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The Theory of Jacobi Forms

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TABLE OF CONTENTS

INTRODUCTION

The functions studied in this monograph are a cross between elliptic functions and modular forms in one variable. Specifically, we define a *Jacobi form on* $\mathrm{SL}_2(\mathbb{Z})$ to be a holomorphic function

$$\phi: \mathcal{H} \times \mathbb{C} \to \mathbb{C} \qquad (\mathcal{H} = \text{upper half-plane})$$

satisfying the two transformation equations

$$(1) \qquad \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k \, e^{\frac{2\pi i m c z}{c\tau+d}} \, \phi(\tau,z) \qquad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\right),$$

$$(2) \qquad \phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)} \, \phi(\tau,z) \qquad ((\lambda\ \mu) \in \mathbb{Z}^2)$$

and having a Fourier expansion of the form

$$(3) \qquad \phi(\tau,z) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm}} c(n,r) \, e^{2\pi i(n\tau+rz)} \, .$$

Here k and m are natural numbers, called the *weight* and *index* of ϕ, respectively. Note that the function $\phi(\tau,0)$ is an ordinary modular form of weight k, while for fixed τ the function $z \to \phi(\tau,z)$ is a function of the type normally used to embed the elliptic curve $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ into a projective space.

If $m = 0$, then ϕ is independent of z and the definition reduces to the usual notion of modular forms in one variable. We give three other examples of situations where functions satisfying (1)-(3) arise classically:

1. Theta series. Let $Q: \mathbb{Z}^N \to \mathbb{Z}$ be a positive definite integer valued quadratic form and B the associated bilinear form. Then for any vector $x_0 \in \mathbb{Z}^N$ the theta series

$$(4) \qquad \Theta_{x_0}(\tau,z) = \sum_{x \in \mathbb{Z}^N} e^{2\pi i(Q(x)\tau + B(x,x_0)z)}$$

is a Jacobi form (in general on a congruence subgroup of $SL_2(\mathbb{Z})$) of weight N/2 and index $Q(x_0)$; the condition $r^2 \leqq 4nm$ in (3) arises from the fact that the restriction of Q to $\mathbb{Z}x + \mathbb{Z}x_0$ is a positive definite binary quadratic form. Such theta series (for N=1) were first studied by Jacobi [10], whence our general name for functions satisfying (1) and (2).

2. Fourier coefficients of Siegel modular forms. Let F(Z) be a Siegel modular form of weight k and degree 2. Then we can write Z as $\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ with $z \in \mathbb{C}$, $\tau, \tau' \in \mathcal{H}$ (and $\mathrm{Im}(z)^2 < \mathrm{Im}(\tau)\mathrm{Im}(\tau')$), and the function F is periodic in each variable τ, z and τ'. Write its Fourier expansion with respect to τ' as

$$F(Z) = \sum_{m=0}^{\infty} \phi_m(\tau, z)\, e^{2\pi i m \tau'} \quad ; \tag{5}$$

then for each m the function ϕ_m is a Jacobi form of weight k and index m, the condition $4nm \geqq r^2$ in (3) now coming from the fact that F has a Fourier development of the form $\sum c(T)\, e^{2\pi i\, \mathrm{Tr}(TZ)}$ where T ranges over positive semi-definite symmetric 2×2 matrices. The expansion (5) (and generalizations to other groups) was first studied by Piatetski-Shapiro [26], who referred to it as the *Fourier-Jacobi expansion* of F and to the coefficients ϕ_m as Jacobi functions, a word which we will reserve for (meromorphic) quotients of Jacobi forms of the same weight and index, in accordance with the usual terminology for modular forms and functions.

3. The Weierstrass $\wp$-function. The function

$$\wp(\tau, z) = z^{-2} + \sum_{\substack{\omega \in \mathbb{Z}+\mathbb{Z}\tau \\ \omega \neq 0}} \left((z+\omega)^{-2} - \omega^{-2}\right) \tag{6}$$

is a meromorphic Jacobi form of weight 2 and index 0; we will see

later how to express it as a quotient of holomorphic Jacobi forms (of index 1 and weights 12 and 10).

Despite the importance of these examples, however, no systematic theory of Jacobi forms along the lines of Hecke's theory of modular forms seems to have been attempted previously.* The authors' interest in constructing such a theory arose from their attempts to understand and extend Maass' beautiful work on the "Saito-Kurokawa conjecture". This conjecture, formulated independently by Saito and by Kurokawa [15] on the basis of numerical calculations of eigenvalues of Hecke operators for the (full) Siegel modular group, asserted the existence of a "lifting" from ordinary modular forms of weight 2k-2 (and level one) to Siegel modular forms of weight k (and also level one); in a more precise version, it said that this lifting should land in a specific subspace of the space of Siegel modular forms (the so-called Maass "Spezialschar", defined by certain identities among Fourier coefficients) and should in fact be an isomorphism from $M_{2k-2}(SL_2(\mathbb{Z}))$ onto this space, mapping Eisenstein series to Eisenstein series, cusp forms to cusp forms, and Hecke eigenforms to Hecke eigenforms. Most of this conjecture was proved by Maass [21,22,23], another part by Andrianov [2], and the remaining part by one of the authors [40]. It turns out that the

* Shimura [31,32] has studied the same functions and also their higher-dimensional generalizations. By multiplication by appropriate elementary factors they become modular functions in τ and elliptic (resp. Abelian) functions in z, although non-analytic ones. Shimura used them for a new foundation of complex multiplication of Abelian functions. Because of the different aims Shimura's work does not overlap with ours. We also mention the work of R.Berndt [3,4], who studied the quotient field (field of Jacobi functions) from both an algebraic-geometrical and arithmetical point of view. Here, too, the overlap is slight since the field of Jacobi functions for $SL_2(\mathbb{Z})$ is easily determined (it is generated over $\mathbb{C}$ by the modular invariant $j(\tau)$ and the Weierstrass $\wp$-function $\wp(\tau,z)$); Berndt's papers concern Jacobi functions of higher level. Finally, the very recent paper of Feingold and Frenkel [Math. Ann. **263**, 1983] on Kac-Moody algebras uses functions equivalent to our Jacobi forms, though with a very different motivation; here there is some overlap of their results and our §9 (in particular, our Theorem 9.3 seems to be equivalent to their Corollary 7.11).

conjectured correspondence is the composition of three isomorphisms

$$
\begin{array}{c}
\text{Maass "Spezialschar"} \subset M_k(Sp_4(\mathbb{Z})) \\
\downarrow\wr \\
\text{Jacobi forms of weight } k \text{ and index } 1 \\
\downarrow\wr \\
\text{Kohnen's "+"-space ([11])} \subset M_{k-\frac{1}{2}}(\Gamma_0(4)) \\
\downarrow\wr \\
M_{2k-2}(SL_2(\mathbb{Z})) \qquad ;
\end{array}
\tag{7}
$$

the first map associates to each F the function ϕ_1 defined by (5), the second is given by

$$
\sum_{n\geq 0} c(n)\, e^{2\pi i n\tau} \longrightarrow \sum_{n\geq 0} \sum_{r^2\leq 4n} c(4n-r^2)\, e^{2\pi i(n\tau+rz)} \quad ,
$$

and the third is the Shimura correspondence [29,30] between modular forms of integral and half-integral weight, as sharpened by Kohnen [11] for the case of forms of level 1.

One of the main purposes of this work will be to explain diagram (7) in more detail and to discuss the extent to which it generalizes to Jacobi forms of higher index. This will be carried out in Chapters I and II, in which other basic elements of the theory (Eisenstein series, Hecke operators, ...) are also developed. In Chapter III we will study the bigraded ring of all Jacobi forms on $SL_2(\mathbb{Z})$. This is much more complicated than the usual situation because, in contrast with the classical isomorphism $M_*(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$, the ring $J_{*,*} = \bigoplus_{k,m} J_{k,m}$ ($J_{k,m}$ = Jacobi forms of weight k and index m) is not finitely generated. Nevertheless, we will be able to obtain considerable information about the structure of $J_{*,*}$. In particular, we will find upper and lower bounds for $\dim J_{k,m}$ which agree for k sufficiently large ($k \geq m$), will prove that $J_{*,m} = \bigoplus_k J_{k,m}$ is a free module of rank $2m$ over the ring $M_*(SL_2(\mathbb{Z}))$, and will describe explicit algorithms for finding

bases of $J_{k,m}$ as a vector space over $\mathbb{C}$ and of $J_{*,m}$ as a module over $M_*(SL_2(\mathbb{Z}))$. The dimension formula obtained has the form

$$\dim J_{k,m} = \sum_{r=0}^{m} \dim M_{k+2r} - N(m) \tag{8}$$

for k even (and sufficiently large), where $N(m)$ is given by

$$N(m) = \sum_{r=0}^{m} \left\lceil \frac{r^2}{4m} \right\rceil \qquad (\lceil x \rceil = \text{smallest integer} \geqq x) .$$

We will show that $N(m)$ can be expressed in terms of class numbers of imaginary quadratic fields and that (8) is equivalent to the formula

$$\dim J^{new}_{k,m} = \dim M^{new}_{2k-2}(\Gamma_0(m))^+ , \tag{9}$$

where $M^{new}_{2k-2}(\Gamma_0(m))^+$ is the space of new forms of weight $2k-2$ on $\Gamma_0(m)$ which are invariant under the Atkin-Lehner (or Fricke) involution $f(\tau) \longrightarrow m^{-k+1}\, \tau^{-2k+2}\, f(-1/m\tau)$ and $J^{new}_{k,m}$ a suitably defined space of "new" Jacobi forms.

Chapter IV, which will be published as a separate work, goes more deeply into the Hecke theory of Jacobi forms. In particular, it is shown with the aid of a trace formula that the equality of dimensions (9) actually comes from an isomorphism of the corresponding spaces as modules over the ring of Hecke operators.

Another topic which will be treated in a later paper (by B.Gross, W.Kohnen and the second author) is the relationship of Jacobi forms to Heegner points. These are specific points on the modular curve $X_0(m) = \mathcal{H}/\Gamma_0(m) \cup \{\text{cusps}\}$ (namely, those satisfying a quadratic equation with leading coefficient divisible by m). It turns out that for each n and r with $r^2 < 4nm$ one can define in a natural way a class $P(n,r) \in \mathrm{Jac}(X_0(m))(\mathbb{Q})$ as a combination of Heegner points and cusps and

that the sum $\sum_{n,r} P(n,r)\, q^n \zeta^r$ is an element of $\mathrm{Jac}(X_0(m))(\mathbf{Q}) \otimes_{\mathbf{Q}} J_{2,m}$.

One final remark. Since this is the first work on the theory of Jacobi forms, we have tried to give as elementary and understandable an exposition as possible. This means in particular that we have always preferred a more classical to a more modern approach (for instance, Jacobi forms are defined by transformation equations in $\mathcal{H} \times \mathbb{C}$ rather than as sections of line bundles over a surface or in terms of the representation theory of Weil's metaplectic group), that we have often given two proofs of the same result if the shorter one seemed to be too uninformative or to depend too heavily on special properties of the full modular group, and that we have included a good many numerical examples. Presumably the theory will be developed at a later time from a more sophisticated point of view.

* * * *

This work originated from a much shorter paper by the first author, submitted for publication early in 1980. In this the Saito-Kurokawa conjecture was proved for modular (Siegel and elliptic) forms on $\Gamma_0(N)$ with arbitrary level N. However, the exact level of the forms in the bottom of diagram (7) was left open. The procedure was about the same as here in §§4-6. The second author persuaded the first to withdraw his paper and undertake a joint study in a much broader frame. Sections 2 and 8-10 are principally due to the second author, while sections 1, 3-7 and 11 are joint work.

The authors would like to thank G. van der Geer for his critical reading of the manuscript.

Notations

We use $\mathbb{N}$ to denote the set of natural numbers, $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. We use Knuth's notation $\lfloor x \rfloor$ (rather than the usual $[x]$) for the greatest-integer function $\max\{n \in \mathbb{Z} \mid n \leqq x\}$ and similarly $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geqq x\} = -\lfloor -x \rfloor$. The symbol $\square$ denotes any square number. By $d \| n$ we mean $d \mid n$ and $(d, \frac{n}{d}) = 1$. In sums of the form $\sum_{d \mid n}$ or $\sum_{ad=\ell}$ it is understood that the summation is over positive divisors only. The function $\sum_{d \mid n} d^{\nu}$ $(d \in \mathbb{N})$ is denoted $\sigma_{\nu}(n)$.

The symbol $e(x)$ denotes $e^{2\pi i x}$, while $e^m(x)$ and $e_m(x)$ $(m \in \mathbb{N})$ denote $e(mx)$ and $e(x/m)$, respectively. In $e(x)$ and $e^m(x)$, x is a complex variable, but in $e_m(x)$ it is to be taken in $\mathbb{Z}/m\mathbb{Z}$; thus $e_m(ab^{-1})$ means $e_m(n)$ with $bn = a \pmod m$, and not $e(a/bm)$.

We use M^t and I_n for the transpose of a matrix and for the $n \times n$ identity matrix, respectively. The symbol $[a,b,c]$ denotes the quadratic form $ax^2 + bxy + cy^2$.

$\mathcal{H}$ denotes the upper half-plane $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. The letters τ and z will always be reserved for variables in $\mathcal{H}$ and $\mathbb{C}$, respectively, with $\tau = u + iv$, $z = x + iy$, $q = e(\tau)$, $\zeta = e(z)$. The group $SL_2(\mathbb{Z})$ will often be denoted by Γ_1 and the space of modular (resp. cusp) forms of weight k on Γ_1 by M_k (resp. S_k). The normalized Eisenstein series $E_k \in M_k$ $(k \geqq 4$ even) are defined in the usual way; in particular one has $M_* := \bigoplus_k M_k = \mathbb{C}[E_4, E_6]$ with $E_4 = 1 + 240 \sum \sigma_3(n) q^n$, $E = 1 - 504 \sum \sigma_5(n) q^n$.

The symbol " := " means that the expression on the right is the definition of that on the left.

Chapter I

BASIC PROPERTIES

§1. Jacobi Forms and the Jacobi Group

The definition of Jacobi forms for the full modular group $\Gamma_1 = SL_2(\mathbb{Z})$ was already given in the Introduction. In order to treat subgroups $\Gamma \subset \Gamma_1$ with more than one cusp, we have to rewrite the definition in terms of an action of the groups $SL_2(\mathbb{Z})$ and $\mathbb{Z}^2$ on functions $\phi: \mathcal{H} \times \mathbb{C} \to \mathbb{C}$. This action, analogous to the action

$$(1) \qquad (f|_k M)(\tau) := (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) \qquad \left(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1\right)$$

in the usual theory of modular forms, will be important for several later constructions (Eisenstein series, Hecke operators). We fix integers k and m and define

$$(2) \qquad \left(\phi|_{k,m}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)(\tau,z) := (c\tau+d)^{-k} e^m\left(\frac{-cz^2}{c\tau+d}\right)\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1\right)$$

and

$$(3) \qquad (\phi|_m[\lambda\ \mu])(\tau,z) := e^m(\lambda^2\tau + 2\lambda z)\,\phi(\tau, z+\lambda\tau+\mu)$$

$$((\lambda\ \mu) \in \mathbb{Z}^2),$$

where $e^m(x) = e^{2\pi i m x}$ (see "Notations"). Thus the two basic transformation laws of Jacobi forms can be written

$$\phi|_{k,m} M = \phi \quad (M \in \Gamma_1), \qquad \phi|_m X = \phi \quad (X \in \mathbb{Z}^2),$$

where we have dropped the square brackets around M or X to lighten the notation. One easily checks the relations

$$(\phi|_{k,m}M)|_{k,m}M' = \phi|_{k,m}(MM') \quad , \qquad (\phi|_m X)|_m X' = \phi|_m(X+X') \quad ,$$

(4)

$$(\phi|_{k,m}M)|_m XM = (\phi|_m X)|_{k,m}M \quad , \qquad (M,M' \in \Gamma_1 , \ X,X' \in \mathbb{Z}^2) \quad .$$

They show that (2) and (3) jointly define an action of the semi-direct product $\Gamma_1^J := \Gamma_1 \ltimes \mathbb{Z}^2$ (= set of products (M,X) with $M \in \Gamma_1$, $X \in \mathbb{Z}^2$ and group law $(M,X)(M',X') = (MM', XM' + X')$; notice that we are writing our vectors as row vectors, so Γ_1 acts on the right), the (full) *Jacobi group*. We will discuss this action in more detail at the end of this section.

We can now give the general definition of Jacobi forms.

Definition. A *Jacobi form* of *weight* k and *index* m ($k,m \in \mathbb{N}$) on a subgroup $\Gamma \subset \Gamma_1$ of finite index is a holomorphic function $\Phi: \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying

i) $\phi|_{k,m}M = \phi \qquad (M \in \Gamma)$;

ii) $\phi|_m X = \phi \qquad (X \in \mathbb{Z}^2)$;

iii) for each $M \in \Gamma_1$, $\phi|_{k,m}M$ has a Fourier development of the form $\sum c(n,r)q^n\zeta^r$ ($q = e(\tau)$, $\zeta = e(z)$) with $c(n,r) = 0$ unless $n \geqq r^2/4m$. (If ϕ satisfies the stronger condition $c(n,r) \neq 0 \Rightarrow n > r^2/4m$, it is called a *cusp form*.)

The vector space of all such functions ϕ is denoted $J_{k,m}(\Gamma)$; if $\Gamma = \Gamma_1$ we write simply $J_{k,m}$ for $J_{k,m}(\Gamma_1)$.

Remarks. 1. The numbers n,r in iii) are in general in $\mathbb{Q}$, not in $\mathbb{Z}$ (but with bounded denominator, depending on Γ and M).

2. We could define Jacobi forms with character, $J_{k,m}(\Gamma,\chi)$, by inserting a factor $\chi(M)$ in i) in the usual way.

3. Also, we could replace $\mathbb{Z}^2$ by any lattice invariant under Γ, e.g. by imposing congruence conditions modulo N if $\Gamma = \Gamma(N)$. It would therefore be more proper to refer to functions satisfying i)-iii)

as Jacobi forms on the Jacobi group $\Gamma^J = \Gamma \ltimes \mathbb{Z}^2$ (rather than on Γ). However, we will not worry about this since most of the time we will be concerned only with the full Jacobi group.

Our first main result is

THEOREM 1.1. *The space* $J_{k,m}(\Gamma)$ *is finite-dimensional.*

This will follow from two other results, both of independent interest:

THEOREM 1.2. *Let* ϕ *be a Jacobi form of index* m. *Then for fixed* $\tau \in \mathcal{H}$, *the function* $z \mapsto \phi(\tau,z)$, *if not identically zero, has exactly* 2m *zeros (counting multiplicity) in any fundamental domain for the action of the lattice* $\mathbb{Z}\tau + \mathbb{Z}$ *on* $\mathbb{C}$.

Proof. It follows easily from the transformation law ii) that $\frac{1}{2\pi i} \oint_{\partial F} \frac{\phi_z(\tau,z)}{\phi(\tau,z)}\, dz = 2m$ ($\phi_z = \frac{\partial \phi}{\partial z}$, F = fundamental domain for $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$), (the expression $\frac{1}{2\pi i}\frac{\phi_z}{\phi}$ is invariant under $z \to z+1$ and changes by 2m when one replaces z by $z+\tau$), and this is equivalent to the statement of the theorem. Notice that the same proof works for ϕ meromorphic (with "number of zeros" replaced by "number of zeros minus number of poles") and any $m \in \mathbb{Z}$. A consequence is that there are no holomorphic Jacobi forms of negative index, and that a holomorphic Jacobi form of index 0 is independent of z (and hence simply an ordinary modular form of weight k in τ).

THEOREM 1.3. *Let* ϕ *be a Jacobi form on* Γ *of weight* k *and index* m *and* λ,μ *rational numbers. Then the function* $f(\tau) = e^m(\lambda^2\tau)\,\phi(\tau, \lambda\tau+\mu)$ *is a modular form (of weight* k *and on some subgroup of* Γ' *of finite index depending only on* Γ *and on* λ,μ).

For $\lambda = \mu = 0$ it is clear that $\tau \to \phi(\tau,0)$ is a modular form of weight k on Γ. We will prove the general case later on in this section when we have developed the formalism of the action of the Jacobi group further. Note that the Fourier development of $f(\tau)$ at infinity is

$$\sum_{n,r} e(r\mu)c(n,r)\ e((m\lambda^2 + r\lambda + n)\tau) \quad ,$$

so that the conditions $n \geqq 0$, $r^2 \leqq 4nm$ in the definition of Jacobi forms are exactly what is required to ensure the holomorphicity of f at ∞ in the usual sense.

To deduce 1.1, we pick any $2m$ pairs of rational numbers $(\lambda_i,\mu_i) \in \mathbb{Q}^2$ with $(\lambda_i,\mu_i) \not\equiv (\lambda_j,\mu_j) \pmod{\mathbb{Z}^2}$ for $i \neq j$. Then the functions $f_i(\tau) = e^m(\lambda_i^2\tau)\phi(\tau, \lambda_i\tau + \mu_i)$ lie in $M_k(\Gamma_i)$ for some subgroups Γ_i of Γ, and the map $\phi \to \{f_i\}_i$ is injective by Theorem 1.2. Therefore $\dim J_{k,m}(\Gamma) \leq \sum_i \dim M_k(\Gamma_i)$; this proves Theorem 1.1 and also shows that $J_{k,m}$ is 0 for $k \leq 0$ unless $k = m = 0$, in which case it reduces to the constants.

To prove Theorem 1.3, we would like to apply (3) to $(\lambda,\mu) \in \mathbb{Q}^2$. However, we find that formula (3) no longer defines a group action if we allow non-integral λ and μ, since

$$(\phi|_m[\lambda\ \mu])|_m[\lambda'\ \mu'](\tau,z) =$$

$$= e^m(\lambda'^2\tau + 2\lambda' z + \lambda^2\tau + 2\lambda(z + \lambda'\tau + \mu'))\phi(\tau, z + \lambda'\tau + \mu' + \lambda z + \mu)$$

$$= e(2m\lambda\mu')(\phi|_m[\lambda+\lambda'\ \mu+\mu'])\ (\tau,z)$$

and $e(2m\lambda'\mu)$ will not in general be equal to 1. Similarly, the third equation of (4) breaks down if X is not in $\mathbb{Z}^2$. Hence if we want to extend our actions to $SL_2(\mathbb{Q})$ (or $SL_2(\mathbb{R})$) and $\mathbb{Q}^2$ (or $\mathbb{R}^2$), we must modify the definition of the group action.

The verification of the third equation in (4) depends on the two

elementary identities

$$\frac{z}{c\tau+d} + \lambda\,\frac{a\tau+b}{c\tau+d} + \mu \;=\; \frac{z+\lambda_1\tau+\mu_1}{c\tau+d} \quad ,$$

$$\lambda^2\,\frac{a\tau+b}{c\tau+d} + 2\lambda\,\frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d} + \lambda\mu \;=\; \lambda_1^2\tau + 2\lambda_1 z - \frac{c(z+\lambda_1\tau+\mu_1)^2}{c\tau+d} + \lambda_1\mu_1 \ ,$$

where $(\lambda_1\ \mu_1) = (\lambda\ \mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Thus to make this equation hold for arbitrary $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $X = (\lambda\mu) \in \mathbb{R}^2$ we should replace (3) by

$$(5) \qquad (\phi|_m[\lambda\ \mu])(\tau,z) \;:=\; e^m(\lambda^2\tau + 2\lambda z + \lambda\mu)\phi(\tau, z+\lambda\tau+\mu) \qquad ((\lambda\ \mu) \in \mathbb{R}^2) \ ;$$

this is compatible with (3) because $e^m(\lambda\mu) = 1$ for $\lambda,\mu \in \mathbb{Z}$. Unfortunately, (5) still does not define a group action; we now find

$$(6) \qquad (\phi|_m X)|_m X' \;=\; e^m(\lambda\mu' - \lambda'\mu)\phi|_m(X+X') \qquad (X = (\lambda\ \mu),\ \ X' = (\lambda'\ \mu') \in \mathbb{R}^2) \ .$$

To absorb the extra factor, we must introduce a scalar action of the group $\mathbb{R}$ by

$$(7) \qquad (\phi|_m[\kappa])(\tau,z) \;:=\; e(m\kappa)\phi(\tau,z) \qquad (\kappa \in \mathbb{R})$$

and then make a central extension of $\mathbb{R}^2$ by this group $\mathbb{R}$; i.e. replace $\mathbb{R}^2$ by the Heisenberg group

$$H_{\mathbb{R}} \;:=\; \{[(\lambda\ \mu),\kappa] \mid (\lambda,\mu) \in \mathbb{R}^2,\ \kappa \in \mathbb{R}\} \quad ,$$

$$[(\lambda\ \mu),\kappa][(\lambda'\ \mu'),\kappa'] \;=\; [(\lambda+\lambda'\ \mu+\mu'),\ \kappa+\kappa'+\lambda\mu'-\lambda'\mu] \ .$$

(This group is isomorphic to the group of upper triangular unipotent 3×3 matrices via

$$[(\lambda\ \mu),\kappa] \longmapsto \begin{pmatrix} 1 & \lambda & \tfrac{1}{2}(\kappa+\lambda\mu) \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix} \quad .)$$

The subgroup $C_{\mathbb{R}} := \{[(0\ 0),\kappa],\ \kappa\in\mathbb{R}\}$ is the center of $H_{\mathbb{R}}$ and $H_{\mathbb{R}}/C_{\mathbb{R}} \cong \mathbb{R}^2$. We can now combine (5) and (7) into an action of $H_{\mathbb{R}}$ by setting

$$(\phi|[(\lambda\ \mu),\kappa])(\tau,z) \;=\; e^{m}(\lambda^2\tau+2\lambda z+\lambda\mu+\kappa)\phi(\tau,\ z+\lambda\tau+\mu)\ ,$$

and this now *is* a group action because the extra factor $e^{m}(\lambda'\mu-\lambda\mu')$ in (6) is compensated by the twisted group law in $H_{\mathbb{R}}$. Because this twist involves $\lambda\mu'-\lambda'\mu = \det\begin{pmatrix}\lambda & \mu\\ \lambda' & \mu'\end{pmatrix}$ and the determinant is preserved by SL_2, the group $SL_2(\mathbb{R})$ acts on $H_{\mathbb{R}}$ on the right by

$$[X,\kappa]M \;=\; [XM,\kappa] \qquad\qquad (X\in\mathbb{R}^2,\quad \kappa\in\mathbb{R},\quad M\in SL_2(\mathbb{R}))\ ;$$

the above calculations then show that all three identities (4) remain true if we now take $M,M'\in SL_2(\mathbb{R})$ and $X,X'\in H_{\mathbb{R}}$ and hence that equations (2), (5) and (7) together define an action of the semidirect product $SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}$.

In the situation of usual modular forms, we write $\mathcal{H}$ as G/K, where $G = SL_2(\mathbb{R})$ contains Γ as a discrete subgroup with $\mathrm{Vol}(\Gamma\backslash G)$ finite and $K = SO(2)$ is a maximal compact subgroup of G. Here we would like to do the same. However, the group $SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}$ contains $\Gamma^J = \Gamma\times\mathbb{Z}^2$ with infinite covolume (because of the extra $\mathbb{R}$ in $H_{\mathbb{R}}$) and its quotient by the maximal compact subgroup $SO(2)$ is $\mathcal{H}\times\mathbb{C}\times\mathbb{R}$ rather than $\mathcal{H}\times\mathbb{C}$. To correct this, we observe that the subgroup $\mathbb{Z}\subset\mathbb{R}$ acts trivially in (7), so that (2), (5) and (7) actually define an action of the quotient group

$$G^J \;:=\; SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}/C_{\mathbb{Z}}\ .$$