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C^∞-Differentiable Spaces

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To Prof. Juan B. Sancho Guimerá

Preface

Differential geometry is traditionally regarded as the study of smooth manifolds, but sometimes this framework is too restrictive since it does not admit certain basic geometric intuitions. On the contrary, these geometric constructions are possible in the broader category of differentiable spaces. Let us indicate some natural objects which are differentiable spaces and not manifolds:

– Singular quadrics. Elementary surfaces of classical geometry such as a quadratic cone or a "doubly counted" plane are not smooth manifolds. Nevertheless, they have a natural differentiable structure, which is defined by means of the consideration of an appropriate algebra of differentiable functions.

For example, let us consider the quadratic cone X of equation $z^2 - x^2 - y^2 = 0$ in $\mathbb{R}^3$. It is a differentiable space whose algebra of differentiable functions is defined by

$$A := \mathcal{C}^\infty(\mathbb{R}^3)/\mathfrak{p}_X = \mathcal{C}^\infty(\mathbb{R}^3)/(z^2 - x^2 - y^2) ,$$

where $\mathfrak{p}_X$ stands for the ideal of $\mathcal{C}^\infty(\mathbb{R}^3)$ of all differentiable functions vanishing on X. In other words, differentiable functions on X are just restrictions of differentiable functions on $\mathbb{R}^3$.

Let us consider a more subtle example. Let Y be the plane in $\mathbb{R}^3$ of equation $z = 0$. Of course this plane is a smooth submanifold. On the contrary, the "doubly counted" plane $z^2 = 0$ makes no sense in the language of smooth manifolds. It is another differentiable space with the same underlying topological space (the plane Y) but a different algebra of differentiable functions:

$$A := \mathcal{C}^\infty(\mathbb{R}^3)/(z^2) .$$

Note that A is not a subalgebra of $\mathcal{C}(Y, \mathbb{R})$, so that elements of A are not functions on Y in the set-theoretic sense.

More generally, any closed ideal $\mathfrak{a}$ of the Fréchet algebra $\mathcal{C}^\infty(\mathbb{R}^n)$ defines a differentiable space (X, A), where

$$X := \{x \in \mathbb{R}^n : \ f(x) = 0 \text{ for any } f \in \mathfrak{a}\}$$

is the underlying topological space and

$$A := \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$$

is the algebra of differentiable functions on this differentiable space. The pair (X, A) is the basic example of a differentiable subspace of $\mathbb{R}^n$ and the quotient map $\mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ is interpreted as the restriction morphism, i.e., for any $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ the equivalence class $[f] \in \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ is said to be the restriction of f to the differentiable subspace under consideration.

– Fibres. Given a differentiable map $\varphi\colon \mathcal{V} \to \mathcal{W}$ between smooth manifolds, it may happen that some fibre $\varphi^{-1}(y)$ is not a smooth submanifold of $\mathcal{V}$, although it admits always a natural differentiable space structure.

– Intersections. Given a smooth manifold, the intersection of two smooth submanifolds may not be a smooth submanifold. On the contrary, intersections always exist in the category of differentiable spaces. For example, given two differentiable subspaces $(X, \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a})$ and $(Y, \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{b})$ of $\mathbb{R}^n$, where $\mathfrak{a}$ and $\mathfrak{b}$ are closed ideals of $\mathcal{C}^\infty(\mathbb{R}^n)$, the corresponding intersection is defined by the differentiable subspace

$$(X \cap Y \, , \, \mathcal{C}^\infty(\mathbb{R}^n)/\overline{\mathfrak{a}+\mathfrak{b}} \,) \, .$$

A more explicit example: The intersection of the parabola $y - x^2 = 0$ and the tangent $y = 0$ is the "doubly counted" origin

$$(\{(0,0)\}, \mathcal{C}^\infty(\mathbb{R}^2)/(y - x^2, y) = \mathbb{R}[x]/(x^2))$$

and the number $2 = \dim \mathbb{R}[x]/(x^2)$ defines the *multiplicity* of the intersection.

More generally, *fibred products* exist in the category of differentiable spaces.

– Quotients. If we have a differentiable action of a Lie group G on a smooth manifold $\mathcal{V}$, it may occur that the topological quotient $\mathcal{V}/G$ admits no smooth manifold structure, even in such a simple case as a linear representation of a finite group. For example, if we consider the multiplicative action of $G = \{\pm 1\}$ on $\mathcal{V} = \mathbb{R}^3$, then the topological quotient $\mathcal{V}/G$ is not a topological manifold (nor a manifold with boundary). This example ruins any hope of a general result on the existence of quotients in the category of smooth manifolds under some reasonable hypotheses. On the contrary, in the category of differentiable spaces, we shall show the existence of quotients with respect to actions of compact Lie groups.

In particular, *orbifolds* usually have a natural structure of differentiable space.

– Infinitesimal neighbourhoods. The notion of an infinitesimal region naturally arises everywhere in differential geometry, but it is only used informally as a suggestive expression, due to the lack of a rigorous definition. Again, the language of differentiable spaces allows a suitable definition: Given a point x in a smooth manifold $\mathcal{V}$, the r-th infinitesimal neighbourhood of x is the differentiable subspace

$$U_x^r(\mathcal{V}) := \left(\{x\}, \mathcal{C}^\infty(\mathcal{V})/\mathfrak{m}_x^{r+1}\right) \, ,$$

where $\mathfrak{m}_x$ stands for the ideal of all differentiable functions vanishing at x.

The restriction of a differentiable function f to $U_x^r(\mathcal{V})$ is just the r-th jet of f at x, i.e., $j_x^r f = [f] \in \mathcal{C}^\infty(\mathcal{V})/\mathfrak{m}_x^{r+1}$. A systematic use of infinitesimal neighbourhoods simplifies and clarifies the theory of jets.

Let us show another application. The tangent bundle of an affine space $\mathbb{A}_n$ has a canonical trivialization $T\mathbb{A}_n = \mathbb{A}_n \times V_n$, where V_n is the vector space of free vectors on $\mathbb{A}_n$. Now let ∇ be a torsionless linear connection on a smooth manifold $\mathcal{V}$. For any point $x \in \mathcal{V}$, the restriction of the tangent bundle $T\mathcal{V}$ to $U_x^1(\mathcal{V})$ inherits a canonical trivialization (induced by parallel transport). In this sense, we may state that $(\mathcal{V}, \nabla)$ has the same infinitesimal structure as the affine space $\mathbb{A}_n$. This is a statement of the geometric meaning of a torsionless linear connection.

In a similar way, first infinitesimal neighbourhoods $U_x^1(\mathcal{V})$ in a Riemannian manifold $(\mathcal{V}, g)$ are always Euclidean, i.e., they have the same metric tangent structure as the first infinitesimal neighbourhood of any point in a Euclidean space. This statement captures the basic intuition about the notion of a Riemannian manifold as an infinitesimally Euclidean space.

With this, let us finish our list of examples of differentiable spaces. It should be sufficient to convince anybody of the necessity for an extension of the realm of differential geometry to include more general objects than smooth manifolds. A similar expansion occurred in the theory of algebraic varieties and analytic manifolds with the introduction of schemes and analytic spaces in the fifties. Following this path, Spallek introduced the category of differentiable spaces [59, 60, 62] which contains the category of smooth manifolds as a full subcategory. Moreover, all the foundational theorems on algebras of $\mathcal{C}^\infty$-differentiable functions are already at our disposal [26, 35, 71, 74]. In spite of this, the theory of general differentiable spaces has not been developed to the point of providing a handy tool in differential geometry. The aim of these notes is to develop the foundations of the theory of differentiable spaces in the best-behaved case: Spallek's ∞-standard differentiable spaces (henceforth simply differentiable spaces, since no other kind will be considered). These foundations will be developed so as to include the most basic tools at the same level as is standard in the theory of schemes and analytic spaces.

We would like to thank our friend R. Faro, who always has patience to attend to any question and to discuss it with us, and Prof. J. Muñoz Masqué, who taught us a course on rings of differentiable functions 25 years ago at the University of Salamanca.

We dedicate these notes to Professor Juan B. Sancho Guimerá, who directed around 1970 two doctoral dissertations [35, 37, 41] on the Localization theorem and always stressed to us its crucial importance in laying the foundations of differential geometry.

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Introduction

We shall develop the theory of differentiable spaces paralleling the theory of schemes introduced by Grothendieck [17] in algebraic geometry. First we choose the rings that should be considered as rings of differentiable functions, which are fixed to be quotients of $\mathcal{C}^\infty(\mathbb{R}^n)$ by some closed ideal $\mathfrak{a}$ (with respect to the Fréchet topology of uniform convergence on compact sets of functions and their derivatives). These Fréchet algebras $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ are named **differentiable algebras**[1], since $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ will be regarded as an algebra of differentiable functions on the closed subset $(\mathfrak{a})_0 := \{x \in \mathbb{R}^n \colon\ f(x) = 0\ \forall f \in \mathfrak{a}\}$ of $\mathbb{R}^n$, even though this algebra may be full of nilpotent elements. Nevertheless, when $\mathfrak{a}$ is the ideal of all $\mathcal{C}^\infty$-functions vanishing on a given closed set $X \subseteq \mathbb{R}^n$, the quotient algebra $A = \mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ may be identified with a ring of real valued functions on X (in the usual set-theoretic sense), and in such a case we say that A is a **reduced** differentiable algebra.

Then we replace each differentiable algebra A by a ringed space (topological space with a sheaf of rings) $\operatorname{Spec}_r A$ called the **real spectrum** of A, since it is analogous to the prime spectrum used in algebraic geometry. Moreover, the analogue of a quasi-coherent sheaf on the prime spectrum is provided in our setting by the sheaf of modules defined by a Fréchet A-module. These ringed spaces $\operatorname{Spec}_r A$ define a category dual to the category of differentiable algebras, but it has the enormous advantage of leaving room for "recollement" procedures, as do any other kind of ringed spaces. Hence, a ringed space is said to be an **affine differentiable space** if it is isomorphic to the real spectrum of some differentiable algebra (the analogue of affine schemes), and **differentiable spaces** are defined to be ringed spaces where every point has an open neighbourhood which is an affine differentiable space (the analogue of schemes in algebraic geometry). Moreover, sheaves of Fréchet modules provide the analogue of quasi-coherent sheaves on schemes.

There has long been perceived the need for an extension of the framework of smooth manifolds in differential geometry, and there are several definitions that attempt to capture the intuitive concept of "non-smooth space with a differentiable structure". Over the years, some categories have appeared that are both large enough to include smooth manifolds and some other geometric objects, and small enough to admit a differential calculus. Therefore, the name *differentiable*

[1] Not to be confused with the notion of differentiable algebra as used in commutative algebra, which is simply an algebra with a derivation.

space and similar terms have been used in a number of quite different senses. Let us discuss briefly some of them:

Spallek's differentiable spaces. Our differentiable spaces coincide with Spallek's differentiable spaces of a particular type (named ∞-standard); hence these notes fit naturally into the theory and applications of such spaces ([48] – [51] and [59] – [66]).

Synthetic differential geometry. At the end of the sixties Lawvere [24] proposed an axiomatic approach to the category of schemes, intended to be used in differential geometry and named "synthetic differential geometry". It was in this context that $\mathcal{C}^\infty$-schemes [9, 31] appeared, providing a model of this axiomatic theory. $\mathcal{C}^\infty$-rings are just $\mathbb{R}$-algebras where the composition $f(a_1, \ldots, a_n)$ with any smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ is defined, satisfying all the equations that hold between these functions; hence differentiable algebras are $\mathcal{C}^\infty$-rings. Then $\mathcal{C}^\infty$-ringed spaces are defined in the obvious way [9] and a $\mathcal{C}^\infty$-ringed space Spec A is associated to any $\mathcal{C}^\infty$-ring A (it coincides with our real spectrum whenever A is a differentiable algebra). Finally $\mathcal{C}^\infty$-schemes are introduced as $\mathcal{C}^\infty$-ringed spaces locally isomorphic to these local building blocks Spec A. Therefore, differentiable spaces in our sense are $\mathcal{C}^\infty$-schemes; but $\mathcal{C}^\infty$-schemes lack a handy theory of sheaves of modules paralleling the useful theory of sheaves of Fréchet modules that we shall develop in the realm of differentiable spaces.

Real algebraic varieties of any kind are, of course, differentiable spaces. For example, up to isomorphisms, affine real algebraic varieties in the sense of Palais [44] are just pairs (X, A) where X is an algebraic set in $\mathbb{R}^n$ (closed subset defined by some polynomial equations) and A is the algebra of all polynomial functions $X \to \mathbb{R}$, so that A is just the quotient of $\mathbb{R}[x_1, \ldots, x_n]$ by the ideal of all polynomials vanishing on X. Hence any such algebraic variety inherits a natural structure of *reduced* affine differentiable space. In order to define non-affine real algebraic varieties, one introduces the sheaf $\mathcal{O}_X$ of regular functions on any algebraic set $X \subseteq \mathbb{R}^n$:

$$\mathcal{O}_X(U) := \{p/q\colon\; p, q \in A,\; q(x) \neq 0 \;\forall x \in U\}\ .$$

These ringed spaces $(X, \mathcal{O}_X)$ are the local building blocks of the definition of real algebraic varieties given by Bochnak, Coste and Roy [2]. Again such algebraic varieties inherit a natural structure of *reduced* differentiable space. They consider only reduced algebraic varieties in spite of the critical role of non-reduced spaces in any infinitesimal consideration.

Orbifolds are defined to be locally isomorphic to $\mathbb{R}^n/G$, where G is some finite group [55, 69, 70]. Since such quotients of $\mathbb{R}^n$ are always differentiable spaces (see theorem 11.14) any differentiable orbifold is a differentiable space in our sense. We refer the reader to example 11.21 for a more detailed discussion of the relation between orbifolds and differentiable spaces.

Sikorski's differential spaces. According to Sikorski [57, 19] a differential space is a pair (X, A) where A is a set of continuous real functions on a topological space X such that:

1. X has the weakest topology such that all functions in A are continuous.
2. A is defined by local conditions: any function on X locally coinciding with functions in A belongs to A.
3. If $f_1, \dots, f_n \in A$ and $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$, then $\phi(f_1, \dots, f_n) \in A$.

Therefore, reduced affine differentiable spaces (in our sense) are differential spaces in this sense, and reduced differentiable spaces are differential spaces in the sense of Mostow [33, 19] which are the sheaf-theoretic version of Sikorski's differential spaces. But non-reduced differentiable spaces are never differential spaces in this sense. See [33] for a comparison of the notion of differential space in Mostow's sense with the definitions of Smith [58] and Chen [6, 7, 8].

Fröhlicher spaces. Fröhlicher and Kriegl [12, 23] defined a differential structure on a set X to be a family $\mathcal{C}$ of curves $\mathbb{R} \to X$ and a family $\mathcal{F}$ of functions $X \to \mathbb{R}$ such that

$$\mathcal{F} = \{f : X \to \mathbb{R} \mid f \circ c \in \mathcal{C}^\infty(\mathbb{R}),\ \forall c \in \mathcal{C}\}\ ,$$
$$\mathcal{C} = \{c : \mathbb{R} \to X \mid f \circ c \in \mathcal{C}^\infty(\mathbb{R}),\ \forall f \in \mathcal{F}\}\ .$$

Hence, by definition, differentiable functions on any $\mathcal{C}^\infty$-space of Fröhlicher and Kriegl are maps $X \to \mathbb{R}$ in the set-theoretic sense, so that non-reduced differentiable spaces are not $\mathcal{C}^\infty$-spaces in this sense. Moreover, such a simple reduced differentiable space as a convergent sequence with the limit point is not a $\mathcal{C}^\infty$-space of Fröhlicher.

Wiener's differential spaces. Our concept has nothing to do with the differential spaces introduced by Wiener [75].

Now let us briefly comment on the plan of these notes:

Chapter 1 presents the elementary theory of smooth manifolds in the spirit of differentiable spaces, so that their connections become clear. We try to take great care with the definitions, while omitting or at best just providing an indication of many proofs, since they are well-known.

Chapter 2 studies differentiable algebras $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$, where $\mathfrak{a}$ is a closed ideal in the usual Fréchet topology of $\mathcal{C}^\infty(\mathbb{R}^n)$. These algebras provide the building blocks for the construction of differentiable spaces since, by definition, the algebra of all differentiable functions on a certain open neighbourhood U of any point of a differentiable space is a differentiable algebra. Note that differentiable functions on U are not a certain kind of map $U \to \mathbb{R}$, since it may occur that $f^2 = 0$ while $f \neq 0$. Then, chapter 3 introduces differentiable spaces as ringed spaces locally modelled on differentiable algebras.

Chapter 4 is devoted to the study of some basic topological properties of differentiable spaces, including the existence of partitions of unity and the equivalence, in the affine case, between locally free sheaves of bounded rank and finitely generated projective modules over the ring of global differentiable functions.

Chapter 5 introduces differentiable subspaces and embeddings, with special attention to infinitesimal neighbourhoods. The main result is an embedding theorem for separated differentiable spaces whose topology has a countable basis.

Chapters 6 and 8 present an interlude of functional analysis, when the introduction of topological modules is unavoidable. Chapter 6 introduces locally convex modules over a Fréchet algebra A and studies topological tensor products $M \otimes_A N$ of these modules. Tensor products provide the basic tool for the theorem of existence of finite direct products and fibred products in the category of differentiable spaces, which is the main result of chapter 7. In particular, we have arbitrary finite intersections of differentiable subspaces and may define the fibre of any morphism of differentiable spaces over a subspace or a point.

In chapter 8 we study modules of fractions $S^{-1}M$ (see [35, 52]), including the Localization theorem for Fréchet modules. Modules of fractions are used in chapter 9 to study finite morphisms. The main result, analogous to Zariski's main theorem for algebraic varieties, states that a morphism of differentiable algebras $A \to B$ is finite if and only if $\mathrm{Spec}_r B \to \mathrm{Spec}_r A$ is a closed separated morphism with finite fibres of bounded degree, which is essentially a reformulation of Malgrange's preparation theorem [26].

In chapter 10 we use topological modules to introduce the module of relative differentials $\Omega_{B/A}$ for any morphism of differentiable algebras $A \to B$, and we study its properties, the main reference being [35]. These modules provide the basic tool for a differential calculus in the realm of differentiable spaces. We use them to define the sheaf of relative differentials $\Omega_{X/S}$ for any morphism of differentiable spaces $X \to S$, and to study smooth morphisms. The main result is the characterization, when the fibres are topological manifolds, of smooth morphisms over a reduced space as open maps with a locally free sheaf of relative differentials. In this chapter we also introduce formally smooth spaces and prove that a differentiable space X is formally smooth if and only if it is locally isomorphic to the Whitney space of a closed set in $\mathbb{R}^n$.

In the last chapter 11 we study quotients of smooth manifolds by compact Lie groups of transformations, which frequently are not smooth manifolds. We show that Schwarz's theorem [56] essentially states that such quotients have a structure of differentiable space, and study a natural stratification. We also briefly consider differentiable groups.

Finally we present two appendices. In the first one we study sheaves of Fréchet modules. For the sake of simplicity, in these notes we have deliberately avoided any topological structure on the sheaves that we introduce along the different chapters. Nevertheless, our sheaves typically have a natural topological structure, and the systematic use of sheaves of Fréchet modules greatly clarifies the theory, enabling us to restate some messy results with a more natural language and a more familiar aspect. In particular we define the inverse image of a sheaf of Fréchet modules.

In appendix B we introduce the space of r-jets of morphisms $X \to Y$, assuming that X is formally smooth. Since the r-jet of a morphism (or a function, a section, etc.) at a point x is just the restriction to the r-th infinitesimal neighbourhood of x, the theory of jets naturally involves non-reduced spaces. This appendix provides an example of the systematic use of the techniques developed in these notes, with a massive utilization of non-reduced differentiable spaces.

As to background, the following are prerequisite:

Commutative algebra: Tensor products, localization (modules of fractions), and completions [1].

Functional analysis: Fréchet spaces, Fréchet algebras, and topological tensor products [16, 18, 27, 29].

Sheaf theory: Basic operations with sheaves, and flabby sheaves [15].

Ideals of differentiable functions: Spectral synthesis of closed ideals , Whitney's ideals, Borel's theorem, Malgrange's preparation theorem, and Schwarz's theorem [26, 45, 71].

Differential geometry: Smooth manifolds, Lie groups, and actions of compact groups on manifolds [4, 73].

1 Differentiable Manifolds

The notion of smooth manifold, as well as those of analytic manifold and scheme, may be expressed appropriately in the language of ringed spaces. In this chapter we shall use this language, reformulating the traditional concepts of smooth manifold, differentiable map, submanifold, etc. Even in this limited context, the use of ringed spaces already presents some conceptual advantages. For example, the artificial concept of "maximal atlas" disappears in the definition of smooth manifold and no coordinate systems are required in the definition of differentiable map.

1.1 Smooth Manifolds

Definitions. Let $\mathcal{C}_X$ be the sheaf of real valued continuous functions on a topological space X. Subsheaves of $\mathbb{R}$-algebras of $\mathcal{C}_X$ are said to be **sheaves of continuous functions** on X. A **reduced ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of continuous functions on X (i.e., $\mathcal{O}_X(U)$ is a subalgebra of the algebra $\mathcal{C}(U, \mathbb{R})$ of all real valued continuous functions on U containing all constant functions, for any open set $U \subseteq X$).

Morphisms of reduced ringed spaces $\varphi\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ are defined to be continuous maps $\varphi\colon Y \to X$ such that $\varphi^* f := f \circ \varphi \in \mathcal{O}_Y(\varphi^{-1}U)$ whenever $f \in \mathcal{O}_X(U)$, so that φ induces a morphism of sheaves

$$\varphi^* : \mathcal{O}_X \longrightarrow \varphi_* \mathcal{O}_Y \ .$$

A morphism φ is said to be an **isomorphism** if there exists a morphism $\psi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $\varphi \circ \psi = Id$ and $\psi \circ \varphi = Id$; that is to say, if φ is a homeomorphism and $\varphi^* : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$ is an isomorphism of sheaves.

The following properties are obvious:

1. Let $(X, \mathcal{O}_X)$ be a reduced ringed space. If U is an open set in X, then $(U, \mathcal{O}_X|_U)$ is a reduced ringed space and the inclusion $U \hookrightarrow X$ is a morphism of reduced ringed spaces.
2. Compositions of morphisms of reduced ringed spaces also are morphisms of reduced ringed spaces.
3. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be reduced ringed spaces and let $\{U_i\}$ be an open cover of Y. A map $\varphi\colon Y \to X$ is a morphism of reduced ringed spaces if and only if so is the restriction $\varphi|_{U_i} : U_i \to X$ for any index i.