Lecture Notes in Mathematics 1588

Claus Scheiderer

Real and Étale Cohomology

$$\begin{array}{ccccc} X_{ret} & \xrightarrow{\nu} & X'_{et} & \xrightarrow{\pi} & X_{et} \\ r_{ret}\,\uparrow\,\downarrow\, q_{ret} & & \downarrow \pi & & q_{et}\,\downarrow\,\uparrow\, r_{et} \\ X_{ret}(G) & \xrightarrow{\nu(G)} & X_{et} = X'_{et}(G) & \xrightarrow{\pi(G)} & X_{et}(G) \end{array}$$

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Introduction

This work wants to reveal some of the intimate connections that exist between the étale site of a scheme X and the orderings of the residue fields of X. The emphasis is laid on cohomological aspects. It is well known that the existence of an ordering on some residue field influences directly the qualitative behavior of étale cohomology. For a basic example take X to be the Zariski spectrum of a number field k. If k is totally imaginary then $H^n_{et}(X,A) = 0$ for $n > 2$ and any torsion coefficients A. But if k is real then $H^n_{et}(X,\mathbb{Z}/2) \neq 0$ for any $n \geq 0$.

More generally, it is often felt that étale cohomology with 2-torsion coefficients has "bad" properties if there exists an ordering on some residue field of X; and so this case has frequently to be excluded or needs special consideration. The existence of an ordering implies $H^n_{et}(X,\mathbb{Z}/2) \neq 0$ for all n, no matter how well-behaved X is otherwise, and no matter what the cohomological 2-dimension $\mathrm{cd}_2(X'_{et})$ of $X' = X[\sqrt{-1}]$ is. The infinity of $\mathrm{cd}_2(X_{et})$ is in some sense accidental, and a more sensible cohomological dimension is exhibited only after killing all real phenomena by adjoining a square root of -1, i.e. passing to X'. But this being said, what then is the significance of the groups $H^n_{et}(X,A)$ for $n \gg 0$ and A 2-torsion?

I propose that there is a very satisfactory answer to this question. The orderings of all residue fields of X form a topological space X_r, the real spectrum of X. It turns out that étale cohomology of X (with 2-primary coefficients) is in high degrees just cohomology of the real spectrum X_r! This is not a very precise formulation, but for the moment it gives the right idea about one of the main results of this treatise.

Before I summarize the contents of this book more systematically, I would like to sketch what was the "point of departure" for this work, and then to highlight some of its main ideas.

I was drawn to the questions treated here when I studied the paper *Real components of algebraic varieties and étale cohomology* by Colliot-Thélène and Parimala [CTP]. Let X be an algebraic variety over the real numbers $\mathbb{R}$, and denote by $\mathcal{H}^n$ the Zariski sheaf on X associated with the presheaf $U \mapsto H^n_{et}(U,\mathbb{Z}/2)$. The main theorem of [CTP] says that if X is smooth, there are canonical isomorphisms $H^0(X,\mathcal{H}^n) \cong H^0(X(\mathbb{R}),\mathbb{Z}/2)$ for $n > \dim X$. More generally, the result is proved over arbitrary real closed fields, where classical topology is replaced by semialgebraic topology. An essential ingredient of the proof is quadratic form theory, and in particular, Mahé's theorem on the separation of real connected components by such forms.

When I was discussing some simplifications and generalizations with Colliot-Thélène, he pointed out to me Cox's paper on the étale homotopy type of $\mathbb{R}$-varieties [Co]. From Cox's results one could easily deduce the main theorem of [CTP] for arbitrary, not necessarily smooth $\mathbb{R}$-varieties. But what was needed from [Co] is obtained there through the complicated machinery of étale homotopy theory; moreover Cox's construction uses transcendental methods. Therefore it became desirable to find a different approach. Before proceeding further I want to recall Cox's theorem. Based on an idea of M. Artin, it says:

Theorem [Co]. — *Let X be a scheme of finite type over $\mathbb{R}$. Let $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$ act on $X(\mathbb{C})$ by conjugation. Then there is a weak homotopy equivalence*

$$\{X_{et}\}^\wedge \simeq \left(X(\mathbb{C})_G\right)^\wedge$$

of the pro-finite completions.

Here $\{X_{et}\}$ denotes the étale homotopy type of X in the sense of Artin-Mazur [AM], and $X(\mathbb{C})_G = EG \times_G X(\mathbb{C})$ is the total space of the fibre bundle over $BG = EG/G$ associated with $X(\mathbb{C})$, where EG is a free contractible G-space.

This can be seen as a real analogue of the comparison theorem of [AM] for the étale homotopy type of complex algebraic varieties. It implies in particular that étale cohomology of X with finite constant coefficients M can be calculated as G-equivariant cohomology on $X(\mathbb{C})$. In this way one derives a long exact sequence [Co, Prop. 1.2]

$$\cdots\ H^n\big(X(\mathbb{C})/G,\, X(\mathbb{R});\, M\big) \longrightarrow H^n_{et}(X, M) \longrightarrow H^n_G\big(X(\mathbb{R}),\, M\big)\ \cdots \qquad (1)$$

to which I will refer as to the "Cox sequence". Here $H^n_G\big(X(\mathbb{R}),\, M\big) = H^n\big(X(\mathbb{R}) \times BG,\, M\big)$, and both this group and the first group in (1) are singular cohomology groups. This sequence shows, in particular, that there are isomorphisms for $n > 2d$ (with $d := \dim X$)

$$H^n_{et}(X,\, \mathbb{Z}/2) \xrightarrow{\sim} \bigoplus_{i=0}^{d} H^i\big(X(\mathbb{R}),\, \mathbb{Z}/2\big) = H^*\big(X(\mathbb{R}),\, \mathbb{Z}/2\big). \qquad (2)$$

The present work grew out of an attempt to understand (1) and its consequences in a more elementary way. Besides I wanted to see whether (1) was special to $\mathbb{R}$-varieties, or rather whether it could be generalized to other situations. It was clear that in the hoped-for generalization the real spectrum of X would have to replace the space $X(\mathbb{R})$. It was less obvious, however, what could play the role of the quotient space $X(\mathbb{C})/G$. Also it wasn't clear initially how to give a purely algebraic construction of the homomorphisms $H^n_{et}(X, M) \to H^n_G(X_r, M)$; Cox's construction for $X/\mathbb{R}$ uses transcendental methods. The idea that (1) or (2) could possibly be generalized was supported by a theorem of Arason ([Ar], completed in [AEJ]), which says: If k is any field, $\operatorname{char} k \neq 2$, then there is a natural isomorphism

$$\lim_{n\to\infty} H^n_{et}(k, \mathbb{Z}/2) \xrightarrow{\sim} H^0(\operatorname{sper} k, \mathbb{Z}/2) = H^*(\operatorname{sper} k, \mathbb{Z}/2), \tag{3}$$

where $\operatorname{sper} k$ is the space of all orderings of k (the real spectrum of k) and the transition maps on the left are cup-product with the class of -1 in $k^*/k^{*2} \cong H^1_{et}(k, \mathbb{Z}/2)$. So one meets the same phenomenon here as for $\mathbb{R}$-varieties, namely that in high degrees, étale cohomology stabilizes against cohomology of the real spectrum.

The key to the desired generalizations, and more generally to a better understanding of the relationship between étale site and real spectrum, is to see the situation as an equivariant one. This is best explained by way of analogy with a space with operators. If T is a topological G-space (G being the group of order two, say), there is a well-developed theory which relates equivariant cohomology of G-sheaves on T to cohomology of both the space of fixpoints T^G and the quotient space T/G. For example, the sequence (1) is just an application of this theory to the G-space $X(\mathbb{C})$.

Let now X be a (general) scheme and consider the G-action on $X' := X \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{-1}]$ over X. If X has no points of characteristic 2, it follows by descent that étale sheaves on X are the same thing as G-equivariant étale sheaves on X'; and so étale cohomology of X can be identified with G-equivariant étale cohomology of X'. Now the real spectrum X_r of X (or rather, the topos $\widetilde{X}_r$ of sheaves on X_r) must be thought of as the "fixobject" (or "fixtopos") of the G-action on X'_{et}, in a similar way as $X(\mathbb{R}) = X(\mathbb{C})^G$ in the case of an algebraic $\mathbb{R}$-variety! Indeed, there is a natural topos morphism ν from $\widetilde{X}_r$ to $\widetilde{X}'_{et}$; although ν is by no ways an embedding, it plays in a precise sense the role of the inclusion of the fixpoints in a G-space. By means of ν one can pull back any étale sheaf A on X to a G-sheaf $\nu(G)^*A$ on the real spectrum X_r. This purely algebraic construction yields in particular homomorphisms in cohomology which generalize the right arrows in (1); and it works for arbitrary schemes and sheaves of coefficients.

To get something which corresponds to the quotient space one has to form a new Grothendieck topology, namely the intersection of the étale and the real étale topology of X. The study of this site, which is denoted by X_b, occupies a considerable part of this book. The topos of sheaves on X_b contains both $\widetilde{X}_{et}$ and $\widetilde{X}_r$ as full subcategories, in such a way that $\widetilde{X}_{et}$ is an open subtopos and $\widetilde{X}_r$ is its closed complement. In this way one arrives at a long exact sequence (Theorem (6.6)) which exists for arbitrary schemes over $\mathbb{Z}[\frac{1}{2}]$ and arbitrary sheaves of coefficients, and which generalizes the Cox sequence.

The obvious question becomes then, what can one say about the cohomological properties of X_b, in particular about its cohomological dimensions? The answer to this question includes also a comparison of the cohomology of X_r and of X'_{et}. Here it is shown that if $\frac{1}{2} \in \mathcal{O}(X)$, the functor ν_* (on abelian sheaves) is always exact; and so all cohomological dimensions of X_r are bounded above by those of X'_{et}. The main result of this paper (Section 7) says (in a simplified form) that X_b has the same cohomological dimension for 2-primary torsion sheaves as X'_{et}, or possible one higher. (There is a similar result for odd torsion sheaves, but this is easy to prove.) All one needs is that the scheme X is quasi-compact and quasi-separated, and that 2 is invertible on X. As a corollary one gets that the homomorphisms

$$H^n(X_{et}, A) \longrightarrow H^n_G(X_r, \nu(G)^* A) \tag{4}$$

are isomorphisms for $n > \mathrm{cd}_2(X'_{et})$ and any 2-primary sheaf A on X_{et}. In this sense, high-dimensional étale cohomology of X is cohomology of the real spectrum X_r. Actually the theorem makes a non-trivial statement also in the case $\mathrm{cd}_2(X'_{et}) = \infty$, since for A annihilated by 2 it asserts that (4) is an isomorphism "in the limit" $n \to \infty$. For example this shows that the localization of the cohomology ring $H^*(X_{et}, \mathbb{Z}/2) = H^*(X_{et}, \mu_2)$ with respect to the class $(-1) \in H^1(X_{et}, \mu_2)$ is isomorphic to the full cohomology ring $H^*(X_r, \mathbb{Z}/2)$ of the real spectrum!

A different line of investigation is taken up in the second part of the paper. The aim is to reach a better understanding of why the real spectrum X_r behaves so much like a fixobject of the G-action on the étale site of X'. For this one needs the notion of G-toposes, which are toposes with a (pseudo) G-action. The analogue of the space of fixpoints for a G-topos is its inverse limit topos (or "fixtopos"). It is characterized by a 2-categorical universal property. In Section 10 this inverse limit of a G-topos is explicitly constructed (G may be any finite group here), and in particular, its existence is shown. Then it is proved for an arbitrary scheme X with $\frac{1}{2} \in \mathcal{O}(X)$ that the real topos $\widetilde{X}_r$ is the inverse limit of the G-topos $\widetilde{X}'_{et}$.

It becomes therefore interesting to take other examples of G-toposes, try to determine their inverse limit toposes and see whether one can prove similar theorems about cohomology. Apart from the basic example of a topological G-space there is one major example studied here, namely group extensions: Any such extension

$$1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \tag{5}$$

makes the category (Δ-sets) into a G-topos. In the case where G is finite and the groups are discrete or profinite, the inverse limit of this G-topos is determined in Section 11.2.

The motivation for the study of this case, and for its inclusion into this paper, is as follows. The proof of the main theorem was reduced in Section 7 to the case of fields; and in Section 9 it was shown that for fields this main result can be deduced from Arason's theorem (3) mentioned above. Now the proof of the latter uses very specific (cohomological) properties of absolute Galois groups of fields. On the other hand, Arason's theorem fits exactly a general pattern which is given by theorems of K.S. Brown for *discrete* groups: If Γ is any discrete group of finite virtual cohomological dimension d, Brown has shown that cohomology of Γ in degrees $n > d$ is Γ-equivariant cohomology of a certain simplicial complex formed by finite subgroups of Γ. It is quite straightforward how a conjectural analogue of Brown's theorem for profinite groups should look like. In the case of the absolute Galois group of a field, this conjectural theorem would become just the main theorem of Section 9! This makes one wonder, of course, whether there is indeed such a profinite analogue of Brown's result. In fact, such a theorem exists. In Section 12, it is proved only under a special hypothesis, namely in the "rank one" case. Nevertheless, this includes absolute Galois groups of fields. In this way an independent and completely different approach to Arason's theorem is given, which does not use any of the special properties of Galois groups. The general profinite version of Brown's theorem will be proved in a forthcoming paper [Sch2].

Now the interesting point is this. Let the group G in (5) be cyclic of *prime* order p, and let $d := \mathrm{cd}_p(\Delta)$ be finite. The groups with which the cohomology groups $H^n(\Gamma)$ for $n > d$ are identified (by Brown's theorem in the discrete case, by Section 12 in the profinite case) are nothing but the G-equivariant cohomology groups of the fixtopos of the G-topos (Δ-sets). This follows from the identification of this fixtopos in 11.2. So from the perspective of G-toposes, the main results of Section 7 (for general schemes) and Brown's (discrete or profinite) theorem (in the case just considered) have *identical formulations*!

It is shown in Section 13 that also the (easier) case of a topological G-space fits the same pattern.

I now give a systematic overview of the contents of this paper. For some more specific information see also the introductory remarks to the respective sections.

In Section 1 it is proved that sheaves on the real spectrum X_r of a scheme are, up to equivalence, the same as sheaves on the real étale site X_{ret}. Although this theorem is well known, at least for affine schemes (proved by Coste-Roy and Coste), the proof given here is new and — as I think — more elementary than the proofs existing before. It uses neither methods from mathematical logic nor the concept of strict real localization. The result is basic for much of what follows, since it allows to switch freely between two quite different descriptions of the real topos of X, each of which has its specific advantages.

In Section 2 the b-topology of a scheme is defined as the intersection of the étale and the real étale topologies. It is shown that $\widetilde{X}_b$ (the category of sheaves on X_b) is the result of glueing $\widetilde{X}_{et}$ and $\widetilde{X}_{ret}$, and some consequences of this fact are obtained. Also the next section presents basic material: Limit theorems for sheaf cohomology, the description of the stalk functors, stalks of (higher) direct images and the like. Here as well as in later sections it happens frequently (but not always) that results for the b-topology are obtained by combining results for the étale and the real (étale) topologies. In most of the cases, the étale part is the harder one, of course.

The next topic is the study of the topos morphism ν from $\widetilde{X}_{ret}$ to $\widetilde{X}'_{et}$ in Section 5. Geometrically ν corresponds to the Weil restriction functor with respect to $X' \to X$. The main result is that ν_* is exact on abelian sheaves, provided that 2 is invertible on X. For this one has to study the real spectrum of the Weil restriction of a strictly henselian local ring, and to show that its sheaf cohomology vanishes. I could not decide whether this remains true when characteristic 2 points are present.

For lack of suitable references I have inserted before this a section in which all facts about Weil restrictions are proved which are used later on (Section 4).

In Section 6 the fundamental long exact sequence is established which relates cohomology of X_{et}, X_r and X_b, and which generalizes the Cox sequence (1). In Section 7 the proof of the main result on cohomology of X_b is taken up. Assuming that the theorem is true for fields, it is first extended to schemes of finite type over spec $\mathbb{Z}$, and then by limit and glueing arguments to general quasi-compact, quasi-separated schemes. The case of fields is treated in Section 9. For this one needs a third description of the real topos of a field k. The absolute Galois group Γ of k acts on the space T of all real closures of k (inside a fixed algebraic closure). If Γ' denotes the subgroup of Γ which fixes $\sqrt{-1}$ then sheaves on the real spectrum of k can be identified with Γ'-equivariant sheaves on T. Here a "continuous" notion of equivariant sheaves is required which takes care of the fact that Γ' carries a topology. Since I do not know any reference for this concept, I have again inserted a preparatory section (Section 8) in which some necessary foundations are laid. As already remarked, the proof of the main theorem is finally reduced to Arason's theorem (3). The proof of the latter is discussed at the end of Section 9, and the question is raised whether the specific cohomological properties of Galois groups which it uses are necessary for the theorem to hold.

In Section 12 it is shown that this is not the case. The approach there is completely different from Arason's. One considers a profinite group Γ which has an open subgroup Δ with $\mathrm{cd}_p(\Delta) < \infty$ (p may be any prime). Let $\mathfrak{T}$ be the (boolean) space of all subgroups of order p in Γ. Under the assumption that Γ contains no subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$, it is proved that the natural homomorphisms

$$H^n(\Gamma, A) \longrightarrow H^n_\Gamma(\mathfrak{T}, A)$$

are isomorphisms for $n > d$ and for any p-primary Γ-module A. As already remarked, this theorem has a counterpart for discrete groups, which is due to K.S. Brown and whose proof is based on Farrell cohomology. If one tries to imitate Brown's proof one runs into difficulties in the profinite case. The proof given here proceeds differently, but is limited to the case where Γ contains no $\mathbb{Z}/p \times \mathbb{Z}/p$ (but see [Sch2] for a proof covering the general case, based on the same ideas). In any case it is essential that one uses *projective* resolutions of modules, instead of injective ones. This cannot be done with discrete Γ-modules alone (there aren't enough projectives in the profinite case), rather one has to consider also profinite Γ-modules. The necessary backgroung material is summarized at the beginning of Section 12.

Before Section 12, however, there are two sections on G-toposes. In Section 10 I give a detailed review of the basic concepts, and then construct the fixtopos of an arbitrary G-topos E in the case where the group G is finite. If F denotes this fixtopos and $\nu: F \to E$ is the corresponding topos morphism, then all composite topos morphisms $g \circ \nu: F \to E$ $(g \in G)$ are "coherently" isomorphic (but not in general equal); and ν is universal for this property. This fixtopos is determined in Section 11 in the two cases which are important for this paper. Namely, if X is a scheme on which 2 is invertible, the fixtopos of $\widetilde{X}'_{et}$ is identified as the real topos $\widetilde{X}_r$. Thus a precise meaning is given here to the feeling that the real spectrum behaves like sort of a fixobject of the G-action on X'_{et}. Second, if (5) is an extension of discrete or profinite groups, and if G is finite, the fixtopos of the G-topos (Δ-sets) is shown to be the category of Δ-equivariant sheaves on the space of splittings of (5). Note how these two examples agree on their common "intersection", namely the spectrum (resp. Galois group) of a field.

The material on G-toposes may appear quite technical and "dry". But there seems no doubt that this perspective is essential if one wants to grasp the right idea about the relations between the sites X'_{et}, X_{et}, X_r and X_b. Moreover, if one assumes this point of view, one is rewarded in several ways: By analogy with topological G-spaces the main results of the first part become very natural, if not to be expected. Besides I found it quite satisfactory to see a common principle at work in so different situations. To describe the basic idea, consider a group G of *prime* order p which acts on a (nice) topological space T. If $Z = T^G$ is the set of fixpoints then the restriction maps $H^n_G(T, A) \to H^n_G(Z, A|_Z)$ in equivariant cohomology are bijective for $n > \dim T$. Alternatively one can read this as a statement on the cohomological dimension of T/G, by a long exact sequence similar to Cox's sequence. Now replace T by a G-topos E and Z by its fixtopos F. The idea is that there should be a good chance for an analogous theorem to hold. Indeed, the results of this paper give three different examples for such a situation: In the case of the G-topos $\widetilde{X}'_{et}$ (X a scheme over $\mathbb{Z}[\frac{1}{2}]$) the above principle corresponds to the main theorem of Section 7. In the case of a group extension (5), with $|G| = p$ and $\mathrm{cd}_p(\Delta) < \infty$, it corresponds to particular cases of Brown's theorem (for discrete groups) resp. of the main result of Section 12 (for profinite groups). Finally, if E is the topos of sheaves on a G-space T then F is the category of sheaves on $Z = T^G$, as shown in Section 13. For a summarizing discussion I refer to the end of Section 14.

Also the notion of quotient is considered for G-toposes. In general there does not seem to be an obvious topos-theoretic construction which corresponds to the quotient space T/G of a topological G-space T. In Section 14 I propose for G-

toposes E, in the case where G is of prime order, to form a glued topos by mimicking the construction of $\widetilde{X}_b$, and to regard this as a substitute for the topological quotient. That this yields the right thing in the situation of a topological G-space is proved in Section 13.

In the third part of this book I return to the study of the three topologies et, b, ret on a scheme X. Section 15 contains comparison results. For example it is shown that, for a (separated) scheme X of finite type over $\mathbb{R}$, the cohomology of X_b with torsion coefficients is the cohomology of the quotient space $X(\mathbb{C})/G$. A similar result holds over arbitrary real closed fields. This shows once more that X_b has to be considered as the "topological" (or "non-free") quotient of X'_{et} mod G. Also a very easy deduction of the Cox exact sequence (1) is given which does not need étale homotopy theory. However it uses the Comparison Theorem between classical and étale cohomology of complex varieties, which of course is also a non-trivial tool.

In Section 16 the proper and smooth base change theorems are proved for the real étale topology, and from this one gets corresponding theorems also for the b-topology (using the étale theorems, of course). Proper base change for real spectra has been proved before by H. Delfs; here a new proof is given. The smooth base change theorem seems to be new in the real setup. The reason why it hasn't been considered so far may be that smoothness doesn't make sense in the context of semi-algebraic spaces and maps (the structure sheaves are simply too large), nor in the more general framework of real closed spaces. I explain however how to weaken the smoothness hypothesis in such a way that one can prove a corresponding base change theorem for real closed spaces.

Section 17 contains finiteness theorems for the real and the b-topology, and for finitely presented proper morphisms. So these theorems are saying that the higher direct image functors of such morphisms preserve constructible abelian sheaves (and similarly for set-valued sheaves). As a corollary one gets a theorem on the smooth specialization of the cohomology of proper schemes. By making use of the étale finiteness theorem one can again reduce the study of the b-topology to the real topology. In the latter case the proof reduces to a semi-algebraic situation in which the finiteness theorem is proved without any properness hypothesis. The main technical tool is semi-algebraic triangulation.

The aim of Section 18 is to show, for a d-dimensional affine variety over a real closed field, that b-cohomology vanishes in degrees $> d$. Unfortunately it seems that there is no way of getting this result as a corollary to the étale case (which is well known) plus the real case (which is obvious). Rather one has to mimic the proof of the corresponding étale theorem. In particular this means to consider more generally a relative affine situation between varieties over a field.

Section 19 relates the three topologies on a scheme X to the Zariski topology. In particular it is proved that the direct image functor of the support map $X_r \to X$ is exact on abelian sheaves. This simple fact is quite useful, as is demonstrated by various applications. Among them are very far reaching generalizations of results of Colliot-Thélène and Parimala from [CTP].

The last section (Section 20) is devoted to some explicit computations and applications. First, smooth curves over a real closed field are considered and their étale and b-cohomology (with coefficients $\mu_n^{\otimes i}$) is determined. Also, classical theorems on real curves by Weichold, Witt and Geyer are reproved by means of results of this work. Then it is studied (on general schemes) what one gets from the results of Part One for some specific étale sheaves. Most interesting is here the case of the multiplicative sheaf $\mathbb{G}_m$; its n-th étale cohomology group, for $n \gg 0$, is the part of $H^*(X_r, \mathbb{Z}/2)$ of the same parity as n. This subsection contains also a variety of side remarks and other complements. Then fields are considered again: Some remarks are made on the b-cohomology of a field, and on the fundamental group of the b-topology. At the end of Section 20 a few historical remarks are made on the relations of this work to work of other authors.

There are two appendices. The first assembles results on spectral spaces which are used in several places throughout the paper and which I could not find in the literature. The second is an application of results of this paper to Artin-Schreier structures. The notion of an Artin-Schreier structure was invented by D. Haran and M. Jarden in the course of their study of absolute Galois groups of *PRC* fields, where it plays a key role [HJ]. For example, every field k gives rise to an Artin-Schreier structure $\mathfrak{A}(k)$, which consists essentially of the absolute Galois group $\mathrm{Gal}(k_s/k)$ together with its distinguished subgroup $\mathrm{Gal}(k_s/k(\sqrt{-1}))$ and its action on the space of real closures of k. In [Ha3] Haran proposes a cohomology theory for these structures. Among its features is that it yields reasonable (finite) cohomological 2-dimensions for real fields, and that it allows a cohomological characterization of real projective groups. (The class of these groups comprises exactly the class of absolute Galois groups of *PRC* fields [HJ].) For non-real fields this cohomology coincides with Galois cohomology. Haran's cohomology is defined by *ad hoc* methods, and the definition is somewhat mysterious, as the author remarks by himself. In Appendix B a natural explanation of this cohomology theory is given in terms of the b-topology. For example, in the case of the Artin-Schreier structure $\mathfrak{A}(k)$ of a field, Haran's cohomology coincides with sheaf cohomology on the site $(\mathrm{spec}\, k)_b$; and a similar characterization is true in general. By the results of Section 9 (in the field case) or Section 12 (in general) it is an easy corollary to determine the cohomological dimensions of arbitrary Artin-Schreier structures. A particular corollary is that a field k is real projective if and only if $k(\sqrt{-1})$ is projective. These latter results were proved independently by Haran, who used different methods [Ha4].

After this introduction the reader can find a summary of the general notations, definitions and conventions valid in this paper. This part is referred to by labels of the form $(0.xx)$. Otherwise the first entry of a label indicates the number of the section in which it can be found.

I should add the remark that, although this paper is devoted to cohomological studies, non-abelian cohomology has not been considered anywhere in it. This mainly in order to save space and time.

This book is a slightly revised version of my Habilitationsschrift with the same title, which was submitted to the Mathematische Fakultät of Universität Regensburg in November 1992. The idea to think about relations between étale cohomology and the real spectrum owes a lot to Louis Mahé, whom I want to thank warmly for stimulating questions and discussions through which he whetted my appetite. He always stressed the similarity between étale cohomology classes and quadratic forms. Although later this work turned into somewhat different directions, it was Louis who had shown me in his friendly way that here is something to think about. I am also grateful to Dan Haran for sending me his papers and preprints; in particular I profited from his exposition of profinite group modules in [Ha2].

Moreover, I am indebted to several people who critically read the whole manuscript or parts of it, for their constructive criticism. Above all, Jean-Louis Colliot-Thélène contributed a great number of detailed suggestions and comments which were incorporated into this revision and improved the presentation. I also received valuable advice from Claudio Casanova, Michel Coste, Manfred Knebusch, Louis Mahé, Manuel Ojanguren and Jean-Pierre Serre. It is a pleasure to thank all of them here.

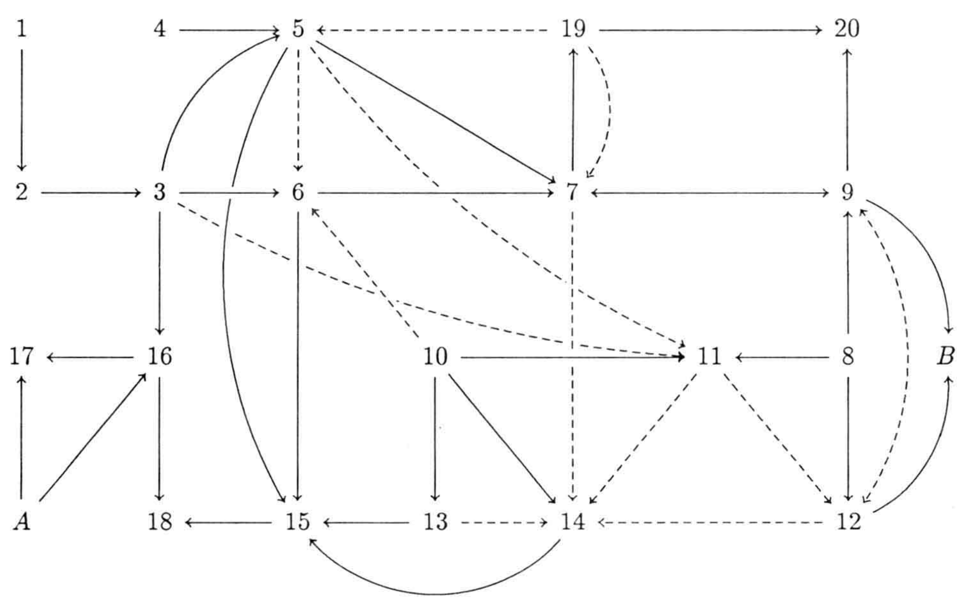
Finally I would like to express my special gratitude to Manfred Knebusch, who has actively supported and encouraged me for many years now.

Leitfaden

The following diagram indicates the main relations between the sections of the book. An arrow from X to Y signalizes that results or techniques of Section X are used in Section Y. A dotted such arrow indicates a weaker sort of dependency.

The reader should start with Sections 1-3, and then proceed with Sections 5, 6, 7 and 9 in order, using Sections 4 resp. 8 as technical references for Sections 5 resp. 9. The lecture of most of Section 5 may be skipped at a first reading, if one is willing to accept in Section 7 that Theorem (5.9) is true.

After the study of Part One, the reader may proceed with either Part Two or Part Three (if he or she hasn't lost any interest at all), or may even pass directly to Sections 19 and 20, where applications of Part One can be found. Generally, Part Two emphasizes topos theoretic techniques; but Section 12 doesn't use such techniques, and can practically be read independently of the rest of the book, perhaps with a glance into Section 8 at some points. (See the Introduction for why this section is placed here.) Sections 15-18 are devoted to fundamental theorems for the real and the b-topology, and do neither depend seriously on Part Two nor on the main results from Section 7.



General notations and conventions

The following serves the purpose of fixing notations and conventions which are used throughout the paper. It also gives the precise sense in which some general concepts are used later on, either by explicitly recalling definitions or by giving references to the literature.

(0.1) Categories and functors

In general I have tried to maintain definitions and notations from [SGA4 I]. Throughout all set-theoretic questions are ignored. In particular, no attempt has been made to keep track of a hierarchy of universes, as is done in [SGA4].

(0.1.1) Let C be a category. Examples are

$$\begin{aligned} \text{(sets)} &= \text{the category of sets,} \\ \text{(Ab)} &= \text{the category of abelian groups,} \\ \text{(Top)} &= \text{the category of topological spaces,} \end{aligned}$$

the morphisms being the obvious ones. I often write "$x \in C$" to indicate that x is an object of C. The set of C-arrows from x to y is denoted $\mathrm{Hom}_C(x,y)$, or simply $\mathrm{Hom}(x,y)$ if C is clear from the context; or occasionally by $y(x)$. If $x \in C$ then C/x is the category of C-objects over x, i.e. the objects of C/x are the C-arrows with target x.

(0.1.2) C° is the opposite category of C (same objects but arrows reversed). If D is a second category then $\underline{\mathrm{Hom}}(C,D)$ is the category of functors $C \to D$ (with morphisms of functors as arrows). Note that all functors are "covariant"; thus a "contravariant" functor from C to D is either a functor $C^\circ \to D$ or a functor $C \to D^\circ$. Let a diagram of categories and functors

$$C' \xrightarrow{\ g\ } C \underset{}{\overset{f,f'}{\rightrightarrows}} D \xrightarrow{\ h\ } D'$$

be given. If $\varphi\colon f \to f'$ is a morphism of functors then $\varphi * g$ denotes the morphism of functors $f \circ g \to f' \circ g$ induced by φ; and $h * \varphi$ is defined similarly. This notation is not used consequently, however.

(0.1.3) One puts

$$\widehat{C} := \underline{\mathrm{Hom}}(C^\circ, \text{(sets)}) \quad \text{and} \quad C^\vee := \underline{\mathrm{Hom}}(C, \text{(sets)});$$