Algorithms and Combinatorics 3

Kazuo Murota

Systems Analysis by Graphs and Matroids

Structural Solvability and Controllability

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Preface

Recent technology involves large-scale physical/engineering systems consisting of hundreds or thousands of interconnected elementary units. Such large-scale systems cannot be expected to function correctly as a whole unless they are well structured in some appropriate sense. Thus, it is natural that there is an increasing demand for systematic procedures for structural analyses of large-scale systems.

It has now been widely recognized that combinatorial mathematics, especially those theories which are accompanied by efficient algorithms, can provide some useful tools for the structural analyses of large-scale systems. In this book, two problems in structural analyses, namely, the structural solvability of a system of linear/nonlinear equations and the structural controllability of a linear time-invariant dynamical system, are treated by means of combinatorial concepts such as graphs and matroids. Special emphasis is laid on the importance of relevant physical observations to successful mathematical modelings.

Related works, theoretical and practical, abound in the literature of various fields; no attempt is made to cover them all. The forthcoming book [Recski 86] by Professor A. Recski seems to have much to do with the present work, sharing the same methodology as ours while focusing on the problems on electrical networks and structural rigidity.

This monograph is primarily based on the author's dissertation [Murota 83c] at the University of Tokyo. It is, however, completely rewritten on this occasion to include the succeeding works done by the author at the University of Tsukuba.

It is a pleasure to express a deep sense of gratitude to those who helped me in writing this book. I owe much to Professor Masao Iri of the University of Tokyo, who introduced me to this field five years ago, gave me a never-failing guidance through penetrating and substantial suggestions, and encouraged me to write this book. My cordial thanks are due to Professor Bernhard Korte of the University of Bonn, who invited me to write a book in ALGORITHMS AND COMBINATORICS. Acknowledgement is given to Junkichi Tsunekawa of the Institute of Japanese Union of Scientists and Engineers, who kindly provided relevant materials on JUSE-GIFS and DPS, to Sueo Abe of Nissan Chemical Industries, Ltd., who made suggestive comments, and to Masaki Ichikawa of the Environment Agency, who, in his graduation thesis, implemented the algorithm for computing the rank of a mixed matrix and prepared the data used in examples. The handy word-processing tool designed excellently by Professor Yoshitsugu Yamamoto of the University of Tsukuba has facilitated preparing the manuscript. Discussions with Professor Satoru Fujishige and Dr. Masaaki Sugihara of the University of Tsukuba have always been helpful and enjoyable. Professor Hiroshi Imai of Kyushu University kindly suggested many improvements in the early draft. Thanks are also due to my colleagues, Dr. Ryo Fuji-Hara, Professor Takeshi Koshizuka and his students of the University of Tsukuba, and Dr. Seiichi Shin of the University of Tokyo. Finally, I am grateful to the staff of Springer-Verlag for their cooperation in all aspects of the production of this book.

I would like to dedicate this book to the late Mr. Hideo Tanaka, who was full of warmth and truth.

Kazuo Murota

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Introduction

Graph theory, as a branch of combinatorial mathematics, has achieved remarkable development in the past several decades, leading to fruitful generalizations and extensions such as network theory and matroid theory, and yielding quite a few mathematical results which are interesting and beautiful in themselves.

Recently, however, it has become widely recongnized that some of the concepts and results in combinatorics serve also as useful mathematical tools for the analysis of engineering systems. In fact, various kinds of graphical representations of systems are now in common use in various fields of engineering; for example, circuit diagrams of electrical networks, flow-charts of computer programs, block diagrams and signal-flow graphs of control systems, process flowsheets of chemical plants, and transportation networks, etc. With these representations, a variety of so-called graphical techniques is employed for the analysis of systems.

One of the most naive graphical techniques would be to draw the graphs, i.e., the figures consisting of circles and arrows, which represent some aspects of systems. Such an approach may certainly be helpful for visualization and hence for analysis by inspection, at least for moderately-sized systems. Modern industries, however, are based on large-scale systems, for which the structural analysis is vital and for which analysis by inspection fails. In order that any graphical technique be of practical use for the analysis of large-scale discrete systems, the graphical representation of systems must be such that it admits systematic analysis based on the mathematical results obtained in graph theory; therefore, graphs must be treated as combinatorial objects, primarily consisting of the incidence relations between vertices and arcs.

For a successful analysis of any kind, it is of ultimate importance to set up a mathematical model of a real-world system so that the relevant aspects of the real situations are represented with sufficient faith, and at the same time, that mathematical rigor and simplicity are incorporated to render it amenable to subsequent mathematical treatments.

The first, and probably the most crucial, step in mathematical modeling would be to select the relevant set of quantities characterizing the problem and to find the description of the system suitable for mathematical analysis. When the structural aspects of a discrete system are in question, a description in terms of a collection of elementary variables is often more appropriate than a sophisticated compact one. In describing a linear time-invariant dynamical system, for example, the so-called descriptor form of state-space equations is more suitable in this respect than the standard form (see §12 for more arguments).

The second is to grasp the nature of the quantities, which is to be reflected in the mathematical structure of the model. For example, even the primitive classification of the quantities into zeros and nonzeros, which usually leads to graph-theoretic models, often yields meaningful results in the analysis of large-scale discrete systems.

Another important aspects, especially from the practical point of view, is that the methods of analysis are to be backed up by efficient algorithms which can be performed on computers. In this regard, the theory of computational complexity [Aho-Hopcroft-Ullman 74], [Garey-Johnson 79] may be useful. But further elaboration on algorithms should be made to enhance the efficiency for individual problems.

The present monograph is devoted to the study of the structural analysis of a system of linear/nonlinear equations and the structural controllability of a linear time-invariant dynamical system. The outline of the contents of this book is as follows.

Chapter 1: Mathematical preliminaries are given along with conventions. A list of symbols is shown in §1. Algebraic concepts concerning algebraic independence and ranks of matrices are introduced in §2. Relevant results in graph theory and matroid theory such as the Dulmage-Mendelsohn decomposition of bipartite graphs and the principal partition of submodular functions, are mentioned in §3 and §4, respectively, with some emphasis on the algorithmic aspects.

Chapter 2: A graph-theoretic method is developed for the structural analysis of a system of equations. In §5, the structural solvability of a system of equations is formulated in algebraic terms. Under a certain "generality assumption" on the functions in the system, a necessary and sufficient condition for the structural solvability is given in §7 in terms of Menger-type linkings on the representation graph introduced in §6. Then in §8 are defined the L-decomposition and the M-decomposition of graphs, which are applied in §9 to the hierarchical decomposition of a system of equations into smaller subsystems. Various graphical techniques are integrated into a systematic procedure for solving a system of equations in §10, followed by examples in §11.

Chapter 3: Graph-theoretic conditions are given to the structural controllability of a linear dynamical system expressed in the descriptor form: F dx/dt = Ax + Bu, where the nonvanishing entries of the coefficient matrices F, A and B are taken for independent parameters. In §12 various descriptions of a dynamical system and the associated natural graph representations are discussed from the viewpoint of structural analysis. Some known results on the controllability condition of a descriptor system are described in §13. Then in §14, the structural controllability of a descriptor system is equivalently expressed in terms of the Dulmage-Mendelsohn decomposition of the associated bipartite graph, and some of the known results on structural controllability are derived therefrom as corollaries. Discussions in §15 conclude this chapter.

Chapter 4: Physical observations are made for providing the physical basis for the more elaborate and faithful mathematical models adopted in subsequent chapters. First in §16 it is observed that two different kinds are to be distinguished among the nonvanishing numbers characterizing real-world systems, and the notion of "mixed matrix" is introduced as a mathematical tool for incorporating this intuitive physical observation in the structural analysis. Next observation, made in §17, may be categorized as a kind of dimensional analysis, pointing out some algebraic implications of the principle of dimensional homogeneity. A novel concept of "physical matrix" is then introduced in §18 as a mathematical model of the matrices encountered in real problems, reflecting the dual viewpoint from structural analysis and dimensional analysis developed in the preceding sections.

Chapter 5: A matroid-theoretic method is developed for the structural analysis of a system of equations under a more realistic setting than in Chapter 2. First, the rank of a mixed matrix is characterized in matroid-theoretic terms in §19, and an efficient algorithm for computing it is described in §20. Matroidal conditions are given in §21 to the structural solvability under the refined formulation, along with some practical examples. Then the canonical forms of mixed matrices are considered in §22 to §24, which unify the LU-decomposition and the Dulmage-Mendelsohn decomposition. They provide in §25 a powerful method for hierarchical decomposition of a system of linear/nonlinear equations into smaller subsystems. Finally miscellaneous results are mentioned in §26.

Chapter 6: Structural controllability of a dynamical system is investigated by means of matroid-theoretic concepts under the mathematical model established in Chapter 4. As a prototype of our approach, the dynamical degree is characterized in §27 in connection with the independent-flow problem. Then the matroidal conditions are derived in §28 for the structural controllability, followed by the description in §29 of the combinatorial algorithm for testing them and by some illustrative examples of §30. Relations to other works are mentioned in §31.

Finally, the results obtained and the problems left unanswered are summarized in the Conclusion.

Chapter 1. Preliminaries

This chapter recapitulates the mathematical preliminaries needed in the subsequent arguments. In particular, brief mention is made of the relevant concepts in algebra, graph theory and matroid theory. Emphasis is laid on the algebraic independence, the Dulmage-Mendelsohn decomposition of bipartite graphs with reference to matchings, and the decomposition principle of submodular functions. Some observations which are not explicit in the literature are also made.

1. Convention and Notation

Expressions are referred to by their numbers; for example, (2.1) designates the expression (2.1) that appears in §2. Similarly for figures and tables.

Some of the symbols used in this book are listed below.

Z : ring of integers
Q : field of rational numbers
R : field of real numbers
R_+ : set of nonnegative real numbers
C : field of complex numbers
K : a commutative field, a subfield of F
F : a commutative field, an extension of K
K( ) : field adjunction to K
K[ ] : ring adjunction to K
$\dim_K F$: degree of transcendency of F over K

$\underline{\underline{D}}$: set of partial derivatives
$\underline{\underline{T}}$: "general part" of $\underline{\underline{D}}$

$\underline{\underline{M}}(F;m,n)$: set of m by n matrices over F
$\underline{\underline{MM}}(F/K;m,n)$: set of m by n mixed matrices over F with respect to K
$\underline{\underline{LM}}(F/K;m_Q,m_T,n)$: set of m_Q+m_T by n layered mixed matrices over F with respect to K
GL(n,K) : set of n by n nonsingular matrices over K (general linear group of degree n over K)
$\underline{\underline{N}}$( ) : collection of the nonvanishing entries of a matrix

r( ) : rank of a matrix
t( ) : term-rank of a matrix
R : row-set of a matrix
C : column-set of a matrix

M|S : restriction of a matroid **M** to a set S
M.S : contraction of a matroid **M** to a set S
M( ) : linear matroid defined by a matrix
M{ } : linear matroid defined by a subspace

∂^+ : initial vertex of an arc
∂^- : terminal vertex of an arc
δ^+ : set of out-going arcs from a vertex
δ^- : set of in-coming arcs to a vertex
(u,v) : an arc directed from initial vertex u to terminal vertex v
$\overset{*}{\longrightarrow}$: reachability on a graph

| | : cardinality of a set

$\prec$: a partial order
$|\!\prec$: relation of "covered by" with respect to a partial order $\prec$, i.e., $x \mathrel{|\!\prec} y$ means that $x \prec y$, $x \neq y$ and there exists no z ($\neq x, y$) such that $x \prec z \prec y$.

2. Algebra

2.1. Algebraic independence ([Baker 75], [Jacobson 64], [Waerden 55])

Let $\mathbf{F} \supset \mathbf{K}$ be fields; $\mathbf{K}$ is a subfield of $\mathbf{F}$, and $\mathbf{F}$ is an extension field of $\mathbf{K}$. For a subset Y of $\mathbf{F}$, we denote by $\mathbf{K}(Y)$ and $\mathbf{K}[Y]$ the field and the ring adjunction, respectively; $\mathbf{K}(\cdot Y)$ is the extension field of $\mathbf{K}$ generated by Y over $\mathbf{K}$, while $\mathbf{K}[Y]$ the ring generated by Y over $\mathbf{K}$.

A polynomial in $X_1, \ldots, X_q$ over $\mathbf{K}$ (i.e., with coefficients from $\mathbf{K}$) is called nontrivial if some of its coefficients are distinct from zero. An element y of $\mathbf{F}$ is called algebraic over $\mathbf{K}$ if there exists a nontrivial polynomial $p(X)$ in one indeterminate X over $\mathbf{K}$ such that $p(y) = 0$. An element of $\mathbf{F}$ is called transcendental over $\mathbf{K}$ if it is not algebraic over $\mathbf{K}$. A subset (more precisely, multiset) $Y = \{y_1, \ldots, y_q\}$ of $\mathbf{F}$ is called algebraically independent if one of the following equivalent conditions holds:

(i) For any i, y_i is transcendental over $\mathbf{K}(Y \setminus y_i)$.

(ii) There exists no nontrivial polynomial $p(X_1, \ldots, X_q)$ in q indeterminates over $\mathbf{K}$ such that $p(y_1, \ldots, y_q) = 0$.

(iii) The degree of transcendency of the extension field $\mathbf{K}(y_1, \ldots, y_q)$ over $\mathbf{K}$ equals q, i.e., $\dim_{\mathbf{K}} \mathbf{K}(y_1, \ldots, y_q) = q$.

Many important properties concerning algebraic independence may be viewed as consequences of the fact that algebraic independence over a fixed base field $\mathbf{K}$ defines a matroid on a finite subset of $\mathbf{F}$, called algebraic matroid, of which mention will be made in §4.2.

Finally we refer to the following theorem.

Theorem 2.1 (Lindemann-Weierstrass Theorem). Let $y_1, \ldots, y_q$ be algebraic numbers over $\mathbf{Q}$ that are linearly independent over $\mathbf{Q}$. Then $\{\exp y_1, \ldots, \exp y_q\}$ is algebraically independent over $\mathbf{Q}$. □

2.2. Rank, term-rank and generic-rank

Let $A=(a_{ij})$ be an m×n matrix over a field $\mathbf{K}$, i.e., $A \in \underline{\underline{M}}(\mathbf{K};m,n)$. $A[I,J]$ means the submatrix of A with rows in I and columns in J. The collection (or multiset, to be more precise) of the nonvanishing entries of a matrix A will be denoted by $\underline{\underline{N}}(A)$, where we understand that with each element $a_{ij} \in \underline{\underline{N}}(A)$, the pair (i,j) of indices is implicitly associated.

The rank of A, in the ordinary sense in linear algebra, is denoted by $r(A)$. The term-rank of A, denoted by $t(A)$, is defined as the maximum of k such that $a_{i(1)j(1)}a_{i(2)j(2)}\cdots a_{i(k)j(k)} \neq 0$ for some suitably chosen distinct rows i(1), i(2), ..., i(k) and columns j(1), j(2), ..., j(k). The term-rank of A is equal to the maximum size of a matching on the bipartite graph B = $(V^+, V^-; \underline{\underline{N}}(A))$ (cf. §3) associated with A; it has the vertex-set $V^+ \cup V^-$ ($V^+ \cap V^- = \emptyset$) with V^+ and V^- corresponding to the column-set and the row-set, respectively, and the arc-set corresponding to the nonvanishing entries of A, that is, an arc (j,i) from $j \in V^+$ to $i \in V^-$ exists in B iff $a_{ij} \neq 0$. By utilizing an efficient algorithm [Hopcroft-Karp 73], [Lawler 76], [Papadimitriou-Steiglitz 82] for finding a maximum matching on a bipartite graph, we can determine the term-rank of a matrix with graph manipulations of complexity $O(mn\ (\min(m,n))^{1/2})$.

Let the entries a_{ij} of A be rational functions over K in q independent parameters, or indeterminates, $X_1, \ldots, X_q$. If the rank of A is uniquely determined except for those parameter values in K^q which lie outside some proper algebraic variety [Jacobson 64], [Waerden 55] in K^q, we call the uniquely determined rank the generic-rank of A with respect to parameters $X_1, \ldots, X_q$.

The generic-rank is smaller than or equal to the term-rank. We are often interested in the cases where these two coincide. If each of the nonvanishing entries of A is an indeterminate by itself, the generic-rank of A agrees with the term-rank of A, regardless of the characteristic of K. A less obvious example is the case of symmetric matrices. Namely, if $A=(a_{ij})$ is a symmetric matrix and K is of characteristic zero, the term-rank of A is equal to the generic-rank of A with $\{a_{ij} \mid a_{ij} \neq 0,\ i \leq j\}$ as the set of independent parameters. Several other classes of matrices are known whose generic-ranks admit combinatorial characterizations; for instance, the generic-rank of a "skew-plus-diagonal" matrix is expressed in terms of a non-bipartite matching in [Anderson 75].

The generic-rank of A, the entries of which are still assumed to be rational functions in $X_1, \ldots, X_q$ over K ($\supset$ Q), is equal to the maximum of the rank of A with particular values (in K) given to the parameters. Suppose that F is an extension field of K such that $\dim_K F$ is infinite, which is the case with K=Q and F=R. Then we can choose an arbitrary number of elements in F which are algebraically independent over K. It is not difficult to observe that the generic-rank of A is equal to the rank of A with parameter values fixed to transcendentals in F which are algebraically independent over K.

3. Graph

3.1. Directed graph

Let $G = (V,A)$ be a directed graph with vertex-set V and arc-set A. For an arc $a \in A$, $\partial^{+}a$ ($\partial^{-}a$) denotes the initial (terminal) vertex of a, while for a vertex $v \in V$, $\delta^{+}v$ ($\delta^{-}v$) is the set of out-going (in-coming) arcs incident to v. For two vertices u and v, we say that v is reachable from u on G, which we denote as "u-*->v on G" (or simply, "u-*->v"), iff there exists a directed path from u to v on G. A vertex is called maximal (minimal) if it has no in-coming (out-going) arcs.

For V' ($\subset V$) the vertex-induced subgraph, or the section graph, on V' is a graph $G' = (V',A')$ with $A' = \{a \in A \mid \partial^{+}a \in V',\ \partial^{-}a \in V'\}$. We also say that G' is obtained from G by deleting the vertices of $V \backslash V'$.

Two vertices u and v belong to the same strongly connected component (or strong component, in short) iff u-*->v and v-*->u. The vertex-set V is partitioned into strong components $\{V_i\}$, each of which determines the vertex-induced subgraph (i.e., section graph) $G_i = (V_i, A_i)$ of G, also called a strong component of G. Partial order $\prec$ can be defined on the family of strong components $G_i = (V_i, A_i)$ of G, or, in other words, on the family of subsets V_i of V, by

$$V_i \prec V_j \iff v_j \text{-*->} v_i \text{ on G for some } v_i \in V_i \text{ and } v_j \in V_j.$$

We also write $G_i \prec G_j$ iff $V_i \prec V_j$. An efficient algorithm of complexity $O(|A|)$ is known for the decomposition of a graph into strong components.

3.2. Bipartite graph ([Ford-Fulkerson 62], [Iri-Han 77], [Lawler 76])

Let $B = (V^{+}, V^{-}; A)$ be a bipartite graph with vertex-set consisting of two disjoint parts V^{+} and V^{-}, and with arc-set A, where arcs are directed from V^{+} to V^{-}. For W^{+} ($\subset V^{+}$) we often write $\Gamma(W^{+}) = \partial^{-}\delta^{+}(W^{+})$ ($\subset V^{-}$), which stands for the set of vertices in V^{-} adjacent to the vertices in W^{+}.

A cover of B is a pair (W^{+}, W^{-}) such that $W^{+} \subset V^{+}$, $W^{-} \subset V^{-}$, and no arcs exist from $V^{+} \backslash W^{+}$ to $V^{-} \backslash W^{-}$. The size of a cover (W^{+}, W^{-}) is defined to be $|W^{+}| + |W^{-}|$ and a cover of minimum size is called a minimum cover.

A matching M on B is a subset of A such that no two arcs in M share a common vertex incident to them. If we denote by $\partial^{+}M$ ($\partial^{-}M$) the set of vertices in V^{+} (V^{-}) incident to arcs in M, this condition is equivalent to $|\partial^{+}M| + |\partial^{-}M| = 2|M|$. A vertex v is said to be covered by M iff $v \in \partial^{+}M \cup \partial^{-}M$. A matching of maximum cardinality is called a