Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Berlin Heidelberg New York London Paris Tokyo

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Mathematics Subject Classification (1980): 34C35, 46L55, 46L80, 57R30

ISBN 3-540-17903-8 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-17903-8 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

Lecture Notes in Mathematics

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TO MY FATHER

I would rather schematize the structure of mathematics by a complicated graph, where the vertices are the various parts of mathematics and the edges describe the connections between them. These connections sometimes go one way, sometimes both ways, and the vertices can act both as sources and sinks. The development of the individual topics is of course the life and blood of mathematics, but, in the same way as a graph is more than the union of its vertices, mathematics is much more than the sum of its parts. It is the presence of those numerous, sometimes unexpected edges, which makes mathematics a coherent body of knowledge, and testifies to its fundamental unity.

——— A. Borel

I have the feeling that we don't understand at all the extraordinary interplay of combinatorics and what I would call "conceptual" mathematics.

——— J. Dieudonné

Contents

§0. Introduction

0.1 In [C1] A. Connes introduced C*-algebras of foliations as an important ingredient in his noncommutative integration theory. Linking topology, geometry and analysis, it has become an important area of research; see [C2] for a more detailed treatment. Many general properties of C*-algebras of foliations have also been studied by T. Fack, G. Skandalis, M. Hilsum and others; see [F-S], [F], [H-S].

A very interesting and natural problem is to determine for various manifolds the structure of the C*-algebras of special types of foliations, or at least their K-theory. This will help us understand how the algebraic and topological invariants of the C*-algebras are related to the geometric and topological properties of foliations. For this aspect, see for instance [T], [N-T1], [N-T2], [N1], [N2] and [Tor].

So far the investigation has been carried out for only a few closed manifolds, i.e., the manifolds which are compact and without boundary. A closed manifold does not necessarily admit a nonsingular foliation. For instance, it is well known that the only closed connected orientable two-manifold which does is the torus. The C*-algebras of foliated tori have been studied by A. M. Torpe in [Tor]. In order to carry the investigation further, especially to explore the potential application of C*-algebras in dynamical systems, it is necessary to consider C*-algebras of foliated *open* manifolds, or what amounts to the same thing, to consider singular foliations of closed manifolds. The purpose of this paper is to initiate a study in this direction.

The most important kind of open two-manifolds are the punctured closed two-manifolds. Thus our study will be focused on foliations of (topological) two-manifolds with isolated singularities. Since all closed two manifolds have the plane as their universal covering space, except the two-sphere which is however the one-point compactification of the plane, the plane is the simplest yet most important open two manifold.

0.2 In this paper we discuss a method which will allow us to compute the "ordinary" foliations of the plane (Definition 1.5.12). A crucial conception involved is separatrices and canonical regions (1.5.1), which were studied in qualitative theory of ordinary differential equations. Roughly, a separatrix is a leaf whose arbitrary small neighborhoods contain leaves behaving differently from it in the large. The canonical regions in a foliation are just the complement regions of the separatrices. By definition, an *ordinary foliation* is topologically conjugate to a foliation in which the set of separatrices has planar measure zero. A foliation is called *with* T_2*-separatrices* if the set of separatrices is Hausdorff with the quotient topology in the leaf space. We give a complete classification of all the C*-algebras of foliations with T_2-separatrices up to isomorphism. This study shows how noncommutative "CW 1-complexes" appear naturally in C*-algebras. The "glueing" construction is used extensively. It suggests that the method of surgery in topology may be adopted in the noncommutative context. Our investigation also establishes a direct contact between C*-algebras, and the qualitative theory of ordinary differential equations. It is shown that the solution of the classification of C*-algebras of C^{∞}-foliations of the plane provides in the meantime the solution of the classification of the C*-algebras of all the 1-parameter continuous transformation groups of the plane (cf. §1.1). In fact C*-algebras of foliations are the prototypes of C*-algebras of transformation groups [E-H]. These two notions are elegantly unified in terms of C*-algebras of groupoids (cf. [Ren]). This approach is essential in our investigation (e.g. the proof of Theorem 4.1.2).

0.3 Our study is based on the remarkable work of Wilfred Kaplan [K1], [K2] and Lawrence Markus [M] which in turn is based on the previous works of I. Bendixon, H. Poincaré, H. Whiteney, E. George, E. Kamke and many others. The Kaplan-Markus theory provides a complete classification of the foliations of the plane up to topological conjugacy in terms of certain configurations known as Kaplan diagrams (see Ch. 11 [Bec]). Since nonconjugate foliation of the plane may very well have isomorphic C*-algebras, the Kaplan diagrams are not an invariant for C*-algebras.

Example. The two foliations illustrated by Fig. 0.1 are not topologically conjugate. Later (Example 4.3.3) we shall see that they have isomorphic C*-algebras.

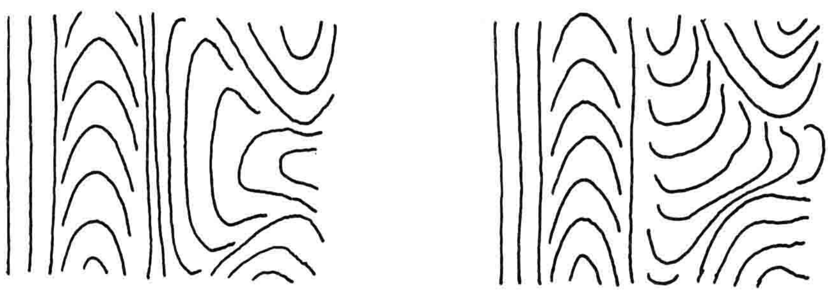

Fig. 0.1

Thus not only do we need to determine the C*-algebra of the foliation, represented by an arbitrary Kaplan diagram, but also to find a new configuration which is a complete intrinsic invariant for these C*-algebras. This invariant for all foliations with T_2-separatrices is what we shall call distinguished trees (Definition 3.3.1).

0.4 From the point of view of noncommutative topology, the way we study the C*-algebras of foliations is a natural analogue to that of studying topological manifolds by triangulation. We decompose a foliated manifold into saturated connected components. What equivalence relation among the leaves we choose to yield the decomposition may depend on the types of manifolds. For instance, one can use Novikov connected components for closed manifolds. For the plane region, our decomposition yields the standard canonical regions and separatrices defined for topological dynamical systems. This step is obvious when one recalls that the C*-algebra of a foliation is the reduced C*-algebra of the holonomy groupoid, which is the desingularization of the leaf space. In fact, such a decomposition divides the leaf space into "homogeneous" components.

We then compute the reduced C*-algebra of the restricted holonomy groupoid of each connected component along with its boundary. These C*-algebras are the "cells". A noncommutative "CW-complex" is then constructed by glueing these cells in a certain way. There are two types of glueing conditions in general: "local" and "global". The "local glueing" condition is fulfilled by taking the fibred product of the two "nearby" C*-algebras. The "global glueing" condition has, however, no analogy in the commutative context. It is fulfilled by imposing conditions on continuous fields of C*-algebras. The distinguished trees are a combinatorial-topological invariant. Their combinatorial structure contains the "local" information while their topological structure contains the "global" information.

In order to illustrate this method without involving too much combinatorial complicacy, we derive only the classification theorems for the C*-algebras of all foliations with T_2-separatrices. Our method may be used to obtain a possible classification of C*-algebras of all ordinary C*-algebras.

Finally, of course, we need to show that the "CW-complex" constructed in this way is indeed isomorphic to the C*-algebra of the foliation we begin with.

0.5 In §1, we start by clarifying some technical conditions concerning smoothness of foliations. Then we introduce the main results of Kaplan-Markus theory in a way convenient for our purpose. The main references are [Bec], [K1], [K2] and [M]. The central concept is that of the Kaplan diagram of a foliation (see 1.2.2), and its intrinsic definition (Definition 1.4.5). The fact that the Kaplan diagram of a foliation satisfies the axioms in Definition 1.4.5, together with Theorems 1.4.8 and 1.4.9 are the main results of the Kaplan-Markus theory.

The abstract formulation of Kaplan diagrams, i.e., "normal chordal systems", which is Kaplan's original approach, is postponed to 3.4.9, after we have clarified the relation between $\mathbb{R}$-trees and abstract chordal systems (Theorem 3.4.4). The classification theorem is reformulated in Theorem 3.4.10 (Theorem, p.11 [K2]).

Distinguished trees are, in particular, a class of $\mathbb{R}$-trees. In §2, we discuss briefly $\mathbb{R}$-trees as defined by John Morgan and Peter Shalen ((II.1.2) [M-S1]). By definition $\mathbb{R}$-trees are metric spaces in which any two points are joined by a unique arc with the distance given by the arc length. As generalized trees, they had been studied in [A-M]. Our emphasis is on regular $\mathbb{R}$-trees (Definition 2.3.3) which are the separable $\mathbb{R}$-trees equipped with "vertices" and "edges". They can be regarded as the "minimal" generalization of trees (see Theorem 2.3.5). We generalize the construction of quotient trees to the context of $\mathbb{R}$-trees in 2.4. In order to study Kaplan-Markus flows in multi-connected regions in a later paper, we introduce in 2.5 the concept of $\mathbb{R}$-graphs in passing.

The essential results of this paper are contained in §3 and §4. Distinguished trees are characterized in §3. They are a special class of regular $\mathbb{R}$-trees in which the vertices and edges are $\mathbb{Z}_2$-graded in a certain way (Definition 3.3.1). We first show how to associate an $\mathbb{R}$-tree to a foliation with T_2-separatrices (Definition 3.2.13). These trees are shown to be distinguished trees (Theorem 3.3.4). Conversely, every distinguished tree is shown to be a tree associated to a foliation with T_2-separatrices (Theorem 3.5.12). The special case where the $\mathbb{R}$-trees are actually trees is given an independent proof (Theorem 3.3.7).

Finally we come back to C*-algebras of foliations in §4. Recall the program laid out in 0.4. There are in general five types of canonical regions which have been given in Corollary 1.5.8. The C*-algebras of parallel foliations of polygonoids are determined explicitly in Theorem 4.1.2. In analogy with the construction of "a tree of groups" in combinatorial group theory, for every distinguished tree T we associate a tree of C*-algebras (T,A,σ) (Definition 4.2.9). In analogy with amalgamated products of trees of groups, here we define for a system (T,A,σ) a C*-algebra $C^*(T,A,\sigma)$, called the global fibred product, which is independent of a particular choice of the set σ of homomorphisms, up to isomorphisms of C*-algebras (Theorem 4.2.19).

Thus we may write $C^*(T)$ for $C^*(T,A,\sigma)$. The correspondence between the distinguished trees and the C*-algebras has the following property (Theorem 4.2.21).

Theorem. *Two distinguished trees* T_1 *and* T_2 *are similar iff the C*-algebras* $C^*(T_1)$ *and* $C^*(T_2)$ *are isomorphic.*

Then we show that the C*-algebra of a foliation (Ω,F) of the plane is isomorphic to $C^*(T(F),A,\sigma)$ for some σ, where $T(F)$ is the distinguished tree of the foliation (Ω,F) (Theorem 4.3.1). Combining with the results obtained in §3, we conclude (Theorem 4.3.2)

Theorem. *The distinguished trees are a complete isomorphism invariant for the C*-algebras of foliations with* T_2*-separatrices.*

0.6 Note that a foliation of the plane is equivalent to a foliation of the two-sphere with a singular point of arbitrary type. Our study can be carried over to flows on the two sphere with isolated singularities. "Blowing up" these singular points, we pass over to the situation of Kaplan-Markus flows on multiconnected open regions. Instead of regular $\mathbb{R}$-trees and distinguished trees, we shall have regular $\mathbb{R}$-graphs and distinguished graphs. This will be discussed elsewhere.

0.7 $\mathbb{R}$-trees can be considered to be degenerate hyperbolic spaces. They play an important role in Morgan and Shalen's remarkable work [M-S1] and [M-S2]. There is a close connection between the construction of an $\mathbb{R}$-tree for a measured lamination on a closed surface and the construction of the $\mathbb{R}$-tree $T(F)$ for a foliation of the plane described in §3. In fact, let M be a closed surface with a measured lamination F, $\chi(M) \leq 0$. Then F can be lifted to a measured lamination $\tilde{F}$ on the plane Ω. The complement of $\tilde{F}$ in Ω is a union of open regions, each of which admits parallel foliations. If jointly this gives a foliation of Ω (this is always the case if we can choose the leaves of the foliation of each open region "parallel" to its boundary), then the construction in §3 defines an $\mathbb{R}$-tree $T(\tilde{F})$, which is homeomorphic to the $\mathbb{R}$-tree defined by the measured lamination. There are also interesting relations between the C*-algebra of a singular foliation of a closed surface and the C*-algebra of its singular covering foliation of the plane. The $\mathbb{R}$-graphs corresponding to these two C*-algebras are related through the actions of the fundamental group of the surface (cf. §4.1). We will carry out the investigation of this relation elsewhere.

0.8 **Conventions**

Throughout this paper unless otherwise stated, we use the open unit disc Ω in $\mathbb{R}^2$ as the topological model for the plane. We denote by $\partial\Omega$ the unit circle in $\mathbb{R}^2$. For any metric space X, $x \in X$, $\delta > 0$, we write $B(x,\delta)$, the open ball of radius δ and centered at x.

For a saturated submanifold V of (Ω,F) (with or without boundary), we shall denote by $\mathbf{G}(V,F)$ the graph of the restricted foliation of V, and by $C^*(V,F)$ the corresponding C^*-algebra (cf. the beginning of 4.1). By an ideal of a C^*-algebra, we always mean a closed two-sided ideal.

The bounded operators and compact operators in a Hilbert space H will be denoted $L(H)$ and $K(H)$ respectively.

The author is grateful to Professor Marc A. Rieffel for his invaluable suggestions and extensive comments during the preparation of this manuscript. The author also wishes to express his gratitude to Professor Adrian Ocneanu for a helpful discussion at an earlier stage and to Professor Jonathan Rosenberg for suggesting a simplified proof of Theorem 4.1.2. Finally, he wishes to thank Professors R. Hermann, Morris Hirsch and Charles Pugh for providing the literature about foliations.

§1. Foliations of the plane

A complete classification of foliations of the plane up to topological conjugacy was carried out by Wilfred Kaplan in [W1] and [W2] following the earlier works of I. Bendixon, E. Kamke, H. Poincaré, H. Whiteney and many others ([Ben], [Kam], [Po], [Wh]). Although the main results of Kaplan's theory are intuitively convincing and quite elementary in appearance, the proofs of this kind of classification theory are very complicated since as a general rule the combinatorial analysis involved is messy.

Kaplan's theory grew out of the qualitative theory of ordinary differential equations which in fact provides its most important and direct application. This aspect was studied by L. Markus from the viewpoint of the geometry of the global structure of solutions of ordinary differential equations, cf. [M].

Our aim in this section is to give a concise description of the main results of Kaplan-Markus theory so the reader may avoid getting into all the technical details of Kaplan's original work. Also we shall fix some notations along the way for the following sections, and this makes our exposition self-contained to a large extent.

We shall discuss some basic properties of foliations of the plane in 1.1 and clarify the points involving orientability of foliations and the possibility of smoothing a C^0-foliation to a C^∞-foliation. The need for smoothness arises from the study of C*-algebras of foliations, which is our main concern in this paper.

The presentation of Kaplan-Markus theory in 1.2-1.4 basically follows the approach of A. Beck (Ch. 11 [Bec]). The proofs of most