SIGNAL FLOW GRAPHS AND APPLICATIONS

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Signal Flow Graphs and Applications

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Prentice-Hall, Inc. Englewood Cliffs, N. J.
1962

Foreword

The content of this monograph on Signal Flow Graphs is a result of research done at Laval University in the Department of Electrical Engineering.

The work of S. J. Mason on this subject suggested to one of the authors, M. Boisvert, material for a course given in 1954 in the Department of Electrical Engineering. The first two chapters resume the material of this course.

One of his research assistants, L. P. A. Robichaud from the Canadian Armament Research and Development Establishment, undertook a more profound study of flow graphs and their applications. His interesting work is presented in the following chapters: Direct Analysis of Electrical Networks, Algebra of Quadripole and *N*-port Networks, Direct Simulation on an Analogue Computer, Algebraic Reduction of Flow Graphs Using a Digital Computer.

In most Electrical Engineering Curricula, several courses are given to power and electronic engineering students in common in order to show them the basic unity of the methods applied to both branches. Flow graphs are a noteworthy example of this unity. The third author, J. Robert, shows in Chapter Six that flow graphs lend themselves well to the study of power engineering: Application of Signal Flow Graphs to Electrical Machine Theory, completed after consultation with L. P. A. Robichaud, who has given the course in Flow Graphs since 1957.

I wish to thank J. C. Gille, visiting professor in our Department, who encouraged this publication.

I congratulate the authors for this contribution to the field of Electrical Engineering.

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Preface

A major preoccupation of modern engineering concerns the study of lumped parameter physical systems. These systems consist of a network of physical elements defined by constant or variable parameters and possess a finite number of degrees of freedom. Electrical networks, linear servomechanisms, mechanical, electro-mechanical and electro-accoustical systems are typical examples. The engineer depicts these systems by schematic diagrams or networks, whose branches represent the physical elements. From such diagrams are derived the equations of the system which are generally integro-differential equations. At least in the linear case, these can be reduced with the help of Fourier and Laplace transforms to a set of algebraic equations.

One then proceeds with the reduction of the equations. The objectives of this reduction are either to obtain reduced systems of equations, which contain only certain variables of interest, or characteristic equations, or else to find certain relationships between the variables at the input and the output of the system, such as gains, input and transfer functions, and so on. Since the time of Cayley, Hamilton and Sylvester, matrix algebra has undoubtedly remained a powerful tool for the analysis of physical systems. Of particular note is the work of Kron (1) who thoroughly systematized the formulation of electrical network problems by using connection matrices to describe the network itself, and branch matrices to represent the elements which are interconnected to form the system.

The fact remains, however, that the equilibrium equations of a system are essentially quantitative relations between the variables and do not show explicitly the structure of the system under study. However, the whole algebra of reduction of the system of equations depends much more on this structure, that is on the interconnections of the elements, than on the nature of the elements themselves. The equations alone do not generally allow one to infer easily certain fundamental relations or to make simplifications, whereas the topology of the diagram suggests these immediately. Indeed, who would think of using Thevenin's theorem to solve a system of algebraic equations for one of the variables if he had not previously seen the network corresponding to these equations?

Special techniques have been developed for the solution of some physical systems. In practice, these techniques are generally restricted to well-defined domains, although in general they could be translated into analogous problems in different fields. One can mention in particular the techniques of analysis of electrical networks. In several problems of this type, one can go from an original network to equivalent networks containing less and less branches, and finally to an equivalent network from which the expression of the desired variable can be obtained directly by inspection. It is then unnecessary to write equations, and the process of algebraic reduction with equivalent circuits frequently represents a considerable economy of effort when compared with matrix methods. If the schematic diagrams show the functional relations in an explicit enough manner, and if there exist techniques to perform the reduction of the diagram while maintaining some relationships invariant, one can use the name 'graphic algebra' to describe the algebraic reduction performed with the help of equivalent networks. Another example of graphic algebra is found in the theory of servomechanisms with its block diagrams. In these diagrams, the functional relations are represented in a manner more explicit even than in electric networks; it is thus not surprising to see Tustin (2), for example, evaluate the determinant of a control system by a simple enumeration of the loops in the block-diagram.

Mason's signal flow graphs (3) constitute certainly the most advanced form of graphic algebra. In creating this type of graph, Mason has made the node and the branch essentially pure representations of the variable and the functional relation. By divorcing the idea of branch from the idea of physical element which was attached to any branch in the diagrams used until now, Mason has obtained a graph of the equations applicable to any field of engineering. In such problems, the independent variables represent excitations which are applied to the system, and the responses of the system are associated with the dependent variables. Signal flow graphs, as the name implies, depict the variables as signals traveling along the branches of the graph. These signals are modified according to the characteristics of the branches traversed. The incoming signals are added at each node to define a dependent variable which is considered as a new signal transmitted along all the branches starting from this node. Mason has shown that the topological transformations performed on linear graphs correspond to algebraic transformations performed on the system of equations. In this way, it is possible either to solve the graph directly, or to transform the graph into a residual form, in which appear only the nodes corresponding to specified variables or the branches representing the input and transfer functions of the system.

The use of signal flow graphs for the study of physical systems, especially systems represented by a schematic diagram, constitutes the objective of this monograph. The authors do not pretend to exhaust the subject or to cover all the applications of flow graphs; they have simply collected under the same title different investigations which were related to the study of linear systems with signal flow graphs.

In general, there are several ways of choosing the variables in a complex system. Corresponding to each choice, a system of equations can be written and each system of equations can be represented in a graph. This formulation of the equations becomes direct and automatic if one has at his disposal techniques which permit the drawing of a graph directly from the schematic diagram of the system under study. The structure of the graphs thus obtained is related in a simple manner to the topology of the schematic diagram, and it becomes unnecessary to consider the equations, even implicity, to obtain the graph. In some cases, one has simply to imagine the flow graph in the schematic diagram and the desired answers can be obtained without even drawing the flow graph.

The techniques developed in this monograph are useful particularly in problems defined by a schematic diagram, and their usefulness increases as the schematic diagram becomes more explicit. For this reason, and also possibly because of professional inclination, the authors have chosen to present this monograph in terms of electrical network theory, and to select as examples almost exclusively networks constructed of ideal transformers, active elements and gyrators, besides passive reciprocal elements. All the physical systems analogous to these networks constitute the domain of application of the techniques developed in the following chapters. Trent (4) has shown that all the physical systems which satisfy the following conditions fall into this category.

1. The finite lumped system is composed of a number of simple parts, each of which has known dynamical properties which can be defined by equations using two types of scalar variables and parameters of the system. Variables of the first type represent quantities which can be measured, at least conceptually, by attaching an indicating instrument to two connection points of the element. Variables of the second type characterize quantities which can be measured by connecting a meter in series with the element. Relative velocities and positions, pressure differentials and voltages are typical quantities of the first class, whereas electric currents, forces, rates of heat flow, are variables of the second type. Firestone (5) has been the first to distinguish these two types of variables with the names "across variables" and "through variables."
2. Variables of the first type must obey a mesh law, analogous to Kirchhoff's voltage law, whereas variables of the second type must satisfy an incidence law analogous to Kirchhoff's current law.
3. Physical dimensions of appropriate products of the variables of the two types must be consistent.

For the systems in which these conditions are satisfied, it is possible to draw a linear graph isomorphic with the dynamical properties of the system as described by the chosen variables. The techniques described in this monograph can be applied directly to these linear graphs as well as to electrical networks, to obtain a signal flow graph of the system.

The first two chapters present the elements of signal flow graph theory as developed by Mason. The method of presentation in Chapt. 2, in which the formal theory of feedback is briefly reviewed, stresses the relation between the closed path in the flow graph and the loop in feedback systems. Chapter 3 is entitled: "Direct analysis of electrical networks through signal flow graphs." It is shown in this chapter that one can consider elementary graphs corresponding to the branches of the network and that the elementary graphs can be interconnected in the same way that the elements of the networks are interconnected, or in the dualistic way, to produce a flow graph of the node or mesh equations. A closed path of two branches constitutes the elementary graph, and the functional relations represented by these branches depend directly on the impedance or the admittance of the corresponding branch in the network. A new formula is also developed for the expansion of the determinant of the graph in terms of the elements of the network. For passive networks without mutual inductances, this formula gives the same results as Kirchhoff's rules with its enumeration of all the trees; the new formula can however be applied to the flow graph of any network. In Chapt. 4, more complex elementary graphs are introduced; these have four branches and correspond to two-terminal-pair networks. Instead of a very detailed graph of the network, one obtains a condensed flow graph for a large network by interconnecting the elementary graphs in the same manner as the two-terminal-pair networks are connected to form the larger network. The description of this chapter remains close to the matrix formalism of quadripole theory. Flow graphs are, however, more flexible than matrices, since graphs corresponding to matrices of different forms can be joined together without first performing the transformations which are required before matrix addition or multiplication. The extension of these techniques to n-port networks is briefly mentioned. The simulation of a physical system on an analogue computer is treated in Chapt. 5. By choosing the elementary flow graphs of the preceeding chapter to correspond to the operational units of a computer, the signal flow graph itself can be used to set up the analogue. Because the graph is obtained directly from the schematic diagram of the physical system, this process can be called direct simulation, since it is not required to write the equations in order to set up the computer model. In Chapt. 6, the methods developed in Chapts. 4 and 5 are used in some examples drawn from the field of electrical machine theory; in particular flow graphs are used exclusively to study the generalized rotating machine. In any complex problem, if numerical values are given for the parameters, the reduction of the flow graph will involve tedious computations. For this reason, Chapt. 7 considers the programming of a digital computer to solve a flow graph. The basis of this technique is described in a general way such that it is possible, with appropriate subroutines, to program a computer so as to get a numerical answer as well as a symbolic expression or a characteristic equation.

In concluding this introduction, the authors have the very agreeable task of expressing their thanks to the persons who have lent their assistance in the preparation of this monograph. We would like to express our deep gratitude to Mr. J. C. Gille, Ingénieur de l'Air, professor of servomechanisms at l'Ecole Supérieure d'Aéronautique (Paris), and visiting professor at l'Université Laval. He has been the promoter of this publication, and an untiring instigator and adviser.

We are grateful to M. Lionel Boulet, head of the Department of Electrical Engineering at l'Université Laval, for his constant help and his inspiring attitude. We acknowledge with pleasure the encouragement and cooperation given by the authorities of the Canadian Armament Research and Development Establishment (CARDE), and in particular by Messrs. E. J. Bobyn and R. S. Mitchell.

Many of our colleagues have participated in technical discussions or have proofread the manuscript; we would like to thank in particular Messrs. J. Dumas, J. Duprat of Université Laval, and Messrs. J. Cummins, Y. Caron, R. Vaillancourt, and F. W. Slingerland of CARDE. We would also like to thank Mr. F. C. Holland formerly of CARDE, who has done the coding for the examples described in Chapt. 7.

We gratefully acknowledge the financial assistance received from the National Research Council of Canada and from the Secretary of the Province of Quebec, for the pursuit of this research, and for the publication of its results.

Université Laval
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PRENTICE-HALL INTERNATIONAL
London • Tokyo • Sydney • Paris
PRENTICE-HALL OF CANADA, LTD.
PRENTICE-HALL DE MEXICO, S.A.

Library of Congress Catalog Card Number 62-15497
Printed in the United States of America
89841—C

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1

Elementary Theory

1-1 INTRODUCTION

"One might say that the problem of system analysis is a problem of handling simultaneous equations. This description is certainly valid but incomplete in that it fails to show how important gross identifications can be made in the system by inference from the equations themselves." To illustrate this remark of Linvill[6] one can refer to the theory of feedback, or to the issue of the IRE transactions on circuit theory devoted to signal theory[7]. It might be added to this first consideration that the similarity of two problems does not depend in the first place on the physical arrangement of the elements or on the dimensions of the variables, but rather on the structure of the relationships between the variables in the two problems. As a proof of such a statement, it is necessary only to recall the use of duality in network theory or the numerous overlappings in the theories of feedback and servomechanism. The frequent use by the electrical engineer of diagrams to represent the structure of the relations between the variables of a system, or certain fundamental relationships, is not surprising.

Although Mason, in developing signal flow graphs, has had constantly in mind the theory of feedback and has been profoundly influenced by Bode[8], these graphs have been extremely useful in all sorts of problems where feedback was not of particular interest. For this reason, only the elementary properties of signal flow graphs will be described in this chapter, leaving all the feedback aspects for a second chapter.

After the flow graph has been defined, it is shown that such a graph can be drawn at least as easily as the equations are formulated for a specific problem. A proper terminology will then be introduced to allow a simple statement of a few rules which will permit either the direct solution of the graph for a given variable; or the reduction of the graph to a residual form showing only certain fundamental relations; or the transformation of the graph by inversion of the relations between one excitation and one response.

1-2 DEFINITION OF SIGNAL FLOW GRAPHS

A **signal flow graph** *is a graph of directed branches, interconnected at certain points called nodes, defining uniquely a system of linear algebraic equations.* The **nodes** represent the variables and the coefficients of the equations are written alongside the branches and called **branch transmittances.** The variable x_j represented by the node j in the graph is then defined as the sum of all the products $t_{ij}x_i$ where t_{ij} is the transmittance of a branch going from node i to node j, and x_i is the variable at the node of origin of that branch. With the following system of equations:

$$\begin{aligned} x_2 &= ax_1 + bx_3 \\ x_3 &= x_0 + ex_1 + dx_2 \end{aligned} \tag{1-1}$$

where x_1 and x_0 are independent variables, the graph of Fig. 1-1a is obtained from the first equation and that of Figure 1-1b from the second. The complete graph for the Eq. (1-1) is shown in Fig. 1-1c.

Mason[3] compares this type of graph to a system of signal transmission in which the nodes of the independent variables are transmitting stations and the other nodes are repeaters. The signals travel along the branches and are modified by the transmittances of the branches traversed. The repeater at a given node combines all the incoming signals and sends the resulting signal along all the branches diverging from that node. From this analogy have come the names

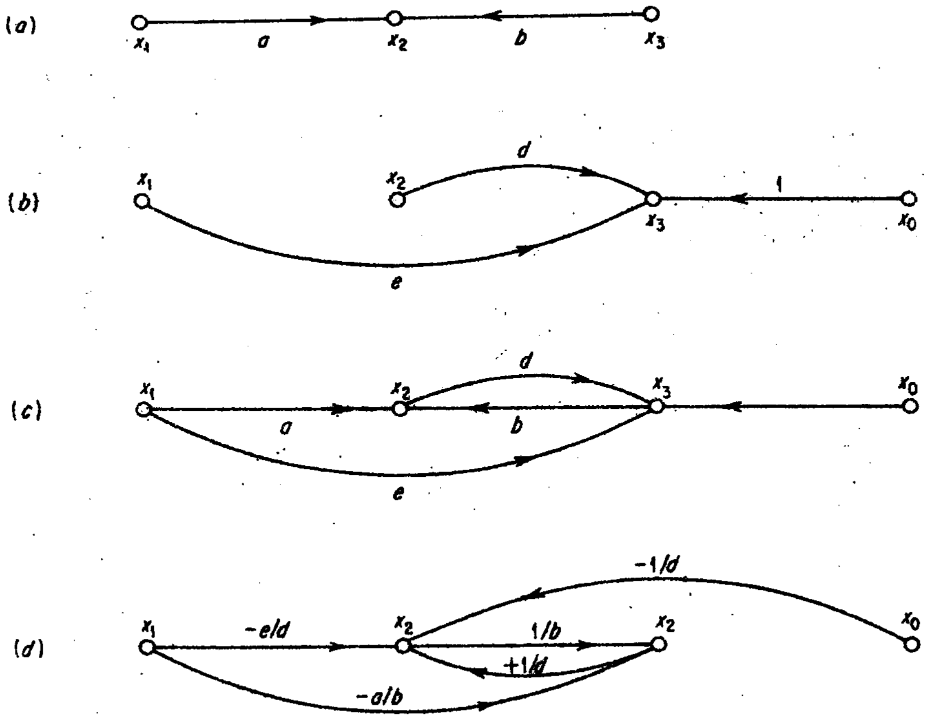

Fig. 1-1. Drawing of a flow graph

"signal flow graph" for this diagram and "transmittance" for the coefficient associated with a branch.

The signal flow graph contains the same information as the equations from which it is derived; but there does not exist a one-to-one correspondence between the graph and the system of equations. One system will give different graphs according to the order in which the equations are used to define the variable written on the left-hand side. As an example, by interchanging the Eq. (1-1) to define x_2 and x_3, one obtains the graph of Fig. 1-1d which contains the same information as that of Fig. 1-1c, but which is topologically different. With respect to the transmission of signals, this flexibility of flow graphs permits a representation very close to the physical system if the graph is drawn in such a way that the signal transmission corresponds directly to a physical phenomenon.

One notes that the signal flow graph is almost the same type of representation as the *block diagram* used in the theory of servomechanisms. However, the idea of branch is now generalized and completely divorced from the idea of block or group of physical elements usually present inside the blocks of a block diagram. In fact, the same physical element can appear in the transmittances of several branches of the flow graph as later examples will show.

The preceding definition of signal flow graph is really the definition of a linear graph. More generally, a flow graph, as defined originally by Mason[3], implies a set of functional relations, linear or not. However, for the applications considered in this monograph (except for a few particular examples) the definition of linear graph is sufficient and is the one used because it allows a simple algebra. The variables of the graph are then defined by linear equations of the form:

$$x_k = f_k(x_1, x_2, x_3, \ldots, x_n) \tag{1-2}$$

or else are independent variables. The matrix T of the branch transmittances is a square matrix of order n equal to the number of nodes of the form:

$$T = \begin{vmatrix} t_{11} & t_{21} & t_{31} & \cdots \\ t_{12} & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \\ t_{1n} & t_{2n} & t_{3n} & \cdots \end{vmatrix} \tag{1-3}$$

and its elements are the derivatives:

$$t_{jk} = \frac{df_k}{dx_j} \tag{1-4}$$

All the elements of a row k will be equal to zero if x_k is an independent variable. If one writes X for the column matrix of the variables $x_1, x_2, \ldots, x_n$, and φ for the column matrix of dependent variables ($\varphi_k = x_k - f_k$), one has for the system of equations represented in the flow graph:

$$(U - T)X = \varphi \tag{1-5}$$

where U is the unit matrix of order n.

1-3 THE DRAWING OF THE FLOW GRAPH

The flow graph can always be obtained from a set of equations. It is necessary only to write the equations in the form shown in Eq. (1-2) with the dependent variables on the left-hand side, taking care to define all the dependent variables and to define them once only.

The usefulness of flow graphs is evidently greater if they can be drawn from a schematic diagram or from an equivalent circuit without the necessity of first writing the equations explicitly. This is