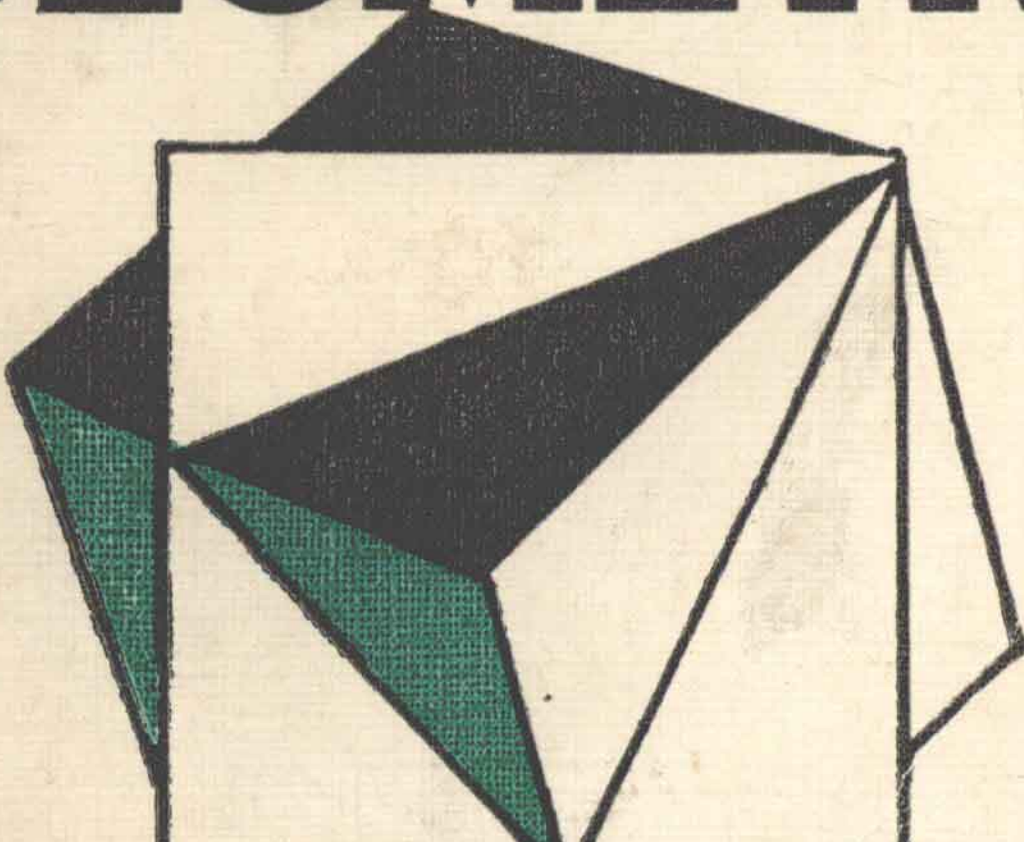
SCIENCE FOR EVERYONE

I.F. SHARYGIN

PROBLEMS IN SOLID GEOMETRY



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Стереометрия

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Preface

This book contains 340 problems in solid geometry and is a natural continuation of *Problems in Plane Geometry*, Nauka, Moscow, 1982. It is therefore possible to confine myself here to those points where this book differs from the first.

The problems in this collection are grouped into (1) computational problems and (2) problems on proof.

The simplest problems in Section 1 only have answers, others, have brief hints, and the most difficult, have detailed hints and worked solutions. There are two reservations. Firstly, in most cases only the general outline of the solution is given, a number of details being suggested for the reader to consider. Secondly, although the suggested solutions are valid, they are not patterns (models) to be used in examinations.

Sections 2-4 contain various geometric facts and theorems, problems on maximum and minimum (some of the problems in this part could have been put in Section 1), and problems on loci. Some questions pertaining to the geometry of tetrahedron, spherical geometry, and so forth are also considered here.

As to the techniques for solving all these problems, I have to state that I prefer analytical computational methods to those associated with plane geometry. Some of the difficult problems in solid geometry will require a high level of concentration from the reader, and an ability to carry out some rather complicated work.

The Author

Section 1

Computational Problems

1. Given a cube with edge a. Two vertices of a regular tetrahedron lie on its diagonal and the two remaining vertices on the diagonal of its face. Find the volume of the tetrahedron.

2. The base of a quadrangular pyramid is a rectangle, the altitude of the pyramid is h. Find the volume of the pyramid if it is known that all five of its faces are equivalent.

3. Among pyramids having all equal edges (each of length a), find the volume of the one which has the greatest number of edges.

4. Circumscribed about a ball is a frustum of a regular quadrangular pyramid whose slant height is equal to a. Find its lateral surface area.

5. Determine the vertex angle of an axial section of a cone if its volume is three times the volume of the ball inscribed in it.

6. Three balls touch the plane of a given triangle at the vertices of the triangle and one another. Find the radii of these balls if the sides of the triangle are equal to a, b, and c.

7. Find the distance between the skew diagonals of two neighbouring faces of a cube with edge a. In what ratio is each of these diagonals divided by their common perpendicular?

8. Prove that the area of the projection of a polygon situated in the plane α on the plane β

is equal to $S \cos \varphi$, where S denotes the plane of the polygon and φ the angle between the planes α and β.

9. Given three straight lines passing through one point A. Let B_1 and B_2 be two points on one line, C_1 and C_2 two points on the other, and D_1 and D_2 two points on the third line. Prove that

$$\frac{V_{AB_1C_1D_1}}{V_{AB_2C_2D_2}} = \frac{|AB_1| \cdot |AC_1| \cdot |AD_1|}{|AB_2| \cdot |AC_2| \cdot |AD_2|}.$$

10. Let α, β, and γ denote the angles formed by an arbitrary straight line with three pairwise perpendicular lines. Prove that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

11. Let S and P denote the areas of two faces of a tetrahedron, a the length of their common edge, and α the dihedral angle between them. Prove that the volume V of the tetrahedron can be found by the formula

$$V = \frac{2SP \sin \alpha}{3a}.$$

12. Prove that for the volume V of an arbitrary tetrahedron the following formula is valid: $V = \frac{1}{6} abd \sin \varphi$, where a and b are two opposite edges of the tetrahedron, d the distance between them, and φ the angle between them.

13. Prove that the plane bisecting the dihedral angle at a certain edge of a tetrahedron divides the opposite edge into parts proportional to the areas of the faces enclosing this angle.

14. Prove that for the volume V of the polyhedron circumscribed about a sphere of radius R

the following equality holds: $V = \frac{1}{3} S_n R$, where S_n is the total surface area of the polyhedron.

15. Given a convex polyhedron all of whose vertices lie in two parallel planes. Prove that its volume can be computed by the formula

$$V = \frac{h}{6}(S_1 + S_2 + 4S),$$

where S_1 is the area of the face situated in one plane, S_2 the area of the face situated in the other plane, S the area of the section of the polyhedron by the plane equidistant from the two given planes, and h is the distance between the given planes.

16. Prove that the ratio of the volumes of a sphere and a frustum of a cone circumscribed about it is equal to the ratio of their total surface areas.

17. Prove that the area of the portion of the surface of a sphere enclosed between two parallel planes cutting the sphere can be found by the formula

$$S = 2\pi Rh,$$

where R is the radius of the sphere and h the distance between the planes.

18. Prove that the volume of the solid generated by revolving a circular segment about a nonintersecting diameter can be computed by the formula

$$V = \frac{1}{6}\pi a^2 h,$$

where a is the length of the chord of this segment and h the projection of this chord on the diameter.

19. Prove that the line segments connecting the vertices of a tetrahedron with the median points of opposite faces intersect at one point (called the centre of gravity of the tetrahedron) and are divided by this point in the ratio 3 : 1 (reckoning from the vertices).

Prove also that the line segments joining the midpoints of opposite edges intersect at the same point and are bisected by this point.

20. Prove that the straight lines joining the midpoint of the altitude of a regular tetrahedron to the vertices of the face onto which this altitude is dropped are pairwise perpendicular.

21. Prove that the sum of the squared lengths of the edges of a tetrahedron is four times the sum of the squared distances between the midpoints of its skew edges.

22. Given a cube $ABCDA_1B_1C_1D_1$* with an edge a, in which K is the midpoint of the edge DD_1. Find the angle and the distance between the straight lines CK and A_1D.

23. Find the angle and the distance between two skew medians of two lateral faces of a regular tetrahedron with edge a.

24. The base of the pyramid $SABCD$ is a quadrilateral $ABCD$. The edge SD is the altitude of the pyramid. Find the volume of the pyramid if it is known that $|AB| = |BC| = \sqrt{5}$, $|AD| =$

* $ABCD$ and $A_1B_1C_1D_1$ are two faces of the cube, AA_1, BB_1, CC_1, DD_1 are its edges.

$|DC| = \sqrt{2}$, $|AC| = 2$, $|SA| + |SB| = 2 + \sqrt{5}$.

25. The base of a pyramid is a regular triangle with side a, the lateral edges are of length b. Find the radius of the ball which touches all the edges of the pyramid or their extensions.

26. A sphere passes through the vertices of one of the faces of a cube and touches the sides of the opposite faces of the cube. Find the ratio of the volumes of the ball and the cube.

27. The edge of the cube $ABCDA_1B_1C_1D_1$ is equal to a. Find the radius of the sphere passing through the midpoints of the edges AA_1, BB_1, and through the vertices A and C_1.

28. The base of a rectangular parallelepiped is a square with side a, the altitude of the parallelepiped is equal to b. Find the radius of the sphere passing through the end points of the side AB of the base and touching the faces of the parallelepiped parallel to AB.

29. A regular triangular prism with a side of the base a is inscribed in a sphere of radius R. Find the area of the section of the prism by the plane passing through the centre of the sphere and the side of the base of the prism.

30. Two balls of one radius and two balls of another radius are arranged so that each ball touches three other balls and a given plane. Find the ratio of the radii of the greater and smaller balls.

31. Given a regular tetrahedron $ABCD$ with edge a. Find the radius of the sphere passing through the vertices C and D and the midpoints of the edges AB and AC.

32. One face of a cube lies in the plane of the base of a regular triangular pyramid. Two vertices of the cube lie on one of the lateral faces of the pyramid and another two on the other two faces (one vertex per face). Find the edge of the cube if the side of the base of the pyramid is equal to a and the altitude of the pyramid is h.

33. The dihedral angle at the base of a regular n-gonal pyramid is equal to α. Find the dihedral angle between two neighbouring lateral faces.

34. Two planes are passed in a triangular prism $ABCA_1B_1C_1$*: one passes through the vertices A, B, and C_1, the other through the vertices A_1, B_1, and C. These planes separate the prism into four parts. The volume of the smallest part is equal to V. Find the volume of the prism.

35. Through the point situated at a distance a from the centre of a ball of radius R ($R > a$), three pairwise perpendicular chords are drawn. Find the sum of the squared lengths of the segments of the chords into which they are divided by the given point.

36. The base of a regular triangular prism is a triangle ABC with side a. Taken on the lateral edges are points A_1, B_1, and C_1 situated at distances $a/2$, a, and $3a/2$, respectively, from the plane of the base. Find the angle between the planes ABC and $A_1B_1C_1$.

37. The side of the base of a regular quadrangular pyramid is equal to the slant height of a lateral face. Through a side of the base a cutting plane is passed separating the surface of the pyra-

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* Here and henceforward, ABC and $A_1B_1C_1$ are the bases of the prism and AA_1, BB_1, CC_1 its lateral edges.

mid into two equal portions. Find the angle between the cutting plane and the plane of the base of the pyramid.

38. The centre of a ball is found in the plane of the base of a regular triangular pyramid. The vertices of the base lie on the surface of the ball. Find the length l of the line of intersection of the surfaces of the ball and pyramid if the radius of the ball is equal to R, and the plane angle at the vertex of the pyramid is equal to α.

39. In a regular hexagonal pyramid $SABCDEF$ (S the vertex), on the diagonal AD, three points are taken which divide the diagonal into four equal parts. Through these division points sections are passed parallel to the plane SAB. Find the ratios of the areas of the obtained sections.

40. In a regular quadrangular pyramid, the plane angle at the vertex is equal to the angle between the lateral edges and the plane of the base. Determine the dihedral angles between the adjacent lateral faces of this pyramid.

41. The base of a triangular pyramid all of whose lateral edges are pairwise perpendicular is a triangle having an area S. The area of one of the lateral faces is Q. Find the area of the projection of this face on the base.

42. $ABCA_1B_1C_1$ is a regular triangular prism all of whose edges are equal to one another. K is a point on the edge AB different from A and B, M is a point on the straight line B_1C_1, and L is a point in the plane of the face ACC_1A_1. The straight line KL makes equal angles with the planes ABC and ABB_1A_1, the line LM makes equal angles with the planes BCC_1B_1 and ACC_1A_1, the line KM also makes equal angles with the planes BCC_1B_1 and ACC_1A_1. It is known that $|KL| = |KM| = 1$. Find the edge of the prism.

43. In a regular quadrangular pyramid, the angle between the lateral edges and the plane of the base is equal to the angle between a lateral edge and a plane of the lateral face not containing this edge. Find this angle.

44. Find the dihedral angle between the base and a lateral face of a frustum of a regular triangular pyramid if it is known that a ball can be inscribed in it, and, besides, there is a ball which touches all of its edges.

45. Each of three edges of a triangular pyramid is equal to 1, and each of three other edges is equal to a. None of the faces is a regular triangle. What is the range of variation of a? What is the volume of this pyramid?

46. The lateral faces of a triangular pyramid are equivalent and are inclined to the plane of the base at angles α, β, and γ. Find the ratio of the radius of the ball inscribed in this pyramid to the radius of the ball touching the base of the pyramid and the extensions of the three lateral faces.

47. All edges of a regular hexagonal prism are equal to a (each). Find the area of the section passed through a side of the base at an angle α to the plane of the base.

48. In a rectangular parallelepiped $ABCDA_1B_1C_1D_1$, $|AB| = a$, $|AD| = b$, $|AA_1| = c$. Find the angle between the planes AB_1D_1 and A_1C_1D.

49. The base of the pyramid $ABCDM$ is a square with base a, the lateral edges AM and BM are also equal to a (each). The lateral edges CM

and DM are of length b. On the face CDM as on the base a triangular pyramid $CDMN$ is constructed outwards, each lateral edge of which has a length a. Find the distance between the straight lines AD and MN.

50. In a tetrahedron, one edge is equal to a, the opposite edge to b, and the rest of the edges to c. Find the radius of the circumscribed ball.

51. The base of a triangular pyramid is a triangle with sides a, b, and c; the opposite lateral edges of the pyramid are respectively equal to m, n, and p. Find the distance from the vertex of the pyramid to the centre of gravity of the base.

52. Given a cube $ABCDA_1B_1C_1D_1$; through the edge AA_1 a plane is passed forming equal angles with the straight lines BC_1 and B_1D. Find these angles.

53. The lateral edges of a triangular pyramid are pairwise perpendicular, one of them being the sum of two others is equal to a. Find the radius of the ball touching the base of the pyramid and the extensions of its lateral faces.

54. The base of a triangular pyramid $SABC$ is a regular triangle ABC with side a, the edge SA is equal to b. Find the volume of the pyramid if it is known that the lateral faces of the pyramid are equivalent.

55. The base of a triangular pyramid $SABC$ is an isosceles triangle ABC ($\hat{A} = 90°$). The angles $\widehat{SAB}$, $\widehat{SCA}$, $\widehat{SAC}$, $\widehat{SBA}$ (in the indicated order) form an arithmetic progression whose difference is not equal to zero. The areas of the faces

SAB, ABC and SAC form a geometric progression. Find the angles forming an arithmetic progression.

56. The base of a triangular pyramid $SABC$ is a regular triangle ABC with side a. Find the volume of this pyramid if it is known that $\widehat{ASC} = \widehat{ASB} = \alpha$, $\widehat{SAB} = \beta$.

57. In the cube $ABCDA_1B_1C_1D_1$ K is the midpoint of the edge AA_1, the point L lies on the edge BC. The line segment KL touches the ball inscribed in the cube. In what ratio is the line segment KL divided by the point of tangency?

58. Given a tetrahedron $ABCD$ in which $\widehat{ABC} = \widehat{BAD} = 90°$. $|AB| = a$, $|DC| = b$, the angle between the edges AD and BC is equal to α. Find the radius of the circumscribed ball.

59. An edge of a cube and an edge of a regular tetrahedron lie on the same straight line, the midpoints of the opposite edges of the cube and tetrahedron coincide. Find the volume of the common part of the cube and tetrahedron if the edge of the cube is equal to a.

60. In what ratio is the volume of a triangular pyramid divided by the plane parallel to its two skew edges and dividing one of the other edges in the ratio $2 : 1$?

61. In a frustum of a regular quadrangular pyramid two sections are drawn: one through the diagonals of the bases, the other through the side of the lower base and opposite side of the upper base. The angle between the cutting planes is