



# CALCULUS

by

G. E. F. SHERWOOD

PROFESSOR OF MATHEMATICS

and

ANGUS E. TAYLOR

PROFESSOR OF MATHEMATICS

UNIVERSITY OF CALIFORNIA  
AT LOS ANGELES

THIRD EDITION

PRENTICE-HALL, INC.  
Englewood Cliffs

*Copyright, 1942, 1946, 1954, by Prentice-Hall, Inc., Englewood Cliffs, N. J. All rights reserved. No part of this book may be reproduced, by mimeograph or any other means, without permission in writing from the publisher.*

First Printing.....August, 1942  
Second Printing.....March, 1943  
Third Printing.....June, 1943  
Fourth Printing.....July, 1943  
Fifth Printing.....November, 1945  
Sixth Printing.....January, 1946  
Seventh Printing...February, 1946  
Eighth Printing.....June, 1946  
Ninth Printing...September, 1946

REVISED EDITION

First Printing.....October, 1946  
Second Printing.....April, 1947  
Third Printing.....January, 1949  
Fourth Printing.....May, 1950  
Fifth Printing.....January, 1952  
Sixth Printing.....December, 1952  
Seventh Printing.....June, 1953

THIRD EDITION

First Printing ..... April, 1954  
Second Printing .... October, 1954  
Third Printing .... August, 1955  
Fourth Printing ..... June, 1956  
Fifth Printing .... January, 1957  
Sixth Printing .... January, 1958  
Seventh Printing .. January, 1959  
Eighth Printing ..... July, 1959

# CALCULUS

PRENTICE-HALL MATHEMATICS SERIES

ALBERT A. BENNETT, EDITOR

## Preface

It is the purpose of this book to set forth in a systematic and thorough manner the fundamental principles, methods, and uses of calculus. The plan of the book is basically the same as it was in earlier editions. The changes arise largely from the experience of the authors, their colleagues, and other users of the book in the classroom. The aim of the changes is to make a clearer, more interesting, and more profitable book for the student, by incorporating various improvements in explanatory material, in diagrams, in arrangement, and in the sets of exercises.

The presentation is designed to give the student a sound understanding of the concepts of calculus, and of their applications, to make him aware of the logical structure of the subject, and to give him experience and training in the formulation and solution of problems. The illustrative examples and the exercises bear not only on mathematics, but on physics, chemistry, engineering, and to a lesser extent on economics and other disciplines.

In calculus the student has his first extensive acquaintance with limit processes. One of the foremost problems confronting the teacher of calculus is that of guiding his students toward an understanding of the basic definitions and theorems about limits. Such an understanding should lead the students to see the logical coherency in the structure of calculus, and develop in them the power to use limit concepts effectively in reasoning. It is not enough to rely entirely on a student's intuitive grasp of limit concepts. Intuitive understanding of limit processes, as they are met in every-day situations in geometry and physics, should be carefully cultivated. But the student should be taught to appreciate that mathematical reasoning in calculus, as in geometry and elsewhere, proceeds from fundamental definitions and assumptions, and systematizes its progress in theorems which can be accurately stated and proved. The student should be conscious of the definition of a derivative as an instance of a limit concept, and he should see that the development of the technique of differentiation is based largely on the theorems about limits of sums, products, and quotients. He should realize that the definite integral concept rests on a notion of limit distinct from that occurring in the definition of the derivative, and that it is by means of a chain of general theorems that we come to an understanding of how to compute the values of integrals by a process inverse to differentiation.

A feature of the present edition is the very early introduction of the inverse of differentiation, in Chapter II, §§16, 17, and 18. The actual

techniques here are limited to polynomials. There is a section on applications to problems of velocity and acceleration in rectilinear motion. This will be advantageous to students who take physics concurrently with the beginning of calculus. It is also possible to deal with simple area problems at an early stage, for Chapter I contains a discussion of the definition of area under a curve and a rudimentary definition of a definite integral (using subdivision of the interval into  $n$  equal parts), and Chapter II has a section on finding areas by antidifferentiation.

The full development of the relation between differentiation and integration comes in Chapter IX. The definite integral is defined as the limit of approximating sums, and the connection between differentiation and integration is worked out analytically. Not until this has been done is the word *integration* used in connection with the inverse differentiation. Adherence to this procedure in treating integration seems to us to be important if the concept of an integral is to be properly understood.

Many geometrical and physical quantities are defined by processes which closely resemble but do not exactly coincide with the definition of a definite integral as the limit of approximating sums. There is a systematic general argument which can usually be employed to show that the quantities in question actually are expressible as definite integrals, even though their definitions are not quite the same in form as the definition of a definite integral. The formalization of this general argument is what we call *Duhamel's principle*. The essential form of the principle, as stated in §93, is due to the late Professor W. F. Osgood. Since uniform continuity is the concept which lies back of this whole matter, no proof is presented for Duhamel's principle. The object is twofold: (1) to show that, when careful attention is paid to definitions, such topics as are length, centroids of solids of revolution, and many others, call for something more than recognition of the definition of a definite integral, and (2) to supply a systematic procedure for attending to the situation.

The greatest changes have been made in Chapters I, II, XIV, XV, and XVI, and there are extensive changes in other chapters. The concept of a derivative has been introduced much sooner. Chapter I in its new form is less formidable for the beginner, and it makes possible more rapid progress into interesting applications of the fundamental concepts of calculus. The fundamental theorems about limits of sums, products and quotients are stated in Chapter I, but the proofs are not given until Chapter XIV. The fundamental limit theory relating to trigonometric functions has been moved from Chapter I to Chapter V. The discussion of  $e$ , both in Chapter V and the Appendix, is entirely revised. Improper integrals have been shifted from Chapter XI to Chapter XIV, and Chapter XIV contains a revised proof of l'Hospital's rule, including the  $\frac{\infty}{\infty}$  case. Use of the rule is taken up earlier, in Chapter VI. The whole

approach to Cauchy's principle of convergence and the convergence of bounded monotonic sequences has been rewritten. The treatment of infinite series and the expansion of functions in power series (Chapters XV and XVI) has been very much rearranged. The present arrangement is more flexible than the previous one if time does not permit the full coverage of both chapters in a single course. We have included the integral form of the remainder in Taylor's formula, and the discussion of operations with power series is somewhat fuller than it was.

The teaching of calculus is not static, fortunately, for both teachers and students. Our notions of what is best, and of what goals are attainable, undergo some changes with the years. Also, we like to try new approaches. Our colleagues have been cooperative and kind, and our students have continued to teach us both our strengths and our weaknesses. We are grateful to them all.

G.E.F.S.

A.E.T.



## Foreword to the Student

You are no doubt wondering what calculus is about. We shall try to give you some idea of the nature of the subject and its importance before you set out to study it in detail.

The most important applications of calculus are to the physical sciences and to geometry. It enables us to study effectively many of the phenomena of physics: the velocity, acceleration, and general character of the motion of objects acted upon by known forces; the work done by known forces under given circumstances; the force exerted by an impounded fluid upon the walls of its container; the gravitational attraction due to material objects of various shapes and composition. It enables us to compute or measure many important things: areas; volumes; the masses of bodies of variable density; the location of the center of gravity of a body; the moment of inertia of a body when it is revolved about an axis. The laws of electricity and magnetism and all of the modern developments of atomic physics require calculus for their elaboration.

There are applications of calculus to chemistry, biology, and others of the natural sciences. Calculus is also being used in increasing measure in the study of economics.

What are the methods of calculus? It draws freely upon the subjects with which you are already familiar: algebra, trigonometry, and analytic geometry. The new concept is that of *limiting processes*. The notion of a limit is the fundamental underlying idea of calculus. It is what gives the subject its power. Accordingly, the first chapter of this book deals with the limit idea. Again, in Chapter XIV, a further development of the notion is set forth. You will not learn about limits from these chapters alone, however; the concept of a limit must be clothed with meaning as you go along, observing its occurrence in each new situation.

To study calculus, or any mathematical discipline, effectively, you must develop proper habits of study. Always read the text with a pencil and paper at hand, ready to elaborate for yourself any step in the reasoning which is not clear. It is especially desirable that you work out each illustrative example yourself, supplying any details that may have been omitted from the presentation in the book. When doing the exercises, remember that you develop your own power over the subject, and implant the method more firmly in mind, if you get along with a minimum of help from the text. You will also, in this way, discover those things that are causing you trouble.

G.E.F.S.  
A.E.T.

# Contents

	PAGE
GREEK ALPHABET; FORMULAS AND TABLES FOR REFERENCE . . . . .	1
CHAPTER	
I. FUNDAMENTAL CONCEPTS. . . . .	11
1. What is calculus? . . . . .	11
2. Study of a motion . . . . .	12
3. Rates of change . . . . .	14
4. Variables. Functions. . . . .	16
5. Definition of the derivative . . . . .	20
6. The limit concept for functions. . . . .	22
7. Areas. . . . .	27
8. The limit concept for sequences . . . . .	31
9. Summary . . . . .	35
II. ELEMENTARY TECHNIQUES AND APPLICATIONS . . . . .	37
10. The $\Delta$ -process. . . . .	37
11. Differentiation of $x^n$ . . . . .	39
12. Geometrical significance of the derivative. . . . .	42
13. Constant factors. Sums. . . . .	45
14. Increasing and decreasing functions . . . . .	48
15. Continuity and differentiability. . . . .	52
16. Finding a function with known derivative . . . . .	55
17. Rectilinear motion. . . . .	57
18. Finding areas by antidifferentiation . . . . .	63
19. Summary. . . . .	67
III. GENERAL FORMULAS OF DIFFERENTIATION: ALGEBRAIC FUNCTIONS . . . . .	69
20. Differentiation of products and quotients. . . . .	69
21. Composite functions. . . . .	73
22. Implicit functions. . . . .	78
23. Fractional exponents. . . . .	82
24. Derivatives of higher order. . . . .	84
25. Summary. . . . .	86
IV. APPLICATIONS. . . . .	89
26. Maximum and minimum values. . . . .	89
27. Significance of the second derivative. . . . .	91
28. Curve tracing; rational functions . . . . .	95
29. Exceptional cases. Vertical tangents . . . . .	101
30. Curve tracing; algebraic curves . . . . .	102

31. Problems involving maxima and minima . . . . .	105
32. The use of auxiliary variables . . . . .	110
33. Rates . . . . .	113
34. Summary . . . . .	117

## V. DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS WITH APPLICATIONS. . . . . 119

35. Preliminary remarks . . . . .	119
36. Trigonometric functions . . . . .	119
37. Differentiation of the sine function . . . . .	122
38. Differentiation of the other trigonometric functions . . . . .	126
39. Exponentials and logarithms . . . . .	130
40. Differentiation of the logarithm function . . . . .	132
41. Natural logarithms. Differentiation technique . . . . .	135
42. Differentiation of exponential functions . . . . .	137
43. Inverse functions . . . . .	141
44. The inverse trigonometric functions . . . . .	143
45. Maxima and minima. Graphing. Rates . . . . .	148
46. Simple harmonic motion . . . . .	157
47. Summary . . . . .	160

## VI. DIFFERENTIALS. THE LAW OF THE MEAN . . . . . 163

48. The differential of a function . . . . .	163
49. Approximation by differentials . . . . .	164
50. Formulas of differentiation . . . . .	167
51. Parametric representation . . . . .	171
52. Rolle's theorem. The law of the mean . . . . .	173
53. L'Hospital's rule . . . . .	177
54. Numerical solution of equations. Newton's method . . . . .	181
55. Summary . . . . .	185

## VII. FURTHER APPLICATIONS . . . . . 188

56. Arc length . . . . .	188
57. Motion in a curve. Vector velocity . . . . .	192
58. Polar coördinates . . . . .	196
59. Vector acceleration . . . . .	202
60. Curvature . . . . .	207
61. Parametric forms. Normal component of acceleration . . . . .	211
62. The center of curvature. The evolute . . . . .	214
63. Roulettes. The cycloid . . . . .	219
64. Summary . . . . .	222

## VIII. THE INVERSE OF DIFFERENTIATION . . . . . 224

65. Antiderivatives . . . . .	224
66. Further practice with substitutions . . . . .	231
67. Differential equations . . . . .	234
68. Applications . . . . .	238
69. Summary . . . . .	245

<b>IX. THE DEFINITE INTEGRAL . . . . .</b>	<b>246</b>
70. The integral concept. . . . .	246
71. Properties of the definite integral . . . . .	251
72. Variable limits of integration . . . . .	253
73. The method of calculating definite integrals. . . . .	255
74. Finding areas by integration . . . . .	257
75. The volume of a solid of revolution . . . . .	261
76. The existence of the integral . . . . .	264
77. Summary. . . . .	265
<b>X. THE TECHNIQUE OF INTEGRATION . . . . .</b>	<b>267</b>
78. The nature of the problem of finding indefinite integrals . . . . .	267
79. Completing the square. A reduction formula. . . . .	268
80. Integration of rational functions. . . . .	273
81. Integration by parts. . . . .	279
82. Certain trigonometric integrals . . . . .	282
83. Trigonometric substitutions. . . . .	289
84. Rationalizing substitutions . . . . .	292
85. Substitution in a definite integral . . . . .	295
86. Further discussion of integration. Summary. . . . .	296
<b>XI. GEOMETRICAL APPLICATIONS OF DEFINITE INTEGRALS . . . . .</b>	<b>302</b>
87. Plane areas: polar coördinates. . . . .	302
88. Plane areas: parametric representation. . . . .	306
89. Space figures . . . . .	307
90. Volumes by slicing. . . . .	311
91. Solids of revolution: shell method . . . . .	314
92. Arc length . . . . .	316
93. The principle of Duhamel . . . . .	321
94. The area of a surface of revolution. . . . .	325
95. Approximate integration . . . . .	328
96. Summary. . . . .	330
<b>XII. PHYSICAL APPLICATIONS OF DEFINITE INTEGRALS . . . . .</b>	<b>333</b>
97. The center of gravity of a system of particles . . . . .	333
98. The center of gravity of continuous masses . . . . .	334
99. The centroid of a solid of revolution . . . . .	335
100. The centroid of a plane area. . . . .	339
101. Fluid pressure . . . . .	341
102. Work. . . . .	344
103. Centroids of surfaces of revolution and of arcs. . . . .	347
104. Material curves. Variable density. . . . .	350
105. Gravitational attraction. . . . .	351
106. Summary . . . . .	354
<b>XIII. THE HYPERBOLIC FUNCTIONS. . . . .</b>	<b>357</b>
107. Definitions and functional relations. . . . .	357
108. Differentiation of hyperbolic functions . . . . .	359

109. The inverse hyperbolic functions . . . . .	360
110. Integration . . . . .	361
111. Summary . . . . .	364
<b>XIV. FURTHER STUDY OF LIMITS . . . . .</b>	<b>365</b>
112. Review of fundamental concepts . . . . .	365
113. Monotonic sequences and Cauchy's principle of convergence . . . . .	368
114. Improper integrals . . . . .	373
115. Proof of l'Hospital's rule . . . . .	378
116. Summary . . . . .	380
<b>XV. INFINITE SERIES AND TAYLOR'S FORMULA . . . . .</b>	<b>381</b>
117. Definitions and motivations . . . . .	381
118. Series for logarithms and inverse tangents . . . . .	385
119. Taylor's formula with integral remainder . . . . .	390
120. Taylor's formula with derivative remainder . . . . .	395
121. Comparison tests. The integral test . . . . .	399
122. Alternating series. . . . .	405
123. Absolute convergence. The ratio test . . . . .	409
124. Summary . . . . .	414
<b>XVI. POWER SERIES . . . . .</b>	<b>417</b>
125. The binomial series. . . . .	417
126. Operations with power series. . . . .	421
127. Applications of power series . . . . .	425
128. Summary . . . . .	427
<b>XVII. ANALYTICAL GEOMETRY OF THREE DIMENSIONS . . . . .</b>	<b>429</b>
129. The distance formula. Spheres . . . . .	429
130. Direction cosines of a line . . . . .	432
131. The angle between two lines. . . . .	434
132. The plane. . . . .	437
133. The straight line . . . . .	441
134. Cylinders . . . . .	445
135. Surfaces of revolution. . . . .	447
136. Quadric surfaces . . . . .	449
137. Systems of coördinates . . . . .	452
138. Summary . . . . .	453
<b>XVIII. PARTIAL DIFFERENTIATION. . . . .</b>	<b>455</b>
139. Functions of several variables. Limits . . . . .	455
140. Partial derivatives . . . . .	456
141. The differential. . . . .	459
142. Maxima and minima . . . . .	464
143. Directional derivatives . . . . .	468
144. Implicit functions . . . . .	473

145. Change of independent variables . . . . .	477
146. Implicit functions . . . . .	481
147. Summary . . . . .	484

## XIX. CURVES AND SURFACES IN SPACE. ENVELOPES . . . . . 487

148. The normal line and tangent plane to a surface . . . . .	487
149. Tangent line and normal plane to a space curve . . . . .	491
150. Length of a curve in three dimensions. . . . .	495
151. The osculating plane . . . . .	497
152. Envelopes. . . . .	500
153. Summary . . . . .	503

## XX. DOUBLE INTEGRALS . . . . . 505

154. Definition and interpretation. . . . .	505
155. The iterated integral . . . . .	507
156. Mass and center of gravity of a lamina . . . . .	513
157. The iterated integral. Second method . . . . .	516
158. The iterated integral in polar coördinates . . . . .	517
159. Moments of inertia. . . . .	523
160. The area of a curved surface. . . . .	527
161. Summary . . . . .	530

## XXI. TRIPLE INTEGRALS . . . . . 533

162. The definition of a triple integral. . . . .	533
163. The iterated integral in rectangular coördinates . . . . .	535
164. Cylindrical coördinates . . . . .	540
165. Spherical coördinates . . . . .	544
166. Summary. Some matters of notation. . . . .	549

## APPENDIX . . . . . 553

The Base of Natural Logarithms . . . . .	553
Table I. Table of Natural Logarithms . . . . .	564
Table II. Exponential and Hyperbolic Functions . . . . .	566
Table III. Natural Functions for Angles in Radians . . . . .	567
Table IV. Values of Trigonometric Functions . . . . .	569
Table V. Degrees and Minutes to Radians . . . . .	574
Table VI. Radians to Degrees and Minutes. . . . .	574

## INDEX . . . . . 575

# Greek Alphabet; Formulas and Tables for Reference

## I. The Greek alphabet

<i>Letters</i>	<i>Names</i>	<i>Letters</i>	<i>Names</i>	<i>Letters</i>	<i>Names</i>
A $\alpha$	alpha	I $\iota$	iota	P $\rho$	rho
B $\beta$	beta	K $\kappa$	kappa	$\Sigma$ $\sigma$	sigma
$\Gamma$ $\gamma$	gamma	$\Lambda$ $\lambda$	lambda	T $\tau$	tau
$\Delta$ $\delta$	delta	M $\mu$	mu	$\Upsilon$ $\upsilon$	upsilon
E $\epsilon$	epsilon	N $\nu$	nu	$\Phi$ $\phi$	phi
Z $\zeta$	zeta	$\Xi$ $\xi$	xi	X $\chi$	chi
H $\eta$	eta	O $\omicron$	omicron	$\Psi$ $\psi$	psi
$\Theta$ $\theta$	theta	$\Pi$ $\pi$	pi	$\Omega$ $\omega$	omega

## II. Elementary geometry

1. Let  $r$  denote radius,  $h$  height,  $B$  area of base,  $\theta$  central angle in radians,  $s$  length of arc subtended by  $\theta$ , and  $l$  slant height.

*Circle.*  $s = r\theta$ . Circumference  $= 2\pi r$ . Area  $= \pi r^2$ . Area of sector  $= \frac{1}{2}r^2\theta$ . Area of segment  $= \frac{1}{2}r^2(\theta - \sin \theta)$ .

*Sphere.* Volume  $= \frac{4}{3}\pi r^3$ . Surface  $= 4\pi r^2$ .

*Segment of sphere.* Volume  $= \pi h^2 \left( r - \frac{h}{3} \right)$ . Surface  $= 2\pi rh$ .

*Prism.* Volume  $= Bh$ .

*Pyramid.* Volume  $= \frac{1}{3}Bh$ .

*Right circular cylinder.* Volume  $= \pi r^2 h$ . Lateral surface  $= 2\pi rh$ .

*Right circular cone.* Volume  $= \frac{1}{3}\pi r^2 h$ . Lateral surface  $= \pi rl$ .

2. *Triangle.* Area  $= \frac{1}{2}ab \sin C$ , where  $a$  and  $b$  are two sides and  $C$  the included angle.

3. *Parallelogram.* Area  $= ab \sin C$ , where  $a$  and  $b$  are two adjacent sides and  $C$  the included angle.

4. *Frustum of pyramid or cone.* Volume  $= \frac{1}{3}(B_1 + B_2 + \sqrt{B_1 \cdot B_2})h$ , where  $h$  is the height and  $B_1$  and  $B_2$  the areas of the parallel bases.

## III. Elementary algebra

1. *Quadratic equations.* If  $ax^2 + bx + c = 0$ ,  $a \neq 0$ ,

then 
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. *General equation of degree  $n$ .* The equation  $a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$ ,  $a_0 \neq 0$ ,  $n$  a positive integer, coefficients  $a_i$  real, has exactly  $n$  roots, multiple roots being counted to degree of multiplicity, and imaginary roots occurring in conjugate pairs.

3. *Logarithms.*  $\log_b N = x$  means  $b^x = N$ .  $\log_b b = 1$ .  $\log 1 = 0$ .  
 $\log xy = \log x + \log y$ .  $\log \frac{x}{y} = \log x - \log y$ .  $\log x^n = n \log x$ .  
 $\log_b x = \frac{\log_c x}{\log_c b}$ ,  $b \neq 1$ ,  $c \neq 1$ .

4. *Binomial theorem.* ( $n$  a positive integer)

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \dots + \frac{n(n-1)(n-2) \dots (n-r+1)}{r!}a^{n-r}b^r + \dots + nab^{n-1} + b^n,$$

where  $\frac{n(n-1)(n-2) \dots (n-r+1)}{r!} = {}_nC_r$ ,

$n! = \underline{n} = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$ , and  $0! = 1$ .

5. *Arithmetic progression.*  $a, a + d, a + 2d, \dots$  where  $a$  is the first term,  $d$  the common difference,  $n$  the number of terms,  $S$  the sum of  $n$  terms, and  $l$  the last or  $n$ th term,  $l = a + (n-1)d$ ,  $S = \frac{n}{2}(a + l)$ ;  
 $A = \frac{a + b}{2}$ , where  $A$  is the *arithmetic mean* of  $a$  and  $b$ .

6. *Geometric progression.*  $a, ar, ar^2, \dots$ , where  $a$  is the first term,  $r$  the common ratio,  $n$  the number of terms,  $S_n$  the sum of  $n$  terms, and  $l$  the last or  $n$ th term,  $l = ar^{n-1}$ ,  $S_n = \frac{a(1 - r^n)}{1 - r}$ ;  $G = \sqrt{ab}$ , where  $G$  is the *geometric mean* of  $a$  and  $b$  ( $a$  and  $b$  positive).

$$S_\infty = \lim_{n \rightarrow \infty} S_n = \frac{a}{1 - r}$$

if and only if  $r^2 < 1$ .

7. *Harmonic progression.*  $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \dots$ ;  $H = \frac{2ab}{a+b}$ , where  $H$  is the *harmonic mean* of  $a$  and  $b$ .  $G^2 = AH$ .

8. *Permutations and combinations.* For a set of  $n$  things taken  $r$  at a time,

$${}_nP_r = n(n-1) \dots (n-r+1) = \frac{n!}{(n-r)!}, \quad 0! = 1.$$

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n(n-1) \dots (n-r+1)}{r(r-1) \dots 1} = \frac{n!}{r!(n-r)!} = {}_nC_{n-r}.$$



9. *Determinants.* The determinant of order  $n$ ,

$$D \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix},$$

has  $n^2$  elements and  $n!$  terms in its expansion.

$$D = \Sigma(\pm)a_{1i}a_{2j}a_{3k} \cdots a_{nl},$$

the plus or minus sign being taken according as the number of inversions of the positive integers  $1, 2, 3, \cdots$  in the sequence  $i, j, k, \cdots, l$  is even or odd. The *minor*  $M_{ij}$  of the element  $a_{ij}$  is the determinant of order  $n - 1$  which remains after deleting the  $i$ th row and  $j$ th column of  $D$ .

The *cofactor* of the element  $a_{ij}$  is  $A_{ij} = (-1)^{i+j}M_{ij}$ .  $D = \sum_{i=1}^n a_{ij}A_{ij}$ ,

$$j = 1, 2, \cdots, n. \quad 0 = \sum_{i=1}^n a_{ij}A_{ik} \text{ if } j \neq k \text{ and } j, k = 1, 2, \cdots, n.$$

*Properties of determinants.* (1) If the corresponding rows and columns of  $D$  be interchanged,  $D$  is unchanged; (2) if any two rows (or columns) of  $D$  be interchanged,  $D$  becomes  $-D$ ; (3) if any two rows (or columns) be identical,  $D = 0$ ; (4) if each element of a row (or column) of  $D$  be multiplied by  $k$ ,  $D$  becomes  $kD$ ; (5) if to each element of a row (or column) be added  $k$  times the corresponding element of another row (or column),  $D$  is unchanged.

*Application to the solution of a simultaneous system of linear equations.*

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & k_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & k_2 \\ \cdots & & \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & k_n \end{array}$$

(1)  $k_i$  not all zero: equations are nonhomogeneous and solution is unique if  $D \neq 0$ . If  $K_i$  is the determinant which results when the elements of the  $i$ th column of  $D$  are replaced by  $k_1, k_2, \cdots, k_n$ , the solution is  $Dx_i = K_i$ ,  $i = 1, 2, \cdots, n$ .

(2)  $k_i$  all zero: equations are homogeneous and solutions other than the obvious  $x_i = 0$  exist if and only if  $D = 0$ .

## IV. Plane trigonometry

An angle is positive if the rotation in its description is counter-clockwise.

The plane angle about a point is 360 degrees or  $2\pi$  radians.

In addition to the ordinary six trigonometric functions of an angle, viz. sine, cosine, tangent, cotangent, secant, and cosecant, the following