the average energy per particle.

Denote by $d\sigma$ the surface element of the energy surface Σ_N in the $6N$-dimensional space Ω of points $\omega = (p,q)$.

The basic assumption of statistical mechanics is now that the motion $\omega(t)$ is ergodic on the energy-surface, i.e.,

$$\lim_{N\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi(\omega_N(t))dt = \lim_{N\to\infty}\frac{\int_{\Sigma_N}\varphi(p,q)d\sigma}{\sigma(\Sigma_N)}$$

at least for simple functions φ depending on a finite number of variables and belonging to C_0^∞. Actually, from a physical point of view it is more natural to assume that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi(\omega(t))dt$$

exists, where T avoids a set of density zero. We then speak of the Gibbs limit. A natural assumption here is also that we are dealing with a bounded number of different particles, and therefore have a corresponding number of symmetries in the function H. I shall not formulate this in more detail; the meaning in concrete cases is quite clear.

Gibbs' contribution here is that he has given a formula for computing the density $d\sigma/\sigma(\Sigma) = d\mu$. Let us observe that

$$d\omega = d\sigma\, dE \quad \text{in} \quad \Omega_N.$$

Let β be a parameter and consider

$$F(\beta) = \int_\Omega e^{-\beta E}\, d\omega = \int_0^\infty e^{-\beta E}\, dV(E)$$

where $V(E_0)$ is the volume $\int_{E<E_0} d\omega$. By partial integration

$$F(\beta) = \beta \int_0^\infty e^{-\beta E} V(E)dE.$$

The dependence on N is now such that

$$E = Ne, \qquad V(E) = v_N(e)^N C_N, \quad \text{where} \quad v_N(e) \longrightarrow v(e),$$

and $v(e)$ is expected to be a smooth function. We are dealing with an integral essentially of the form

$$I_N = C_N \int_0^\infty e^{-N[\beta t - \psi(t)]}\, dt$$

where $\psi(t)$ is an increasing function bounded from above. If we define

(2) $$-\psi^*(\beta) = \sup_t(\psi(t) - \beta t)$$

we realize that

$$I_N(\beta) \le e^{-N\psi^*(\beta)} \cdot \text{Const.}$$

and

$$I_N(\beta) \ge C_N \int_{t_0}^\infty e^{N\psi(t_0)} \cdot e^{-N\beta t} dt = \text{Const.}\ \frac{e^{-N\psi^*(\beta)}}{N}.$$

Hence I_N and so $F(\beta)$ get their essential contribution from the surface t where the supremum is taken.

$\psi^*(\beta)$ is the Legendre transform of $\psi(t)$. Observe that

$$\psi(t) + \psi^*(\beta) \le \beta t.$$

Hence

$$\psi^{**}(t) \ge \psi(t)$$

and ψ^{**} is the smallest convex majorant of ψ.

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Only those values of β which correspond to linear pieces in ψ^{**} give ambiguous values of t in (2). We have

$$-\psi^{*}(\beta) = \psi(t) - \beta t \quad \text{and} \quad \psi'(t) = \beta$$

so that

$$\psi^{*\prime}(\beta) = t \quad \text{if} \quad \psi'' \neq 0.$$

If the graph of ψ^{**} contains a straight line then ψ^{*} shows a corner. Hence, if ψ^{*} is smooth, then ψ^{**} is strictly convex.

Going back to $F_N(\beta)$, the proper definition is

$$f(\beta) = \lim_{N\to\infty} \frac{\log F_N(\beta) - \log C_N}{N}.$$

Unless the energy surface is one of the exceptional values for which we have ambiguity in t we can choose β so that the integral in the definition of $F(\beta)$ is carried out essentially on the right energy surface. If

$$\mu_{\beta,N} = \frac{e^{-\beta E}d\omega_N}{F_N(\beta)}$$

then it follows that

$$\int \varphi(p,q)\,d\mu = \lim_{N\to\infty} \int \varphi(p,q)\,d\mu_{\beta,N},$$

and this is Gibbs' rule. We also see that we can expect exceptional results if $f(\beta)$ has a singularity—in these cases it is not clear that the formula gives the correct result.

In the case of the simple Hamiltonian,

$$\frac{1}{2}\sum p_i^2 + \sum \Phi(\tilde{q}_i - \tilde{q}_j),$$

the first integral over p gives

$$C^N \beta^{-\frac{3}{2}N}.$$

Classical thermodynamics tells us that we should interpret β as an inverse temperature. The second part is

$$\int \underset{\Lambda_N}{\cdots} \int e^{-\beta\Sigma\Phi(\tilde{q}_i-\tilde{q}_j)}\,dq_1\ldots dq_{3N}.$$

It was proved only rather recently by Ruelle that $f(\beta)$ does indeed exist in a case like this. The problem of regularity of f is, however, still unsolved. Here we shall give some related results, noting that $f(\beta)$ is always analytic for small β.

The problems we have dealt with are closely related to a problem in probability, viz., the problem of large deviations. This was studied by Cramér and Feller and the following results which we need later are well known.

Let $X_1, X_2, \ldots, X_N$ be real stochastic variables with identical distribution and assume

$$E(e^{\lambda X}) = F(\lambda) < \infty .$$

We are interested in

$$\mathrm{Prob}\left(\sum_1^N X_j > tN\right) = e^{-N\mu_N(t)}, \qquad t > E(X).$$

Clearly,

$$F(\lambda)^N = -\int_{-\infty}^{\infty} e^{\lambda t N} d\left(e^{-N\mu_N(t)}\right) = \lambda N \int_{-\infty}^{\infty} e^{N(\lambda t - \mu_N(t))} dt .$$

Hence,

$$e^{N \log F(\lambda)} \geqq \lambda N \exp\{N(\inf_t(\lambda t - \mu_N(t)))\} \int_{-\infty}^{0} e^{N\lambda t} dt \simeq e^{N\mu_N^*(\lambda)}$$

and

$$e^{N \log F(\lambda)} \leqq e^{N\mu_N^*(\lambda)} \int_0^{N^2} dt = N^2 e^{N\mu_N^*(\lambda)} .$$

Therefore,

$$\mu_N^*(\lambda) = \log F(\lambda) + 0\left(\frac{\log N}{N}\right).$$

Since $\log F(\lambda)$ is smooth it follows that

$$\lim_{N\to\infty} \mu_N(t) = \sup_\lambda (\lambda t - \log F(\lambda)).$$

In a similar way one can compute high moments

$$E\left(\left(\frac{X_1 + \cdots + X_N}{N}\right)^{aN}\right) \sim e^{bN}, \qquad E(X) > 0.$$

One finds that

$$b = a \log a - a \log \lambda - a + \log E(e^{\lambda X})$$

$$\lambda \frac{E(Xe^{\lambda X})}{E(e^{\lambda X})} = a .$$

2. In the general case, the motion described by (1) is extremely complicated. Boltzmann introduced a random element in the description of the motion, which is highly plausible but just as difficult to verify. The classical theory concerns elastic collisions between particles assumed to occur in a random fashion. Here we shall present an extremely simple but at the same time very general model which contains some of the characteristics of the Boltzmann theory.

Suppose we have a system of n particles, each in one of N states, $N \ll n$. We should think of the state as a given position and velocity. At each time the particles in states (i,j) can interact and go over to the states (ν,μ). The proportion of particles (ν,μ) which arise in this way is

$$A_{ij}^{\nu\mu} p_i(t) p_j(t) \Delta t,$$

where $p_i(t)$ is the proportion of particles in state i at time t. The matrix $A_{ij}^{\nu\mu}$ is assumed to safisfy

$$(3) \qquad A_{ij}^{\nu\mu} = A_{ji}^{\mu\nu} = A_{\nu\mu}^{ij} \geqq 0, \qquad (i,j) \neq (\nu,\mu).$$

We set

$$(4) \qquad A_{\nu\mu}^{\nu\mu} = -\sum_{(i,j)\neq(\nu,\mu)} A_{ij}^{\nu\mu}.$$

For $p_\nu(t)$ we obtain in this way the differential equations

$$p_\nu'(t) = \sum_{i,j,\mu} A_{ij}^{\nu\mu} p_i(t) p_j(t),$$

which is a general discrete Boltzmann equation. It has many features of the usual equation, and the proofs are, of course, all very easy.

$$\text{(A)} \qquad \sum_1^N p_\nu(t) \equiv 1.$$

<u>Proof</u>. By (4) it follows that

$$\sum_1^N p_\nu'(t) = \sum_{i,j,\nu,\mu} A_{ij}^{\nu\mu} p_i(t) p_j(t) \equiv 0.$$

$$\text{(B)} \qquad p_\nu(t) \geqq 0.$$

<u>Proof</u>. Suppose first that $\alpha_\nu = p_\nu(0) > 0$ for all ν and that if $p_\nu(t) = 0$ then $p_\mu(t) \neq 0$, $\mu \neq \nu$. By analyticity this set of α_ν' s is dense. Suppose now that $p_\nu(t_0) = 0$ and $p_i(t) > 0$ for $0 \leq t < t_0$ and for all i. Then

$$p'_\nu(t) = p_\nu(t) \sum_{j,\mu} (A^{\nu\mu}_{\nu j} + A^{\nu\mu}_{j\nu})\, p_j + \sum_{i,j\neq\nu,\mu} A^{\nu\mu}_{ij} p_i p_j ,$$

i.e., an equation $p'_\nu = \varphi p_\nu + f$, where $f \geqq 0$ on $(0,t_o)$. Hence,

$$g(t) = p_\nu \exp\left\{-\int_0^t \varphi\, d\tau\right\}$$

is non-decreasing on $(0,t_o)$. Since $g(0) > 0$, it follows that $p_\nu(t_o) > 0$ which is a contradiction. The general case follows from density.

(C) $H(t) \equiv -\sum_1^N p_\nu(t) \log p_\nu(t)$ is non-decreasing.

Proof. $H'(t) = -\sum_{i,j,\nu,\mu} A^{\nu\mu}_{ij} p_i p_j \log p_\nu$

$$= -\frac{1}{2} \sum A^{\nu\mu}_{ij} p_i p_j (\log p_\nu + \log p_\mu)$$

$$= -\frac{1}{2} \sum A^{\nu\mu}_{ij} p_i p_j (\log p_\nu + \log p_\mu - \log p_i - \log p_j)$$

$$= -\frac{1}{4} \sum A^{\nu\mu}_{ij} (p_i p_j - p_\nu p_\mu) \log \frac{p_\nu p_\mu}{p_i p_j} \geqq 0 .$$

There is equality if and only if $p_i p_j = p_\nu p_\mu$ whenever $A^{\nu\mu}_{ij} \neq 0$.

(D) Let Λ be the linear space of vectors $\lambda = \{\lambda_\nu\}_1^N$ such that

$$\sum_{\nu=1}^N \lambda_\nu p_\nu(t) = \sum_{\nu=1}^N \lambda_\nu p_\nu(0)$$

for any choice of initial values $p_\nu(0)$. Λ is called the invariants of the motion. In classical theory they are the moments and the energy. Here we first have the trivial invariant $\lambda = \{1\}$.

$$\lambda \in \Lambda \text{ if and only if } A^{\nu\mu}_{ij} \neq 0 \Rightarrow \lambda_i + \lambda_j = \lambda_\nu + \lambda_\mu .$$

We can therefore interpret Λ as an additive invariant under possible interactions.

Proof. Assume Λ satisfies the condition. Then

$$\sum_1^N \lambda_\nu p'_\nu(t) = \sum A^{\nu\mu}_{ij} \lambda_\nu p_i p_j = \frac{1}{2} \sum A^{\nu\mu}_{ij} (\lambda_\nu + \lambda_\mu - \lambda_i - \lambda_j) p_i p_j = 0 .$$

Assume, conversely, that $\sum A^{\nu\mu}_{ij} \lambda_\nu p_i p_j \equiv 0$ for all $p_i \geq 0$ for which $\sum_1^N p_i \equiv 1$. We may also assume that $\sum \lambda_\nu = 0$. It follows that the quadratic form has to be a constant multiple of $(\Sigma p_i)^2$, i.e.,

$$\sum_{\nu,\mu} A_{ij}^{\nu\mu}(\lambda_\nu + \lambda_\mu) = C.$$

Consider

$$\sum A_{ij}^{\nu\mu}(\lambda_\nu + \lambda_\mu - \lambda_i - \lambda_j)^2 = \sum A_{ij}^{\nu\mu}[(\lambda_\nu + \lambda_\mu)^2 + (\lambda_i + \lambda_j)^2]$$

$$- 2 \sum A_{ij}^{\nu\mu}(\lambda_i + \lambda_j)(\lambda_\nu + \lambda_\mu).$$

The first sum vanishes. The second equals

$$-C \sum_{i,j} (\lambda_i + \lambda_j) = 0.$$

Hence,

$$\lambda_i + \lambda_j = \lambda_\nu + \lambda_\mu \quad \text{if} \quad A_{ij}^{\nu\mu} \neq 0.$$

(E) Let us now assume that the system is "ergodic" in the following sense. Let E be any set of indices. Let $\overline{E} = \{\nu \mid \exists\, \mu$ and $i,j \in E$ with $A_{ij}^{\nu\mu} \neq 0\}$. Then the system is called ergodic if for any set E, $E_1 = \overline{E}$, $E_2 = \overline{E}_1, \ldots, E_k = \overline{E}_{k-1}$, and E_k = all indices for k large enough. We choose $t_n \to \infty$ so that $p_\nu(t_n) \to \pi_\nu$. By (C),

(5) $$\pi_i\pi_j = \pi_\nu\pi_\mu \quad \text{if} \quad A_{ij}^{\nu\mu} \neq 0.$$

Let E be the set where $\pi_i \neq 0$. If $i,j \in E$ and $A_{ij}^{\nu\mu} \neq 0$ it follows that $\nu,\mu \in \overline{E}$. Hence, $E = \overline{E}$ and it follows that E = all indices, i.e., $\pi_i \neq 0$ for all i.

We have

$$-\sum \pi_\nu \log \pi_\nu = H(\infty)$$

and

$$\sum \pi_\nu\lambda_\nu = \sum p_\nu(0)\lambda_\nu.$$

By (5), $\log \pi_\nu$ is an invariant, i.e.,

$$\pi_\nu = \exp\{-\sum_\lambda c(\lambda)\lambda_\nu\}.$$

Finally, let x_ν solve the extremal problem

$$\sup(-\sum x_\nu \log x_\nu), \quad \sum x_\nu\lambda_\nu = \sum \pi_\nu\lambda_\nu \text{ and } \lambda \in \Lambda.$$

By the Lagrange theory we have

$$x_\nu = \exp\{-\sum_\lambda d(\lambda)\lambda_\nu\},$$

and x_ν is unique by Jensen's inequality. We have

$$\sum x_\nu \log \pi_\nu = \sum \pi_\nu \log \pi_\nu \quad \text{and} \quad \sum \pi_\nu \log x_\nu = \sum x_\nu \log x_\nu$$

since $\log \pi_\nu$ and $\log x_\nu$ are invariants. Hence,

$$0 = \sum(\pi_\nu \log x_\nu - x_\nu \log x_\nu + x_\nu \log \pi_\nu - \pi_\nu \log \pi_\nu)$$
$$= \sum(x_\nu - \pi_\nu)\log \frac{\pi_\nu}{x_\nu} \leq 0,$$

which gives $\pi_\nu = x_\nu$.

Let us summarize the result in a theorem.

Theorem. Let $(A_{ij}^{\nu\mu})$ be an ergodic transition matrix. The limits,

$$\lim_{t\to\infty} p_\nu(t) = \pi_\nu,$$

exist and $\pi_\nu > 0$. $\{\log \pi_\nu\}$ is an invariant and $\{\pi_\nu\}$ maximizes the entropy H for all distributions with given invariants.

II. The Harmonic Oscillator

1. We consider a model where a particle P_ν is placed at each point of a lattice. Many results would be true in the several dimensional case but for simplicity, let us assume that the lattice is $\mathbb{Z}$. The particles make small oscillations and the movement is governed by the Hamiltonian

$$H_N(p,q) = \frac{1}{2}\sum_1^N p_\nu^2 + \sum_1^N a_{\nu-\mu} q_\nu q_\mu = \frac{1}{2}|p|^2 + U(q).$$

We assume $a_\nu = a_{-\nu}$ and that $U(q) \geq 0$, i.e.,

$$A_N(x) = \sum_{-N}^N a_\nu e^{i\nu x} \geqq 0.$$

When $N \to \infty$, $A_N(x) \to A(x)$, and we assume $a_\nu \to 0$ sufficiently rapidly.

The Gibbs' theory is in this case trivial. The free energy is

$$N \log F_N(\beta) = \log\left\{\int e^{-\frac{\beta}{2}|p|^2} dp \int e^{-\beta U(q)} dq\right\} = -N \log \beta + C_N$$

so that $F(\beta) = C\beta$. The connection between energy and β is simple. We write

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