SET THEORY and THE STRUCTURE of ARITHMETIC

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PREFACE

This book—the first in a series of three volumes—evolved from lecture notes for a course intended primarily for high school mathematics teachers. The purposes of the course were, first, to answer the question "What is a number?" and, of greater importance, to provide a foundation for the study of abstract algebra, elementary Euclidean geometry and analysis. The second and third volumes in this series will deal respectively with some of the elements of abstract algebra and the study of elementary geometry.

The question "What is a number?" is usually ignored in the elementary school curriculum, and perhaps rightly. However, regardless of whether this question is best avoided, the feeling is becoming widespread that secondary, or even primary, school students should be taught early to recognize that numbers are abstract entities as distinguished from the concrete entities—marks on paper—which are used to denote them. Thus, "1," "$\frac{1}{3} + \frac{2}{3}$," "$\pi/\pi$," and "$2\int_0^1 x\, dx$" all denote the number one. If children are taught this concept, the teacher will then want to know whether these marks can be assigned denotations and, if so, what the denotations may be.

Many working mathematicians have come to hold that much of mathematics, including the classical number systems, can be best based on set theory. Certainly, the language and concepts of set theory have become indispensable to mathematicians as a vehicle for the communication of his ideas. Thus, it is natural to base everything upon set theory. Taking this point of view, one must start (Chapter 1) with enough of the rudiments of set theory upon which to build. The rest of the text leads the reader along a path starting with a construction of the natural number system and ending with a construction of the real numbers. En route the basic properties of the several number systems are developed. On finishing the text the reader should be prepared for first courses in abstract algebra and in real variables.

We have paid little attention to the logical foundations of set theory. We operate with naive, intuitive set theory, being careful to insure that all proofs are easily carried out within the framework of an adequately axiomatized set theory. The one exception that the expert will note is in Chapter 2 where it is casually asserted that N is a set. In axiomatic set theory the proof requires an axiom of infinity (e.g., the statement itself). Although the question of antinomies in set theory is not treated within this book, it has usually been raised at some point in the course and the students seem to enjoy a bit of discussion of the topic.

There are two decisions that anyone writing a text on this subject matter must make. For the natural numbers, he must choose between Peano's postulates and von Neumann's construction. For the real numbers, the question involves Dedekind cuts versus Cantor sequences. In each case, we have taken the second alternative. In the first case, we feel that the difficulties the student faces are about the same either way, provided there is no cheating with, say, recursive definition. Also, having adopted the von Neumann alternative, the instructor can, if time permits, mention the Peano postulates and point out that the class has, in effect, been given an existence proof for them. By this time, most students seem to appreciate the point. As for the question of Dedekind versus Cantor, we have perhaps adopted the more difficult alternative. However, many students do continue with a study of real variables and for this experience the Cantor sequences provide a better preparation.

We are greatly indebted to Professors Robert G. Bartle, Pierce W. Ketchum, Echo D. Pepper and Wilson M. Zaring of the Mathematics Department, University of Illinois, who have taught from various earlier drafts of this book and who have given us both useful criticisms of the text and the benefits of their classroom experiences. We also wish to thank Professors William W. Boone and Herbert E. Vaughan, of the same department, who gave valuable suggestions for Chapters 1 and 2. We owe a particular debt of gratitude to Professor Zaring for his detailed and careful comments on every aspect of the next-to-last draft.

Finally, we are grateful to the members of the Academic Year Institutes at the University of Illinois from the years 1957 to date who attended the courses in which earlier versions of this book were taught. Whatever pedagogical merits the book may possess are due to our attempts to meet the high standards of our colleagues who are dedicated teachers of mathematics in secondary schools and colleges.

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1

THE ELEMENTS OF THE THEORY OF SETS

1.1. INTRODUCTION

At a first and casual thought the word "set" fails to conjure up any familiar mental associations in the mind of a novice at the Theory of Sets. Yet, the set concept is so much a part of our culture and our daily lives that the language we speak contains many special words to denote particular kinds of sets. For instance:

1. A *herd* is a collection or *set* of cattle.
2. A *flock* of sheep is a *set* of sheep.
3. A *bevy* is a *set* of quail.
4. A *clutch* is a *set* of eggs in a nest.
5. A *legal code* is a *set* of laws.

Similarly, there is a *school* of fish, a *pride* of lions, a *brace* of ducks, a *moral code*, and so on.

In elementary mathematics the use of set-theoretic concepts occurs with great frequency, albeit in a hidden way. Consider a few examples from elementary algebra and geometry.

6. The solutions, 1 and 2, of the quadratic equation $x^2 - 3x + 2 = 0$ comprise the *set* of solutions of the given quadratic equation.

7. The locus of the equation $x^2 + y^2 = 1$, a circle, is the *set* of all points whose coordinates satisfy this equation.

8. In algebra school books we find statements such as:

$$\text{In general, } a(b + c) = ab + ac.$$

The meaning of this statement is that for every replacement of a, b, c by names of real (and, also, of complex) numbers the statement resulting from $a(b + c) = ab + ac$ is true. Thus, the "general" statement is a statement concerning the members of the *set* of all real (or complex) numbers.

The list of examples of the concealed use of the set concept in the statements of elementary mathematics can be extended indefinitely since *all of them are really statements concerning sets or about the totality of members of certain sets.* The use of set-theoretic language in mathematics has the advantages of clarity and precision in the communication of mathematical ideas. But if these were the only advantages, one might argue: "Clarity and precision can be obtained by care in speaking and writing ordinary English (or whatever language is used in the school) without bothering to develop a special language for this purpose." Although this thesis is debatable, we do not join the debate at this point. Our reason is that the use of set-theoretic concepts goes deeper than the introduction of clarifying terminology. Most mathematical disciplines can be regarded as branches of set theory. Thus the theory of sets provides a mechanism for unifying and simplifying substantial parts of mathematics. In the course of the present book, it will be seen that the few simple set-theoretic ideas presented in this chapter are adequate for the development of much of elementary arithmetic (Volume I), algebra (Volume II), and elementary (Euclidean) plane geometry (Volume III). The same few basic ideas of set theory will be used time and again in each of these disciplines. And every concept in each of the above-named disciplines will be expressed exclusively in terms of the concepts studied in this chapter.

Although the ideas presented in Chapter 1 are truly simple, they may appear strange to the uninitiated reader. He may find himself asking, "What does this have to do with the mathematics with which I am familiar?" The strangeness will disappear as he progresses further into the text. Its vanishing can be accelerated by constructing numerous examples of the concepts introduced. The connection between this chapter and the more familiar aspects of elementary mathematics will require time to expound. Indeed, this is the subject matter of our book. We urge the reader to have a little patience and read on.

What prior knowledge is required to read this book? In the strictest sense one need only know how to read carefully and to write; little previous mathematical experience is needed. However, we shall, on occasion, rely upon the reader's acquaintance with some of the simplest facts of elementary arithmetic, algebra and geometry. These facts will not be used directly in the development of the subjects under consideration. Their sole uses will be to illustrate certain concepts, to motivate others and, in general, to act as a source of inspiration for what we do here.

This book should not be read as a novel or a newspaper; a sharp pencil and a pad of paper are essential tools for a comprehension of what follows. Careful attention to details will be rewarded.

1.2. THE CONCEPT OF SET

It is beyond the scope of this book to attempt a formal (axiomatic) development of set theory, and therefore we begin by describing the concept of set in a heuristic way.

> By a *set* we mean any collection of objects; the nature of the objects is immaterial. The important characteristic of all sets is this: Given any set and any object, then exactly one of the two following statements is true:
> (a) The given object is a member of the given set.
> (b) The given object is not a member of the given set.

The above description of the concept of set is by no means the last word on the subject. However, it will suffice for all the purposes of this book. A deeper study of the basic ideas of set theory usually requires an introduction such as the present one. Moreover, it would take us in a direction different from our proposed course—the study of elementary arithmetic, algebra and geometry.

EXAMPLES

1. The set of all men named "Sigmund Smith" residing in the United States at 1:00 P.M., June 22, 1802.

2. The set of all unicorns that are now living or have ever lived in the Western Hemisphere.
3. The set of all points in the coordinate plane on the graph of $x^2 + y^2 = 1$.
4. The set of all points in the coordinate plane on the graph of $|x| \geq 1$.
5. The set of all points in the coordinate plane common to the graphs of $x^2 + y^2 < 1$ and $x > 1$.
6. The set Z of all integers.
7. The set E of all even integers.
8. The set of all tenor frogs now living in the Mississippi River.
9. The set of all tenor frogs now living in the Mississippi River and of all points in the coordinate plane on the graph of $x > 1$.

Before continuing with the technicalities of set theory, a few preliminary ideas are required. These will be discussed in Sections 1.3 and 1.4.

1.3. CONSTANTS

No doubt the reader is aware that the language in which this book is written—American English—possesses many ambiguities. Were it not so, the familiar and occasionally amusing linguistic trick known as the "pun" would be a rare phenomenon. Although there is no objection to being funny, any mathematical text should resist strenuously all tendencies to ambiguity and confusion. We shall try to minimize such tendencies by describing carefully the uses of several crucial terms and expressions. Foremost among such terms are the words "constant," "variable" and "equals." These terms are familiar to the reader from his earliest study of high-school algebra. But our uses of these words may differ from those he is accustomed to. Therefore it is suggested that he read this section as well as Section 1.4 with care.

Definition 1. A *constant* is a proper name. In other words, a constant is a name of a particular thing. We say that a constant *names* or *denotes* the thing of which it is a name.

EXAMPLES

1. "Calvin Coolidge" is a constant. It is a name of a president of the United States.

2. "2" is a constant. It is a name of a mathematical object—a number—which will be described in detail in Chapter 2.

Of course, a given object may have different names, and so distinct constants may denote the same thing.

3. During his political life, Calvin Coolidge earned the sobriquet "Silent Cal," because of his extraordinary brevity of speech. Thus "Silent Cal" is a constant and denotes Calvin Coolidge.

4. The expressions "$1 + 1$" and "$-2 + 5 - \frac{6}{3} + \frac{6}{3} + 1$" are constants and both denote the number two.

It may come as a surprise that some constants are built of parts which are themselves constants. Thus "$2 + 1$" is a constant built of "2" and "1", both of which are constants. In ordinary English, there are analogous situations. For instance, the name "Sam Jones" is composed of the two names "Sam" and "Jones."

Constants which denote the same thing are *synonyms* of each other. "Calvin Coolidge" and "Silent Cal" are synonyms; similarly, "2" and "$1 + 1$" are synonyms. Observe that a sentence which is true remains true if it is altered by replacing a name by a synonym. Similarly, if the original sentence is false, then the sentence so altered is likewise false. For example, consider the paragraph

> Calvin Coolidge was the third president of the United States. Calvin Coolidge was also, at one time, a governor of the State of Massachusetts.

The first sentence is false and the second one is true. If "Calvin Coolidge" is replaced throughout by "Silent Cal," we obtain

> Silent Cal was the third president of the United States.
> Silent Cal was also, at one time, a governor of the State of Massachusetts.

Again, the first sentence is false, the second is true.

In ordinary, daily conversation it happens rarely, if at all, that a name of a thing, i.e., a constant, and the thing denoted are confused with each other. No one would mistake the *name* "Silent Cal" for the *person* who was the thirtieth president of the United States. In mathematical discourse, on the other hand, confusions between names and the things named do arise. It is not at all uncommon for the *constant* "2" to be regarded as the *number* two which it names. Let us make the convention that enclosing a name in quotation marks makes a name of the name so enclosed. To illustrate this convention, consider the expressions

Silent Cal

and

"Silent Cal"

written *inside* the two boxes. The expression inside the upper box is a name for the thirtieth president of the United States. The expression inside the lower box is a name for the expression inside the upper box. Similarly, the expression inside

" "Silent Cal" "

is a name for the expression inside the box printed five lines above.

Now consider the sentence

Silent Cal was famous for his brevity of speech.

This sentence *mentions* (or, refers to) the thirtieth president of the United States but it *uses* the name "Silent Cal." The name "Silent Cal" occurs in the sentence, while the thirtieth president *in the flesh* is not sitting on the paper. The sentence

"Silent Cal" has nine letters

mentions a name, and it *uses* a name of the name mentioned, to wit " "Silent Cal"." In referring to, or mentioning, the name "Silent Cal," we no more put that name in the sentence than we put Calvin Coolidge himself into the sentence referring to the thirtieth president. Notice that the sentence

"Silent Cal" was famous for his brevity of speech

is not only false, but even downright silly. For it asserts that a *name* was famous for a property attributable only (as far as we know) to a person.

1.4. VARIABLES AND EQUALITY

Variables occur in daily life as well as in mathematics. We may clarify their use by drawing upon experiences shared by many people, even non-mathematicians.

Official documents of one kind or another contain expressions such as

(1.1) I, ________, do solemnly swear (or affirm) that . . .

What is the purpose of the "________" in (1.1)? Obviously, it is intended to hold a place in which a name, i.e., a constant, may be inserted. The variable in mathematics plays exactly the same role as does the "________" in (1.1); it holds a place in which constants may be inserted. However, devices such as a "________" are clumsy for most mathematical purposes. Therefore, the mathematician uses an easily written symbol, such as a letter of some alphabet, as a place-holder for constants. The mathematician would write (1.1) as, say,

(1.2) I, x, do solemnly swear (or affirm) that . . .

and the "x" is interpreted as holding a place in which a name may be inserted.

Definition 2. A *variable* is a symbol that holds a place for constants.

Suppose a variable occurs in a discussion. What are the constants that are permitted to replace it? Usually an agreement is made, in some manner, as to what constants are admissible as replacements for the variable. If an expression such as (1.1) (or (1.2)) occurs in an official document, the laws under which the document is prepared will specify the persons who may execute it. These, then, are the individuals who are entitled to replace the variable by their names. Thus, with this variable is associated a *set* of persons and the names of the persons in the set are the allowable replacements for the variable. In general:

> With each variable is associated a set; the names of the elements in the set are the permitted replacements for the given variable. The associated set is the *range* of the variable.

The range of a variable in a mathematical discussion is usually determined by the requirements of the problem under discussion.

Variables occur frequently together with certain expressions called *quantifiers*. As one might judge from the word itself, quantifiers deal with "how many." We use but two quantifiers and illustrate the first as follows:

Let x be a variable whose range is the set of all real numbers. Consider the sentence

(1.3) *For each* x, if x is not zero, then its square is positive.

The meaning of (1.3) is

> *For each* replacement of x by the name of a real number, if the number named is not zero, then its square is positive.

The quantifier used here is the expression "for each." Clearly, the intention is, when "for each" is used, to say something concerning each and every member of the range of the variable. For this reason, "for each" is called the *universal quantifier*. It is a common practice to use the expressions "for all" and "for every" as synonymous with "for each," and these three expressions will be used interchangeably in this text.

Observe that if in place of (1.3) we write

(1.4) For each y, if y is not zero, then its square is positive.

where the range of y is also the set of all real numbers, then the meanings of (1.3) and (1.4) are the same. Similarly, y can be replaced by z or some other suitably chosen symbol without any alteration of meaning. Such replacement allows us considerable freedom in the choice of symbols for variables.

The use of the second quantifier is illustrated by the sentence

(1.5) *There exists* an x such that x is greater than five and smaller than six

where the range of x is the set of all real numbers. The meaning of (1.5) is

> There is *at least one* replacement of x by the name of a real number such that the number named is greater than five and smaller than six.

The expression "there exists" is the *existential quantifier*. The expression "there is" is regarded as synonymous with "there exists." Again, the reader may observe that if the variable x is replaced throughout (1.5) by y or some other properly chosen symbol, the range being the same, then the meaning of the new sentence is the same as that of (1.5).