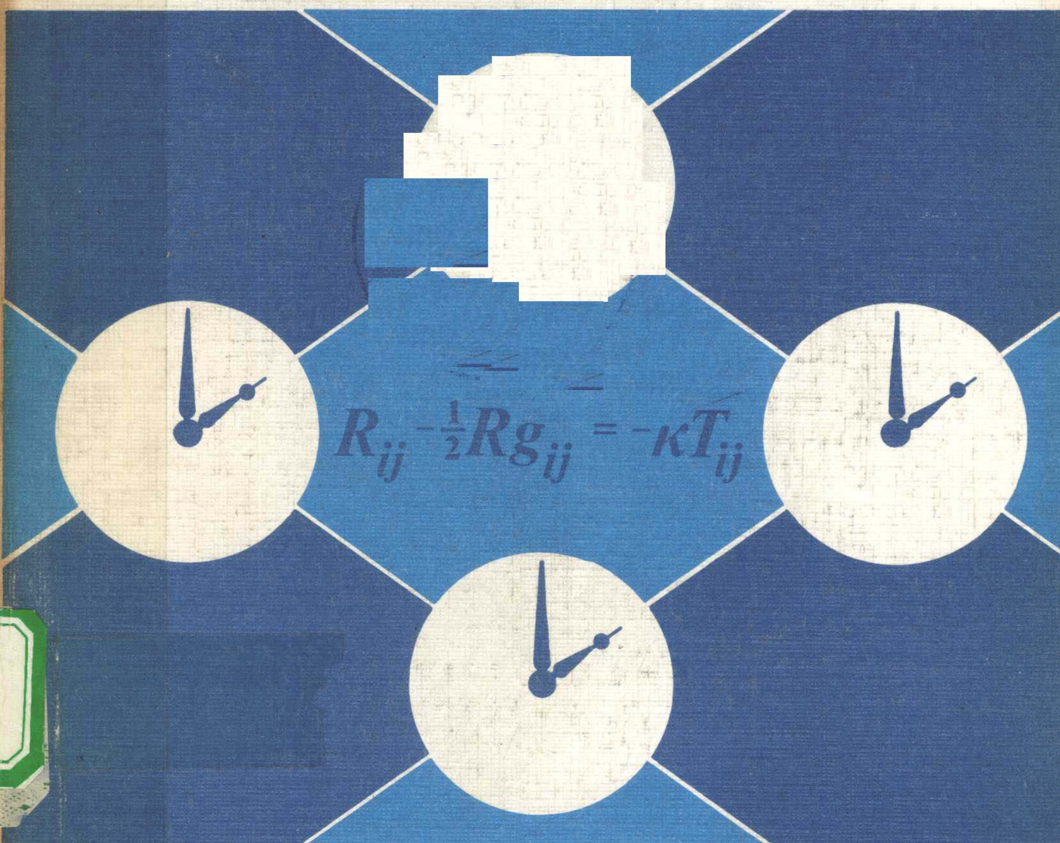
An Introduction to Tensor Calculus, Relativity and Cosmology

3rd Edition



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An Introduction to Tensor Calculus, Relativity and Cosmology

Third Edition

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JOHN WILEY & SONS
Chichester · New York · Brisbane · Toronto · Singapore

First published 1962
by Methuen & Co Ltd
Second edition 1968
Reprinted twice
First published as a Science Paperback 1967
by Chapman and Hall Ltd
11 *New Fetter Lane, London EC4P 4EE*
Reprinted 1975

Library of Congress Cataloging in Publication Data:

Lawden, Derek F.
An introduction to tensor calculus, relativity and cosmology.—3rd ed.

Rev. ed. of: An introduction to tensor calculus and relativity. 2nd ed.
Bibliography: p.
Includes index.
1. Relativity (Physics) 2. Calculus of tensors. 3. Cosmology. I. Title.
QC173.55.L38 1982 530.1'1 81-14801
ISBN 0 471 10082 X (cloth) AACR2
ISBN 0 471 10096 X (paper)

British Library Cataloguing in Publication Data:

Lawden, D. F.
An introduction to tensor calculus, relativity and cosmology.—3rd ed.
1. Relativity (Physics)
2. Calculus of tensors
3. Cosmology—Mathematics
I. Title
530.1'1 QC173.55

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Typeset by Macmillan India Ltd, Bangalore, and printed in the United States of America, by Vail-Ballou Press Inc., Binghamton, N.Y.

Preface

The revolt against the ancient world view of a universe centred upon the earth, which was initiated by Copernicus and further developed by Kepler, Galileo and Newton, reached its natural termination in Einstein's theories of relativity. Starting from the concept that there exists a unique privileged observer of the cosmos, namely man himself, natural philosophy has journeyed to the opposite pole and now accepts as a fundamental principle that all observers are equivalent, in the sense that each can explain the behaviour of the cosmos by application of the same set of natural laws. Another line of thought whose complete development takes place within the context of special relativity is that pioneered by Maxwell, electromagnetic field theory. Indeed, since the Lorentz transformation equations upon which the special theory is based constitute none other than the transformation group under which Maxwell's equations remain of invariant form, the relativistic expression of these equations discovered by Minkowski is more natural than Maxwell's. In the history of natural philosophy, therefore, relativity theory represents the culmination of three centuries of mathematical modelling of the macroscopic physical world; it stands at the end of an era and is a magnificent and fitting memorial to the golden age of mathematical physics which came to an end at the time of the First World War. Einstein's triumph was also his tragedy; although he was inspired to create a masterpiece, this proved to be a monument to the past and its very perfection a barrier to future development. Thus, although all the implications of the general theory have not yet been uncovered, the barrenness of Einstein's later explorations indicates that the growth areas of mathematical physics lie elsewhere, presumably in the fecund soil of quantum and elementary-particle theory.

Nevertheless, relativity theory, especially the special form, provides a foundation upon which all later developments have been constructed and it seems destined to continue in this role for a long time yet. A thorough knowledge of its elements is accordingly a prerequisite for all students who wish to understand contemporary theories of the physical world and possibly to contribute to their expansion. This being universally recognized, university courses in applied mathematics and mathematical physics commonly include an introductory course in the subject at the undergraduate level, usually in the second and third years, but occasionally even in the first year. This book has been written to provide a suitable supporting text for such courses. The author has taught this type of class for the past twenty-five years and has become very familiar with the difficulties regularly experienced by students when they first study this subject;

the identification of these perplexities and their careful resolution has therefore been one of my main aims when preparing this account. To assist the student further in mastering the subject, I have collected together a large number of exercises and these will be found at the end of each chapter; most have been set as course work or in examinations for my own classes and, I think, cover almost all aspects normally treated at this level. It is hoped, therefore, that the book will also prove helpful to lecturers as a source of problems for setting in exercise classes.

When preparing my plan for the development of the subject, I decided to disregard completely the historical order of evolution of the ideas and to present these in the most natural logical and didactic manner possible. In the case of a fully established (and, indeed, venerable) theory, any other arrangement for an introductory text is unjustifiable. As a consequence, many facets of the subject which were at the centre of attention during the early years of its evolution have been relegated to the exercises or omitted entirely. For example, details of the seminal Michelson–Morley experiment and its associated calculations have not been included. Although this event was the spark which ignited the relativistic tinder, it is now apparent that this was an historical accident and that, being implicit in Maxwell's principles of electromagnetism, it was inevitable that the special theory would be formulated near the turn of the century. Neither is the experiment any longer to be regarded as a crucial test of the theory, since the theory's manifold implications for all branches of physics have provided countless other checks, all of which have told in its favour. The early controversies attending the birth of relativity theory are, however, of great human interest and students who wish to follow these are referred to the books by Clark, Hoffmann and Lanczos listed in the Bibliography at the end of this book.

A curious feature of the history of the special theory is the persistence of certain paradoxes which arose shortly after it was first propounded by Einstein and which were largely disposed of at that time. In spite of this, they are rediscovered every decade or so and editors of popular scientific periodicals (and occasionally, and more reprehensibly, serious research journals) seem happy to provide space in which these old battles can be refought, thus generating a good deal of acrimony on all sides (and, presumably, improving circulation). The source of the paradoxes is invariably a failure to appreciate that the special theory is restricted in its validity to inertial frames of reference or an inability to jettison the Newtonian concept of a unique ordering of events in time. Complete books based on these misconceptions have been published by authors who should know better, thus giving students the unfortunate impression that the consistency of this system of ideas is still in doubt. I have therefore felt it necessary to mention some of these 'paradoxes' at appropriate points in the text and to indicate how they are resolved; others have been used as a basis for exercises, providing excellent practice for the student to train himself to think relativistically.

Much of the text was originally published in 1962 under the title *An Introduction to Tensor Calculus and Relativity*. All these sections have been thoroughly revised in the light of my teaching experience, one or two sections have been discarded as containing material which has proved to be of little importance for an understanding of the basics (e.g. relative tensors) and a number of new sections have been added (e.g. equations of motion of an elastic fluid, black holes, gravitational waves, and a more detailed account of the relationship between the metric and affine connections). But the main improvement is the addition of a chapter covering the application of the general theory to cosmology. As a result of the great strides made in the development of optical and, particularly, radio astronomy during the last twenty years, cosmological science has moved towards the centre of interest for physics and very few university courses in the general theory now fail to include lectures in this area.

It is a common (and desirable) practice to provide separate courses in the special and general theories, the special being covered in the second or third undergraduate year and the general in the final year of the undergraduate course or the first year of a postgraduate course. The book has been arranged with this in mind and the first four chapters form a complete unit, suitable for reading by students who may not progress to the general theory. Such students need not be burdened with the general theory of tensors and Riemannian spaces, but can acquire a mastery of the principles of the special theory using only the unsophisticated tool of Cartesian tensors in Euclidean (or quasi-Euclidean) space. In my experience, even students who intend to take a course in the general theory also benefit from exposure to the special theory in this form, since it enables them to concentrate upon the difficulties of the relativity principles and not to be distracted by avoidable complexities of notation. I have no sympathy with the teacher who, encouraged by the shallow values of the times, regards it as a virtue that his lectures exhibit his own present mastery of the subject rather than his appreciation of his students' bewilderment on being led into unfamiliar territory. All students should, in any case, be aware of the simpler form the theory of tensors assumes when the transformation group is restricted to be orthogonal.

As a consequence of my decision to develop the special theory within the context of Cartesian tensors, it was necessary to reduce the special relativistic metric to Pythagorean form by the introduction of either purely imaginary spatial coordinates or a purely imaginary time coordinate for an event. I have followed Minkowski and put $x_4 = ict$; thus, the metric has necessarily been taken in the form

$$ds^2 = dx_1{}^2 + dx_2{}^2 + dx_3{}^2 + dx_4{}^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$$

and ds has the dimension of length. I have retained this definition of the interval between two events observed from a freely falling frame in the general theory; this not only avoids confusion but, in the weak-field approximation, permits the distinction between covariant and contravariant components of a tensor to be eliminated by the introduction of an imaginary time. A disadvantage is that ds is imaginary for timelike intervals and the interval parameter s accordingly takes imaginary values along the world-line of any material body. Thus, when writing down the equations for the geodesic world-line of a freely falling body, it is usually convenient to replace s by τ, defined by the equation $s = ic\tau$, τ being called the proper time and $d\tau$ the proper time interval. However, it is understood throughout the exposition of the general theory that the metric tensor for space-time g_{ij} is such that $ds^2 = g_{ij}\,dx^i\,dx^j$; a consequence is that the cosmical constant term in Einstein's equation of gravitation has a sign opposite to that taken by some authors.

References in the text are made by author and year and have been collected together at the end of the book.

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May, 1981.

List of Constants

In the SI system of units:

Gravitational constant = $G = 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2}
Velocity of light = $c = 2.998 \times 10^8$ ms^{-1}
Mass of sun = 1.991×10^{30} kg
Mass of earth = 5.979×10^{24} kg
Mean radius of earth = 6.371×10^6 m
Permittivity of free space = $\varepsilon_0 = 8.854 \times 10^{-12}$
Permeability of free space = $\mu_0 = 1.257 \times 10^{-6}$

Contents

CHAPTER 1

Special Principle of Relativity. Lorentz Transformations

1. Newton's laws of motion

A proper appreciation of the physical content of Newton's three laws of motion is an essential prerequisite for any study of the special theory of relativity. It will be shown that these laws are in accordance with the fundamental principle upon which the theory is based and thus they will also serve as a convenient introduction to this principle.

The first law states that *any particle which is not subjected to forces moves along a straight line at constant speed.* Since the motion of a particle can only be specified relative to some coordinate frame of reference, this statement has meaning only when the reference frame to be employed when observing the particle's motion has been indicated. Also, since the concept of force has not, at this point, received a definition, it will be necessary to explain how we are to judge when a particle is 'not subjected to forces'. It will be taken as an observed fact that if rectangular axes are taken with their origin at the centre of the sun and these axes do not rotate relative to the most distant objects known to astronomy, viz. the extragalactic nebulae, then the motions of the neighbouring stars relative to this frame are very nearly uniform. The departure from uniformity can reasonably be accounted for as due to the influence of the stars upon one another and the evidence available suggests very strongly that if the motion of a body in a region infinitely remote from all other bodies could be observed, then its motion would always prove to be uniform relative to our reference frame irrespective of the manner in which the motion was initiated.

We shall accordingly regard the first law as asserting that, in a region of space remote from all other matter and empty save for a single test particle, a reference frame can be defined relative to which the particle will always have a uniform motion. Such a frame will be referred to as an *inertial frame.* An example of such an inertial frame which is conveniently employed when discussing the motions of bodies within the solar system has been described above. However, if S is any inertial frame and $\bar{S}$ is another frame whose axes are always parallel to those of S but whose origin moves with a constant velocity $\mathbf{u}$ relative to S, then $\bar{S}$ also is inertial. For, if $\mathbf{v}$, $\bar{\mathbf{v}}$ are the velocities of the test particle relative to S, $\bar{S}$ respectively, then

$$\bar{\mathbf{v}} = \mathbf{v} - \mathbf{u} \tag{1.1}$$

and, since $\mathbf{v}$ is always constant, so is $\bar{\mathbf{v}}$. It follows, therefore, that a frame whose origin is at the earth's centre and whose axes do not rotate relative to the stars can, for most practical purposes, be looked upon as an inertial frame, for the motion of the earth relative to the sun is very nearly uniform over periods of time which are normally the subject of dynamical calculations. In fact, since the earth's rotation is slow by ordinary standards, a frame which is fixed in this body can also be treated as approximately inertial and this assumption will only lead to appreciable errors when motions over relatively long periods of time are being investigated, e.g. Foucault's pendulum, long-range gunnery calculations. A frame attached to a non-rotating spaceship, whose rocket motor is inoperative and which is moving in a negligible gravitational field (e.g. in interstellar space), provides another example of an inertial frame. Since the stars of our galaxy move uniformly relative to one another over very long periods of time, the frames attached to them will all be inertial provided they do not rotate relative to the other galaxies.

Having established an inertial frame, if it is found by observation that a particle does not have a uniform motion relative to the frame, the lack of uniformity is attributed to the action of a *force* which is exerted upon the particle by some agency. For example, the orbits of the planets are considered to be curved on account of the force of gravitational attraction exerted upon these bodies by the sun and when a beam of charged particles is observed to be deflected when a bar magnet is brought into the vicinity, this phenomenon is understood to be due to the magnetic forces which are supposed to act upon the particles. If $\mathbf{v}$ is the particle's velocity relative to the frame at any instant t, its acceleration $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$ will be non-zero if the particle's motion is not uniform and this quantity is accordingly a convenient measure of the applied force $\mathbf{f}$. We take, therefore,

$$\mathbf{f} \propto \mathbf{a}$$

or

$$\mathbf{f} = m\mathbf{a} \tag{1.2}$$

where m is a constant of proportionality which depends upon the particle and is termed its *mass*. The definition of the mass of a particle will be given almost immediately when it arises quite naturally out of the third law of motion. Equation (1.2) is essentially a definition of force relative to an inertial frame and is referred to as the *second law of motion*. It is sometimes convenient to employ a non-inertial frame in dynamical calculations, in which case a body which is in uniform motion relative to an inertial frame and is therefore subject to no forces, will nonetheless have an acceleration in the non-inertial frame. By equation (1.2), to this acceleration there corresponds a force, but this will not be attributable to any obvious agency and is therefore usually referred to as a 'fictitious' force. Well-known examples of such forces are the centrifugal and Coriolis forces associated with frames which are in uniform rotation relative to an inertial frame, e.g. a frame rotating with the earth. By introducing such 'fictitious' forces, the second law of motion becomes applicable in all reference frames. Such forces are called *inertial forces* (see Section 44).

According to the third law of motion, *when two particles P and Q interact so as to influence one another's motion, the force exerted by P on Q is equal to that exerted by Q on P but is in the opposite sense.* Defining the *momentum* of a particle relative to a reference frame as the product of its mass and its velocity, it is proved in elementary textbooks that the second and third laws taken together imply that the sum of the momenta of any two particles involved in a collision is conserved. Thus, if m_1, m_2 are the masses of two such particles and $\mathbf{u}_1, \mathbf{u}_2$ are their respective velocities immediately before the collision and $\mathbf{v}_1, \mathbf{v}_2$ are their respective velocities immediately afterwards, then

$$m_1 \mathbf{u}_1 + m_2 \mathbf{u}_2 = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 \tag{1.3}$$

i.e.

$$\frac{m_2}{m_1}(\mathbf{u}_2 - \mathbf{v}_2) = \mathbf{v}_1 - \mathbf{u}_1 \tag{1.4}$$

This last equation implies that the vectors $\mathbf{u}_2 - \mathbf{v}_2$, $\mathbf{v}_1 - \mathbf{u}_1$ are parallel, a result which has been checked experimentally and which constitutes the physical content of the third law. However, equation (1.4) shows that the third law is also, in part, a specification of how the mass of a particle is to be measured and hence provides a definition for this quantity. For

$$\frac{m_2}{m_1} = \frac{|\mathbf{v}_1 - \mathbf{u}_1|}{|\mathbf{u}_2 - \mathbf{v}_2|} \tag{1.5}$$

and hence the ratio of the masses of two particles can be found from the results of a collision experiment. If, then, one particular particle is chosen to have unit mass (e.g. the standard kilogramme), the masses of all other particles can, in principle, be determined by permitting them to collide with this standard and then employing equation (1.5).

2. Covariance of the laws of motion

It has been shown in the previous section that the second and third laws are essentially definitions of the physical quantities force and mass relative to a given reference frame. In this section, we shall examine whether these definitions lead to different results when different inertial frames are employed.

Consider first the definition of mass. If the collision between the particles m_1, m_2 is observed from the inertial frame $\bar{S}$, let $\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2$ be the particle velocities before the collision and $\bar{\mathbf{v}}_1$, $\bar{\mathbf{v}}_2$ the corresponding velocities after the collision. By equation (1.1),

$$\bar{\mathbf{u}}_1 = \mathbf{u}_1 - \mathbf{u}, \qquad \text{etc.} \tag{2.1}$$

and hence

$$\bar{\mathbf{v}}_1 - \bar{\mathbf{u}}_1 = \mathbf{v}_1 - \mathbf{u}_1, \quad \bar{\mathbf{u}}_2 - \bar{\mathbf{v}}_2 = \mathbf{u}_2 - \mathbf{v}_2 \tag{2.2}$$

It follows that if the vectors $\mathbf{v}_1 - \mathbf{u}_1, \mathbf{u}_2 - \mathbf{v}_2$ are parallel, so are the vectors $\bar{\mathbf{v}}_1 - \bar{\mathbf{u}}_1$, $\bar{\mathbf{u}}_2 - \bar{\mathbf{v}}_2$ and consequently that, in so far as the third law is experimentally verifiable, it is valid in all inertial frames if it is valid in one. Now let $\bar{m}_1, \bar{m}_2$ be the particle masses as measured in $\bar{S}$. Then, by equation (1.5),

$$\frac{\bar{m}_2}{\bar{m}_1} = \frac{|\bar{\mathbf{v}}_1 - \bar{\mathbf{u}}_1|}{|\bar{\mathbf{u}}_2 - \bar{\mathbf{v}}_2|} = \frac{|\mathbf{v}_1 - \mathbf{u}_1|}{|\mathbf{u}_2 - \mathbf{v}_2|} = \frac{m_2}{m_1} \tag{2.3}$$

But, if the first particle is the unit standard, then $m_1 = \bar{m}_1 = 1$ and hence

$$\bar{m}_2 = m_2 \tag{2.4}$$

i.e. the mass of a particle has the same value in all inertial frames. We can express this by saying that mass is an *invariant* relative to transformations between inertial frames.

By differentiating equation (1.1) with respect to the time t, since $\mathbf{u}$ is constant it is found that

$$\bar{\mathbf{a}} = \mathbf{a} \tag{2.5}$$

where $\mathbf{a}, \bar{\mathbf{a}}$ are the accelerations of a particle relative to $S, \bar{S}$ respectively. Hence, by the second law (1.2), since $\bar{m} = m$, it follows that

$$\bar{\mathbf{f}} = \mathbf{f} \tag{2.6}$$

i.e. the force acting upon a particle is independent of the inertial frame in which it is measured.

It has therefore been shown that equations (1.2), (1.4) take precisely the same form in the two frames, $S, \bar{S}$, it being understood that mass, acceleration and force are independent of the frame and that velocity is transformed in accordance with equation (1.1). When equations preserve their form upon transformation from one reference frame to another, they are said to be *covariant* with respect to such a transformation. Newton's laws of motion are covariant with respect to a transformation between inertial frames.

3. Special principle of relativity

The special principle of relativity asserts that *all physical laws are covariant with respect to a transformation between inertial frames.* This implies that all observers moving uniformly relative to one another and employing inertial frames will be in agreement concerning the statement of physical laws. No such observer, therefore, can regard himself as being in a special relationship to the universe not shared by any other observer employing an inertial frame; there are no privileged observers. When man believed himself to be at the centre of creation both physically and spiritually, a principle such as that we have just enunciated would have been rejected as absurd. However, the revolution in attitude to our physical environment initiated by Copernicus has proceeded so far that today the principle is accepted as eminently reasonable and very strong evidence contradicting the principle would have to be discovered to disturb it as the foundation upon which theoretical physics is based. It is this principle which guarantees that observers inhabiting distant planets, belonging to stars whose motions may be very different from that of our own sun, will nevertheless be able to explain their local physical phenomena by application of the same physical laws we use ourselves.

It has been shown already that Newton's laws of motion obey the principle. Let us now transfer our attention to another set of fundamental laws governing non-mechanical phenomena, viz. Maxwell's laws of electrodynamics. These are more complex than the laws of Newton and are most conveniently expressed by the equations

$$\text{curl } \mathbf{E} = -\partial \mathbf{B}/\partial t \tag{3.1}$$

$$\text{curl } \mathbf{H} = \mathbf{j} + \partial \mathbf{D}/\partial t \tag{3.2}$$

$$\text{div } \mathbf{D} = \rho \tag{3.3}$$

$$\text{div } \mathbf{B} = 0 \tag{3.4}$$

where **E**, **H** are the electric and magnetic intensities respectively, **D** is the displacement, **B** is the magnetic induction, **j** is the current density and ρ is the charge density (SI units are being used). Experiment confirms that these equations are valid when any inertial frame is employed. The most famous such experiment was that carried out by Michelson and Morley, who verified that the velocity of propagation of light waves in any direction is always measured to be c ($= 3 \times 10^8 \text{ m s}^{-1}$) relative to an apparatus stationary on the earth. As is well known, light has an electromagnetic character and this result is predicted by equations (3.1)–(3.4). However, the velocity of the earth in its orbit at any time differs from its velocity six months later by twice the orbital velocity, viz. 60 km/s and thus, by taking measurements of the velocity of light relative to the earth on two days separated by this period of time and showing them to be equal, it is possible to confirm that Maxwell's equations conform to the special principle of relativity. This is effectively what Michelson and Morley did. However, this interpretation of the results of their experiment was not accepted immediately, since it was thought that electromagnetic phenomena were supported by a medium called the *aether* and that Maxwell's equations would prove to be valid only in an inertial frame stationary in this medium, i.e. the special principle of relativity was denied for electromagnetic phenomena. It was supposed that an 'aether wind' would blow through an inertial frame not at rest in the aether and that this would have a disturbing effect on the propagation of electromagnetic disturbances through the medium, in the same way that a wind in the atmosphere affects the spread of sound waves. In such a frame, Maxwell's equations would (it was surmised) need correction by the inclusion of terms involving the wind velocity. That this would imply that terrestrial electrical machinery would behave differently in winter and summer does not appear to have raised any doubts!

After Michelson and Morley's experiment, a long controversy ensued and, though this is of great historical interest, it will not be recounted in this book. The special principle is now firmly established and is accepted on the grounds that the conclusions which may be deduced from it are everywhere found to be in conformity with experiment and also because it is felt to possess *a priori* a high degree of plausibility. A description of the steps by which it ultimately came to be appreciated that the principle was of quite general application would therefore be superfluous in an introductory text. It is, however, essential for our future development of the theory to understand the prime difficulty preventing an early acceptance of the idea that the electromagnetic laws are in conformity with the special principle.

Consider the two inertial frames S, $\bar{S}$. Suppose that an observer employing S measures the velocity of a light pulse and finds it to be $\mathbf{c}$. If the velocity of the same light pulse is measured by an observer employing the frame $\bar{S}$, let this be $\bar{\mathbf{c}}$. Then, by equation (1.1),

$$\bar{\mathbf{c}} = \mathbf{c} - \mathbf{u} \tag{3.5}$$

and it is clear that, in general, the magnitudes of the vectors $\bar{\mathbf{c}}$, $\mathbf{c}$ will be different. It appears to follow, therefore, that either Maxwell's equations (3.1)–(3.4) must be modified, or the special principle of relativity abandoned for electromagnetic phenomena. Attempts were made (e.g. by Ritz) to modify Maxwell's equations, but certain consequences of the modified equations could not be confirmed experimentally. Since the special principle was always found to be valid, the only remaining alternative was to reject equation (1.1) and to replace it by another in conformity with the experimental result that the speed of light is the same in all inertial frames. As will be shown in the next section, this can only be done at the expense of a radical revision of our intuitive ideas concerning the nature of space and time and this was very understandably strongly resisted.

4. Lorentz transformations. Minkowski space–time

The argument of this section will be founded on the following three postulates:

Postulate 1. A particle free to move under no forces has constant velocity in any inertial frame.

Postulate 2. The speed of light relative to any inertial frame is c in all directions.

Postulate 3. The geometry of space is Euclidean in any inertial frame.

Let the reference frame S comprise rectangular Cartesian axes $Oxyz$. We shall assume that the coordinates of a point relative to this frame are measured by the usual procedure and employing a measuring scale which is stationary in S (it is necessary to state this precaution, since it will be shown later that the length of a bar is not independent of its motion). It will also be supposed that standard