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Viscosity Solutions and Applications

Montecatini Terme, 1995

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Lectures given at the 2nd Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Montecatini Terme, Italy, June 12–20, 1995

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Preface

The C.I.M.E. School on Viscosity Solutions and Applications held in Montecatini from June 12 to June 20, 1995 was designed with the aim to provide a rather comprehensive and up-to-date account of the theory of viscosity solutions and some of its applications.
The School comprised the following five series of lectures:

M.G. Crandall: General Theory of Viscosity Solutions

M. Bardi: Some Applications of Viscosity Solutions to Optimal Control and Differential Games

L.C. Evans: Regularity for Fully Nonlinear Elliptic Equations and Motion by Mean Curvature

M.H. Soner: Controlled Markov Processes, Viscosity Solutions and Applications to Mathematical Finance

P.E. Souganidis: Front Propagation: Theory and Applications

as well as seminars by:

L. Ambrosio, M. Arisawa, G. Bellettini, P. Cannarsa, M. Falcone, S. Koike, G. Kossioris, M. Motta, A. Siconolfi and A. Tourin.

The present volume is a record of the material presented in the above listed courses. It is our belief that it will serve as a useful reference for researchers in the fields of fully nonlinear partial differential equations, optimal control, propagation of fronts and mathematical finance.

It is our pleasure here to thank the invited lecturers, the colleagues who contributed seminars, all the participants for their active contribution to the success of the School and the Fondazione CIME for the support in the organization.

I. Capuzzo Dolcetta, P.L. Lions

Contents

VISCOSITY SOLUTIONS: A PRIMER

by

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0. Introduction

These lectures present the most basic theory of "viscosity solutions" of fully nonlinear scalar partial differential equations of first and second order. Other contributions to this volume develop some of the amazing range of applications in which viscosity solutions play an essential role and various refinements of this basic material.

In this introductory section we describe the class of equations which are treated within the theory and then our plan of presentation.

The theory applies to scalar second order partial differential equations

$$\text{(PDE)} \qquad F(x, u, Du, D^2u) = 0$$

on open sets $\Omega \subset \mathbb{R}^N$. The unknown function $u : \Omega \to \mathbb{R}$ is real-valued, Du corresponds to the gradient $(u_{x_1}, \ldots, u_{x_N})$ of u and D^2u corresponds to the Hessian matrix (u_{x_i,x_j}) of second derivatives of u. Consistently, F is a mapping

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N) \to \mathbb{R}$$

where $\mathcal{S}(N)$ is the set of real symmetric $N \times N$ matrices. We say that Du (D^2u) "corresponds" to the gradient (respectively, the Hessian) because, as we shall see, solutions u may not be differentiable, let alone twice differentiable, and still "solve" (PDE). We write $F(x, r, p, X)$ to indicate the value of F at $(x, r, p, X) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$. (PDE) is said to be *fully nonlinear* to emphasize that $F(x, r, p, X)$ need not be linear in any argument, including the X in the second derivative slot.

F is called *degenerate elliptic* if it is nonincreasing in its matrix argument:

$$F(x, r, p, X) \le F(x, r, p, Y) \quad \text{for} \quad Y \le X.$$

The usual ordering is used on $\mathcal{S}(N)$; that is $Y \le X$ means

$$\langle X\xi, \xi\rangle \le \langle Y\xi, \xi\rangle \quad \text{for} \quad \xi \in \mathbb{R}^N$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product. If F is degenerate elliptic, we say that it is *proper* if it is also nondecreasing in r. That is, F is proper if

$$F(x, s, p, X) \le F(x, r, p, Y) \quad \text{for} \quad Y \le X, \quad s \le r.$$

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As a first example, F might be of first order

$$F(x,r,p,X) = H(x,r,p);$$

every first order F is obviously (very) degenerate elliptic, and then proper if it is nondecreasing in r. For an explicit example, the equation $u_t + (u_x)^2 = 0$ with $(t,x) \in \mathbb{R}^2$ is a proper equation (we are thinking of (t,x) as (x_1, x_2) above). On the other hand, the Burger's equation $u_t + uu_x = 0$ is not proper, for it is not monotone in u. We refer to proper first order equations $H(x,u,Du) = 0$ and $u_t + H(x,u,Du) = 0$ as "Hamilton-Jacobi" equations.

Famous second order examples are given by $F(x,r,p,X) = -\text{Trace}\,(X)$ and $F(x,r,p,X) = -\text{Trace}\,(X) - f(x)$ where f is given; the pdes are then Laplace's equation and Poisson's equation:

$$F(D^2u) = -\sum_{i=1}^{N} u_{x_i x_i} = -\Delta u = 0 \quad \text{and} \quad -\Delta u = f(x)$$

The equations are degenerate elliptic since $X \to \text{Trace}\,(X)$ is monotone increasing on $\mathcal{S}(N)$. We do not rule out the linear case! Incorporating t as an additional variable as above, the heat equation $u_t - \Delta u = 0$ provides another famous example. The convention used here, that Du, D^2u stand for the spatial gradient and spatial Hessian, will be in force whenever we write "$u_t + F(x,u,Du,D^2u)$".

Note the preference implied by these examples; we prefer $-\Delta$ to Δ. A reason is that (in various settings), $-\Delta$ has an order preserving inverse. This convention is not uniform; for example, Souganidis [35] does not follow it and reverses the inequality in the definition of degenerate ellipticity.

More generally, the linear equation

$$-\sum_{i,j=1}^{N} a_{i,j}(x)u_{x_i,x_j} + \sum_{i=1}^{N} b_i(x)u_{x_i} + c(x)u - f(x) = 0$$

may be written in the form $F = 0$ by setting

$$F(x,r,p,X) = -\text{Trace}\,(A(x)X) + \langle b(x), p\rangle + c(x)r - f(x) \tag{0.1}$$

where $A(x)$ is a symmetric matrix with the elements $a_{i,j}(x)$ and $b(x) = (b_1(x), \ldots, b_N(x))$. This F is degenerate elliptic if $0 \le A(x)$ and proper if also $0 \le c(x)$.

In the text we will pose some exercises which are intended to help readers orient themselves (and to replace boring text with pleasant activities). We violate all conventions by doing so even in this introduction. Some exercises are "starred" which means "please do it now" if the fact is not familiar.

Exercise 0.1.* Verify that F given in (0.1) is degenerate elliptic if and only if $A(x)$ is nonnegative.

The second order examples given above are associated with the "maximum principle". Indeed, the calculus of the maximum principle is a fundamental idea in the entire theory.

Exercise 0.2.* Show that F is proper if and only if whenever $\varphi, \psi \in C^2$ and $\varphi - \psi$ has a nonnegative maximum (equivalently, $\psi - \varphi$ has a nonpositive minimum) at $\hat{x}$, then

$$F(\hat{x}, \psi(\hat{x}), D\psi(\hat{x}), D^2\psi(\hat{x})) \leq F(\hat{x}, \varphi(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})).$$

So far, we have presented nonlinear first order examples and linear second order examples. However, the class of proper equations is very rich. Indeed, if F, G are both proper, then so is $\lambda F + \mu G$ for $0 \leq \lambda, \mu$. More interesting is the following simple fact: if $F_{\alpha,\beta}$ is proper for $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ (some index sets), then so is

$$F = \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} F_{\alpha,\beta}$$

provided only it is finite. This generality is essential to applications of the theory in differential games (see Bardi [2]), while applications in control theory correspond to the case "F_α" in which there is only one index (see Bardi [2] and Soner [34]).

For example, $\max(u_t + |Du|^2 - g(x), -\Delta u - f(x)) = 0$ is a proper equation. The other lecture series will present many examples of scientific significance. We have only attempted here to indicate that that class of proper equations is broad and interesting.

Here we aim at a clear and congenial presentation of the most basic elements of the theory of viscosity solutions of proper equations $F = 0$. These are the notion of a viscosity solution, maximum principle type comparison results for viscosity solutions, and existence results for viscosity solutions via Perron's method. We do not aim at completeness or technical generality, which often distract from ideas.

The text is organized in sections, many of which are quite brief. The descriptions below contain remarks about the logic of the presentation. By the numbers, the topics are:

Section 1: An illustration of the need to be able to consider nondifferentiable functions as solutions of proper fully nonlinear equations is given using first order examples.

Section 2: The notions of viscosity subsolutions, supersolutions and solutions are presented. The convention that the modifier "viscosity" will be dropped thereafter in the text is introduced. It is essential to deal with semicontinuous functions in the theory, and this generality appears here.

Section 3: Striking general existence and uniqueness theorems are presented without proof to indicate the success of viscosity solutions in this arena. The contrast with the examples in Section 1 is dramatic.

Sections 4, 5, 6: A primary test of a notion of generalized solutions is whether or not appropriate uniqueness results can be obtained (when suitable side conditions - boundary conditions, growth conditions, initial conditions, etc. - are satisfied). Actually, one wants a bit more here, that is the sort of *comparison* theorems which follow from the maximum principle. Basic arguments needed in proofs of comparison results for viscosity solutions of first order stationary problems (those without "t") are presented here and typical results are deduced. Section 4 concerns the Dirichlet problem, Section 5 concerns bounded solutions of a problem in $\mathbb{R}^N$, and Section 6 provides an example of treating unbounded solutions. The second order case is more complex and is not taken up until Section 10. However, nothing is wasted, and all the arguments presented in these sections are invoked in the second order setting.

Section 7: The notions of Section 2 are recast in a form convenient for use in the next section and in the comparison theory in Sections 8 and 9.

Section 8: Two related results, each an important tool, are established. One states roughly that the supremum of a family of subsolutions is again a subsolution, and the other that the limit of a sequence of viscosity subsolutions (supersolutions, solutions) of a converging sequence of equations (meaning the F_n's converge) is a subsolution (respectively, a supersolution, solution) of the limiting equation. We call this last theme "stability" of the notion; it is one of the great tools of the theory in applications. The mathematics involved is elementary with a "point-set" flavor.

Section 9: Existence is proved via Perron's Method using a result of the previous section. The existence theory presupposes "comparison". At this stage, comparison has only been treated in the first order case, and is simply assumed for the second order case. This does not affect either clarity or the basic argument. At this juncture, the most basic ideas have been presented with the exception of comparison for second order equations.

Section 10: The primary difference between the first and second order cases is explained. Then the rather deep result which is used here to bridge the gap, called here "the Theorem on Sums" (an analytical result about semicontinuous functions), is stated without proof. An example is given to show how this tool theorem renders the second order case as easy to treat as the first order case.

Section 11: The Theorem on Sums is proved.

Section 12: In the preceding sections comparison was only demonstrated for various equations of the form $F(x,u,Du,D^2u) = 0$. Here the main additional points needed to treat $u_t + F(x,u,Du,D^2u) = 0$ are sketched.

Regarding notation, we use standard expressions like "$C^2(\Omega)$" (the twice continuously differentiable functions on Ω) and "$|p|$" (the Euclidean length of p) without further comment when it seems reasonable. With some exceptions, we minimize distracting notation.

Regarding the literature, it is too vast to try to summarize in a work like this, which aims at presenting basic ideas and not at technical generality or great precision. We will basically rely on the big brother to this work, the more intense (and reportedly less friendly) [12] for its extensive references, together with those in the other contributions to this volume. (We recommend the current work as preparation for reading [12], especially the topics therein not taken up here.) We do give some references corresponding to the original works initiating the themes treated here. A few more recent papers are cited as appropriate. All references appear at the ends of sections. In addition, we mention the books by Cabré and Caffarelli [7] and Dong [17] for recent expositions of regularity theory of solutions, which is not treated here, as well as the classic text of Gilbarg and Trudinger [23]. Regularity theory is also one of the themes of Evans [20]. The recent book of Barles [3] presents a complete theory of the first order case (which itself fills a book that contains 154 references!). The book of Fleming and Soner [22, Chapters II and V] also nicely covers the basic theory. There are alternative theories for first order equations; see, e.g., [9] and [36]. Of course, MathSciNet now allows one to become nearly current regarding the state of the literature relatively easily, and one can profitably search on any of the leads given above.

A significant limitation of our presentation is that only the Dirichlet boundary condition is discussed at any length, and this in its usual form rather than the generalized version. Other boundary conditions appear in the contributions of Bardi [2] and Soner [34] in an essential way. In addition to the references they give, the reader may refer for example to [12, Section 7] for a discussion in the spirit of this work. Another limitation is that singular equations are not treated at all. Equations with singularities appear in contributions of Evans [20] and Souganidis [35]. See also [12, Section 9]. Finally, only continuous solutions are discussed here, while within applications one meets the discontinuous solutions. The contribution of Bardi [2, Section V] treats this issue, and discontinuous functions appear quickly in the exposition of Souganidis [35].

1. On the Need for Nonsmooth Solutions

The fact is that it is difficult to give examples of solutions (in any sense) of equations $F = 0$ which are not classical solutions unless the equation is pretty "degenerate" (roughly, the monotonicity of $X \to F(x, r, p, X)$ is not strong enough) or "singular" (that is, F may have discontinuities or other types of singularities). (A "classical" solution of an equation $F(x, u(x), Du(x), D^2u(x)) = 0$ is a twice continuously differentiable function which satisfies the equation pointwise; if the equation is first order classical solutions are once continuously differentiable; if the equation has the form $u_t + F(x, u, Du, D^2u) = 0$, then a classical solution will possess the derivatives u_t, Du, D^2u in the classical sense. Similar remarks apply to subsolutions and supersolutions.) The reason is that the regularity theory of sufficiently nondegenerate and nonsingular equations is still unsettled. In particular, it may be that nondegenerate nonsingular equations $F = 0$ with smooth F admit only classical solutions, although some suspect that this is not so.

However, if the equation is first order (so very degenerate), then examples are easy. The next exercise gives a simple problem without classical solutions and for which there are solutions slightly less regular than "classical"; however allowing less regular solutions generates "nonuniqueness".

Exercise 1.1.* Put $N = 1$, $\Omega = (-1, 1)$ and $F(x, r, p, X) = |p|^2 - 1$. Verify that there is no classical (here this means $C^1(-1, 1) \cap C([-1, 1])$) solution u of $F(u') = (u')^2 - 1 = 0$ on $(0, 1)$ satisfying the Dirichlet conditions $u(-1) = u(1) = 0$. Verify that $u(x) = 1 - |x|$ and $v(x) = |x| - 1$ are both "strong" solutions: in this case, they are Lipschitz continuous and the equation is satisfied pointwise except at $x = 0$ (so almost everywhere).

Of course, the problem in Exercise 1.1 has a unique solution within our theory, as we will see later (it is $u(x) = 1 - |x|$).

To further establish the desirability of allowing nondifferentiable solutions, we recall the classical method of characteristics as it applies to the Cauchy problem for a Hamilton-Jacobi equation $u_t + H(Du) = 0$:

$$\begin{cases} u_t + H(Du) = 0 & \text{for} \quad x \in \mathbb{R}^N, \ t > 0 \\ u(0,x) = \psi(x), & \text{for} \quad x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

Suppose that H is smooth and that u is a smooth solution of $u_t + H(Du) = 0$ on $t \geq 0$, $x \in \mathbb{R}^N$. Define $Z(t) \in \mathbb{R}^N$ to be the solution of the initial value problem

$$Z'(t) = \frac{d}{dt}Z(t) = DH(Du(t, Z(t))), \quad Z(0) = \hat{x}$$

over the largest interval for which this solution exists. A computation yields

$$\begin{aligned} \frac{d}{dt}Du(t, Z(t)) &= D\frac{\partial u}{\partial t}(t, Z(t)) + D^2u(t, Z(t))Z'(t) \\ &= D\frac{\partial u}{\partial t}(t, Z(t)) + D^2u(t, Z(t))DH(Du(t, Z(t)) \\ &= 0 \end{aligned}$$

where the last equation arises from differentiating $u_t + H(Du) = 0$ with respect to x.

Remark 1.1. In calculations such as the above, one has to decide whether the the gradient Dv of a scalar function v is to be a column vector or a row vector. There is no ambiguity about D^2v, for it is to be square and symmetric in any case. In the introduction we wrote the gradient as a row vector, but above interpret it as a column vector. This is consistent with interpreting points of $\mathbb{R}^N$ as column vectors while writing row vectors, and with these sloppy conventions the above is correct.

We conclude that Du is constant on the curve $t \to (t, Z(t))$. It then would follow that $Z(t) = \hat{x} + tDH(D\psi(\hat{x}))$. However, the resulting equation $Du(t, \hat{x} + tDH(D\psi(\hat{x})) \equiv D\psi(\hat{x})$ yields contradictions as soon as we have characteristics crossing, that is $y \neq z$ but $t > 0$ such that $y + tDH(D\psi(y)) = z + tDH(D\psi(z))$. In this case, one says that "shocks form" and there are no smooth solutions u defined for all $t \geq 0$ in general.

Exercise 1.2. (i) Continue the analysis above to find

$$u(t, Z(t)) = \psi(\hat{x}) + t(\langle D\psi(\hat{x}), DH(D\psi(\hat{x}))\rangle - H(D\psi(\hat{x})))$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.

(ii) If $N = 1$, then shocks will form unless $x \to H'(\psi'(x))$ is monotone.

Under reasonable assumptions, as is shown in elementary courses, analysis by characteristics provides a smooth solution of (1.1) until shocks form. When classical solutions break down, in this area and others, one is led to think of the problem of finding a way to continue past the breakdown with a less regular solution. However, one can also immediately think of the problem of finding solutions in cases where the data does not allow the classical analysis. E.g., what does one do if H and/or ψ above is not smooth? The "breakdown" idea is not central in this view.

Just as in the case of Exercise 1.1, relaxing the regularity requirement for a solution just a tiny bit leads to nonuniqueness for (1.1). One does not expect uniqueness in general for stationary problems, but one does expect uniqueness for initial-value problems.

Exercise 1.3. Consider the equation $u_t + (u_x)^2 = 0$ for $t > 0$, $x \in \mathbb{R}$ coupled with the initial condition $u(0, x) \equiv 0$. Verify that the function

$$\begin{aligned} v(t,x) &\equiv 0 \quad \text{for} \quad 0 < t \le |x|, \\ v(t,x) &= -t + |x| \quad \text{for} \quad |x| \le t, \end{aligned}$$

satisfies the initial condition, is continuous and has all the regularity one desires off the lines $x = 0$, $t = |x|$, and satisfies the equation off these lines. Thus $u \equiv 0$ and v are *distinct* nearly classical - even piecewise linear - solutions of the Cauchy problem.

We have not given second order examples. However, here is a model equation which will be covered under the theory to be described and for which the issue of how smooth solutions are is unsettled. Let $A_i \in \mathcal{S}(N)$, $i = 1, 2, 3$ satisfy $I \le A_i \le 2I$ for $i = 1, 2, 3, 4$ and

$$F(X) = -\max(\text{Trace}\,(A_1X)\,, \min(\text{Trace}\,(A_2X)\,, \text{Trace}\,(A_3X))).$$

This is a uniformly elliptic equation - here this means that there are constants $0 < \lambda < \Lambda$ such that

$$F(X + P) \le F(X) - \lambda \text{Trace}\,(P) \quad \text{and} \quad |F(X) - F(Y)| \le \Lambda \|X - Y\|$$

for $X, Y, P \in \mathcal{S}(N)$, $P \ge 0$. Here $\|X\|$ can be any reasonable matrix norm of X; a good one is the sum of the absolute values of the eigenvalues of X, as it coincides with the trace on nonnegative matrices.

Exercise 1.4. Determine λ, Λ which work above.

It is known that solutions of uniformly elliptic equations typically have Hölder continuous first derivatives, but it is not known if these solutions are necessarily C^2. If the equation is uniformly elliptic and convex in X, regularity is known. See Evans [20], Cabré and Caffarelli [7], Dong [17], the references therein, as well as Trudinger [39] and Święch [37] for a recent result concerning Sobolev rather than Hölder regularity.

2. The Notion of Viscosity Solutions

As we will see, the theory will require us to deal with semicontinuous functions, there is no escape. Therefore, let us recall the notions of the *upper semicontinuous envelope* u^* and the *lower semicontinuous envelope* u_* of a function $u : \Omega \to \mathbb{R}$:

$$(2.1) \qquad \begin{cases} u^*(x) = \limsup_{r\downarrow 0} \{u(y) : y \in \Omega, |y-x| \le r\} \\ u_*(x) = \liminf_{r\downarrow 0} \{u(y) : y \in \Omega, |y-x| \le r\}. \end{cases}$$

Recall that u is upper semicontinuous if $u = u^*$ and lower semicontinuous if $u = u_*$; equivalently, u is upper semicontinuous if $x_k \to x$ implies $u(x) \ge \limsup_{k\to\infty} u(x_k)$, etc. Of course, u^* is upper semicontinuous and u_* is lower semicontinuous.

Exercise 2.1.* In the above definition Ω could be replaced by an arbitrary metric space $\mathcal{O}$ if $|y-x|$ is replaced distance between $x, y \in \mathcal{O}$. Show in this generality that u is upper semicontinuous if and only if $v = -u$ is lower semicontinuous if and only if $\{x \in \mathcal{O} : u(x) \le r\}$ is closed for each $r \in \mathbb{R}$. Show that a function which is both upper semicontinuous and lower semicontinous is continuous. Show that if $\mathcal{O}$ is compact and u is upper semicontinuous on $\mathcal{O}$, then u has a maximum point $\hat{x}$ such that $u(x) \le u(\hat{x})$ for $x \in \mathcal{O}$.

Motivation for the following definition is found in Exercise 0.2; see also Exercise 2.4 below. The semicontinuity requirements in the definition are partly explained by the last part of Exercise 2.1 and the fact that we will want to produce the maxima associated with subsolutions, etc., in proofs.

Definition 2.1. *Let F be proper, Ω be open and $u : \Omega \to \mathbb{R}$. Then u is a viscosity subsolution of $F = 0$ in Ω if it is upper semicontinuous and for every $\varphi \in C^2(\Omega)$ and local maximum point $\hat{x} \in \Omega$ of $u - \varphi$, we have $F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \le 0$. Similarly, $u : \Omega \to \mathbb{R}$ is a viscosity supersolution of $F = 0$ in Ω if it is lower semicontinuous and for every $\varphi \in C^2(\Omega)$ and local minimum point $\hat{x} \in \Omega$ of $u - \varphi$, we have $F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \ge 0$. Finally, u is a viscosity solution of $F = 0$ in Ω if is both a viscosity subsolution and a viscosity supersolution (hence continuous) of $F = 0$.*

Remark 2.2. Hereafter we use the following conventions: "supersolution", "subsolution" and "solution" mean "viscosity supersolution", "viscosity subsolution" and "viscosity solution" – other notions will carry the modifiers (e.g., classical solutions, etc.). Moreover, the phrases "subsolution of $F = 0$" and "solution of $F \le 0$" mean the same (and similarly for supersolutions).

Remark 2.3. Explicit subsolutions and supersolutions which are semicontinuous and not continuous will not appear in these lectures. They intervene abstractly in proofs, however, in an essential way.

Exercise 2.2.* Reconcile Definition 2.1 with Exercise 0.2 in the following sense: Show that if F is proper, $u \in C^2(\Omega)$ and

$$F(x, u(x), Du(x), D^2u(x)) \le 0$$

$(F(x, u(x), Du(x), D^2u(x)) \ge 0)$ for $x \in \Omega$, then u is a solution of $F \le 0$ (respectively $F \ge 0$) in the above sense.

Exercise 2.3.* With F as in Exercise 1.1, verify that $u(x) = 1 - |x|$ is a solution of $F = 0$ on (-1,1), but that $u(x) = |x| - 1$ is not. Attempt to show that $u(x) = 1 - |x|$ is the only solution of $F = 0$ in (-1,1) which vanishes at $x = -1, 1$. Verify that $u(x) = |x| - 1$ is a solution of $-(u')^2 + 1 = 0$. In general, verify that if F is proper then u is a solution of $F \le 0$ if and only if $v = -u$ is a solution of $G \ge 0$ where $G(x, r, p, X) = -F(x, -r, -p, -X)$ and that G is proper. Thus any result about subsolutions provides a dual result about supersolutions.

Exercise 2.4. In general, if Ω is bounded and open in $\mathbb{R}^N$, verify that $u(x) = \text{distance}(x, \partial\Omega)$ is a solution of $|Du| = 1$ in Ω.

We mention that the idea of putting derivatives on test functions in this maximum principle context was first used to good effect in Evans [18, 19]. The full definitions above in all their semicontinuous glory, evolved after the uniqueness theory was initiated in [14], [15]. The definition in these works was equivalent to that above, but was formulated differently and all functions were assumed continuous. The paper [16] comments on equivalences and writes proofs more similar to those given today. Ishii's introduction of the Perron method in [24] was a key point in establishing the essential role of semicontinuous functions in the theory. Ishii in fact defines a "solution" to be a function u such that u^* is a subsolution and u_* is a supersolution. See Bardi's lectures [2] in this regard.

3. Statements of Model Existence - Uniqueness Theorems

Recalling the discussion of classical solutions of the Cauchy problem (1.1) and Exercise 1.3, the following results are a striking affirmation that the solutions introduced in Definition 2.1 are appropriate.

For Hamilton-Jacobi equations we have:

Theorem 3.1. *Let $H : \mathbb{R}^N \to \mathbb{R}$ be continuous and $\psi : \mathbb{R}^N \to \mathbb{R}$ be uniformly continuous. Then there is a unique continuous function $u : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$ with the following properties: u is uniformly continuous in x uniformly in t, u is a solution of $u_t + H(Du) = 0$ in $(0, \infty) \times \mathbb{R}^N$ and u satisfies $u(0, x) = \psi(x)$ for $x \in \mathbb{R}^N$.*

Even more striking is the following even more unequivocal generalization to include second order equations:

Theorem 3.2. *Let $F : \mathbb{R}^N \times \mathcal{S}(N) \to \mathbb{R}$ be continuous and degenerate elliptic. Then the statement of Theorem 3.1 remains true with the equation $u_t + H(Du) = 0$ replaced by the equation $u_t + F(Du, D^2u) = 0$.*

The analogue of 3.2 for the stationary problem (i.e., without "t") is

Theorem 3.3. *Let $F : \mathbb{R}^N \times \mathcal{S}(N) \to \mathbb{R}$ be continuous and degenerate elliptic and $f : \mathbb{R}^N \to \mathbb{R}$ be uniformly continuous. Then there is a unique uniformly continuous $u : \mathbb{R}^N \to \mathbb{R}$ which is a solution of $u + F(Du, D^2u) - f(x) =$ in $\mathbb{R}^N$.*

Moreover, the solutions whose unique existence is asserted above are the ones which are demanded by the theories developed in the other lectures in this volume. In Bardi [2] and Soner [34] formulas are given for potential solutions of various problems, in control theoretic and differential games settings, and it is a triumph of the theory that the functions given by the formulas can be shown to be the unique solutions given by the theory.

All of the heavy lifting needed to prove these results is done below. However, some of the details are left for the reader's pleasure. The proof of Theorem 3.1 is indicated at the end of Section 9, the proof of Theorem 3.3 is completed in Exercise 10.3 and the proof of Theorem 3.2 is completed in Exercise 12.1.

4. Comparison for Hamilton-Jacobi Equations: the Dirichlet Problem

The technology of the proof of comparison in the second order case is more complex than in the first order case, so at this first stage we offer some sample first order comparison proofs. As a pedagogical device, we present a sequence of proofs illustrating various technical concerns. We begin with simplest case, that is the Dirichlet problem. The next two sections concern variants. Arguments are the main point, so we do not package the material as "theorems", etc. All of the arguments given are invoked later in the second order case so no time is wasted by passing through the first order case along the way.

Let Ω be a bounded open set in $\mathbb{R}^N$. The Dirichlet problem is:

$$\text{(DP)} \qquad H(x, u, Du) = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial\Omega.$$

Here H is continuous and proper on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ and $g \in C(\partial\Omega)$. We say that $u : \overline{\Omega} \to \mathbb{R}$ is a subsolution (supersolution) of (DP) if u is upper semicontinuous (respectively, lower semicontinuous), solves $H \le 0$ (respectively, $H \ge 0$) in Ω and satisfies $g \le u$ on $\partial\Omega$ (respectively, $u \ge g$ on $\partial\Omega$).

Exercise 4.1. One does not expect (DP) to have solutions in general. Show that if $N = 1$, $\Omega = (0, 1)$, the Dirichlet problem $u + u' = 1$, $u(0) = u(1) = 0$ does not have solutions (in the sense of Definition 2.1!).

We seek to show that if u is a subsolution of (DP) and v is a supersolution of (DP), then $u \le v$. We will not succeed without further conditions on H. Indeed, choose Ω to be the unit ball and let $w(x) \in C^1(\overline{\Omega})$ be any function which vanishes on $\partial\Omega$ but does not vanish identically. Then w and $-w$ are distinct classical solutions (and hence viscosity solutions, via Exercise 2.4) of (DP) with $H(x, u, p) = |p|^2 - |Dw|^2$, $g = 0$. We will discover sufficient conditions to guarantee the comparison theorem along the way.