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lecture notes in pure and applied mathematic[s]

topology and its applications

S. Thomeier

TOPOLOGY AND ITS APPLICATIONS

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PREFACE

This volume contains the proceedings of the Conference on Topology and its Applications held at Memorial University of Newfoundland from May 7 to 11, 1973. It consists of the papers given by the invited main speakers P.J. Hilton, E. Klein, A. Liulevicius and R. Thom, and of some of the shorter contributed papers presented by other participants. In those cases where a contributed paper has not been included in these proceedings, its abstract is included instead. The manuscript of René Thom's three one-hour lectures on Catastrophe Theory was prepared from audio tapes of his lectures, and the editorial work there was limited to making absolutely necessary changes only in order to preserve the flavour of the oral presentation; for valuable assistance in this task I want to thank Richard L.W. Brown as well as René Thom himself.

As the organizer and chairman of the Conference, I wish to express my appreciation to the National Research Council of Canada, to A.P.I.C.S. (the Atlantic Provinces Inter-University Committee on the Sciences) and to Memorial University for financial support of the Conference, and I want to thank all those, faculty members, graduate students and members of the secretarial staff, who gave manifold assistance before and during the Conference. My special thanks go to the Invited Speakers and the many participants whose contributions made the Conference a success.

As the editor of these proceedings, I want to express my thanks to all colleagues who assisted in the editorial tasks and the refereeing, in particular to Peter Hilton and Arunas Liulevicius. Finally, I want to thank Mrs. H. Tiller for typing a major part of the final typed copy for offset printing.

S. Thomeier

St. John's
February 1974

TABLE OF CONTENTS

CHARACTERISTIC NUMBERS

Arunas Liulevicius

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The lectures are organized as follows: in the first lecture the algebra of unoriented cobordism is described, the second lecture applied cobordism techniques to the geometric question of immersing closed manifolds into Euclidean space, and the third introduces equivariant Stiefel-Whitney classes and numbers and applies these notions to the study of equivariant immersions of G-manifolds into representations of G. Lecture 1 is intended to popularize the work of Thom [19], Newton and VandeVelde [20], lecture 2 -- the work of Brown [4] and Liulevicius [16], lecture 3 - of tom Dieck [8], Stong [18] and Bix [2]. Only lectures 2 and 3 were presented at the conference since fog delayed my arrival.

I am grateful to the organizers of the conference for the opportunity to participate. Thanks also go to Air Canada for a fascinating tour: it was an offer which I couldn't refuse.

Lecture 1. The algebra of cobordism

We shall prove a theorem about Hopf algebras due to Newton (who worked in the context of symmetric polynomials). We prove it for a general ring A (and use the notation of Chern classes) and then specialize it to $A = Z_2$ and show how it applies to cobordism and characteristic numbers. Finally we describe the homotopy part of results of Thom [19] on the structure of the unoriented cobordism ring.

Let A be a commutative ring with unit, C a graded Hopf algebra over A which is described as follows: as an algebra $C = A[c_1,\ldots,c_n,\ldots]$ is a polynomial algebra on an infinite family of generators $c_n \in C^{dn}$

where $d = 1$ if the characteristic of A is 2, $d = 2$ if char $A \neq 2$. The diagonal map $\psi: C \to C \otimes C$ is given by $\psi(c_n) = \sum_{i+j=n} c_i \otimes c_j$, where of course $c_0 = 1$, the augmentation $\varepsilon : C \to A$ is described by letting it be a homomorphism of algebras over A and setting $\varepsilon(c_0) = 1$, $\varepsilon(c_n) = 0$ if $n \geq 1$; the unit map $\eta: A \to C$ is the map of algebras specified by $\eta(1) = 1 = c_0$. Define the graded dual C_* of C by setting $C_{*k} = \mathrm{Hom}_A(C^k, A)$, then $(C_*, \psi_*, \phi_*, \eta_*, \varepsilon_*)$ is a Hopf algebra. If $E = (e_1, \ldots, e_r, \ldots)$ is a sequence of natural numbers such that all but a finite number of the e_r are zero, let $c^E = c_1^{e_1} \ldots c_r^{e_r} \ldots$ and define $y_n \in C_{*dn}$ by the condition $\langle y_n, c^E \rangle = 1$ if $E = (n, 0, 0, \ldots)$, that is $c^E = c_1^n$, and $\langle y_n, c^E \rangle = 0$ if $E \neq (n, 0, 0, \ldots)$.

Theorem 1 (I. Newton). The algebra map $f: C \to C_*$ defined by $f(c_n) = y_n$ is an isomorphism of Hopf algebras.

The main application of this result is to homology of classifying spaces. Let $U = \lim_n U(n)$ be the infinite unitary group, BU its classifying space, then $H^*(BU;Z) = Z[c_1, \ldots, c_n, \ldots]$ as a polynomial algebra where $c_n \in H^{2n}(BU;Z)$ are the Chern classes. Whitney sum of vector bundles gives a map $\psi: BU \times BU \to BU$ which makes BU into an associative H-space, $\psi^* c_n = \sum_{i+j=n} c_i \otimes c_j$, so in this notation $C = H^*(BU;Z)$ for $A = Z$, $d = 2$. We have:

Corollary 2. The homology $H_*(BU;Z)$ is a polynomial algebra on classes $y_n \in H_{2n}(BU;Z)$ coming from $H_{2n}(CP^\infty;Z)$, moreover the coproduct ϕ_* is given by $\phi_*(y_n) = \sum_{i+j=n} y_i \otimes y_j$.

Proof. Only the remark about y_n coming from the homology of CP^∞ needs explanation. The standard inclusion $U(1) \xrightarrow{i} U$ induces a map $CP^\infty = BU(1) \xrightarrow{Bi} BU$, and $(Bi)^* c_n = 0$ if $n \neq 0, 1$, $(Bi)^* c_1 = y$, the fundamental class of CP^∞. Let $y_n \in H_{2n}(CP^\infty;Z)$ be the class in homology dual to y^n, then under the monomorphism $(Bi)_*$ this y_n corresponds to $y_n \in C_{*2n}$ given by Theorem 1.

Motivated by this example we can ask about the filtration of $H_*(BU;Z)$ by means of the images of $H_*(BU(n);Z)$ under the standard maps $BU(n) \to BU$. This corresponds in the setting of Theorem 1 to the following: let $\Delta_i: C_* \to C_{*-di}$ be the map dual to multiplication by c_i. Define $F_n C_*$ by: $F_n C_* = \{x \in C_* \mid \Delta_i(x) = 0 \text{ for } i > n\}$. The structure of

F_nC_* is given by:

Theorem 3. The subgroup F_nC_* is a free A-module with basis y^E $E = (e_1,\ldots,e_r,\ldots)$, $e_1 + \ldots + e_r + \ldots \leq n$.

Our first order of business is the proof of Theorem 1. We do this in a sequence of lemmas.

Lemma 4. If the element $y_n \in C_{*dn}$ is defined by $\langle y_n, c^E\rangle = 1$ if $E = (n,0,0,\ldots)$, $= 0$ if $E \neq (n,0,0,\ldots)$, then

$$\phi_*(y_n) = \sum_{i+j=n} y_i \otimes y_j .$$

Proof. We have $\langle\phi_*(y_n),\ c^{E_1} \otimes c^{E_2}\rangle = \langle y_n, c^{E_1+E_2}\rangle = 0$ unless $E_1 = (i,0,0,\ldots)$, $E_2 = (n-i,0,0,\ldots)$ in which case the value is 1.

Remark. Lemma 4 says that the algebra map $f:C \to C_*$ defined by $f(c_n) = y_n$ is a map of Hopf algebras.

Let $P(C)$ be the primitives of C: these are elements $x \in C^k$, $k > 0$ such that $\psi(x) = x \otimes 1 + 1 \otimes x$.

Lemma 5. $P(C)^{dn}$ is A-free and a direct summand of C^{dn} on one generator s_n defined recursively by

$$s_n - c_1s_{n-1} + c_2s_{n-2} - \ldots + (-1)^{n-1}c_{n-1}s_1 + (-1)^n nc_n = 0.$$

Proof. Consider the homomorphism of graded algebras $h:C \to A[x_1,\ldots,x_n]$ where x_i have grade d, defined by $h(c_k) = \sigma_k$, the k-th elementary symmetric function of $x_1,\ldots,x_n$. Of course, $h(c_k) = 0$ if $k > n$, but h is a monomorphism in gradings less than or equal to dn, and the image of h is precisely the subalgebra of all symmetric polynomials in $x_1,\ldots,x_n$. An A-free basis for the image of h is given by the elements $X(\omega)$, where $\omega = (e_1,\ldots,e_s)$ is a partition, $n \geq s \geq 0$, $e_1 > \ldots > e_s > 0$. and $X(\omega)$ is the sum of the distinct monomials obtained from $x_1^{e_1}\ldots x_s^{e_s}$ by applying permutations of $1,\ldots,n$ to the subscripts. Under the homomorphism h the diagonal $\psi:C \to C \otimes C$ corresponds to the map $\psi X(\omega) = \sum_{(\omega',\omega'')=\omega} X(\omega') \otimes X(\omega'')$, the sum ranging over all subpartitions of ω. In particular, the primitive elements under h correspond to the direct summand generated by $X((k))$, since for $k > 0$ the only partition of k having only trivial subpartitions is (k). By definition $X((k)) = x_1^k + \ldots + x_n^k$. Let

$p(x) = (x-x_1)\ldots(x-x_n) = x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \ldots + (-1)^n \sigma_n$, then $p(x_i) = 0$, $i = 1,\ldots,n$ and so

$$\begin{aligned} 0 &= p(x_1) + \ldots + p(x_n) \\ &= X((n)) - \sigma_1 X((n-1)) + \ldots + (-1)^{n-1}\sigma_{n-1}X((1)) + (-1)^n n\sigma_n. \end{aligned}$$

Let $s_k \in P(C)^{dk}$ be defined by $h(s_k) = X((k))$, then $0 = s_n - c_1 s_{n-1} + \ldots + (-1)^{n-1}c_{n-1}s_1 + (-1)^n nc_n$, and the lemma is proved.

Corollary 6. For each n, $\langle y_n, s_n \rangle = 1$.

Proof. True for $n = 1$, and if true for $n-1$ then the recursion relation for s_n shows that $s_n = c_1^n + \sum a_E c^E$, $E \neq (n,0,0,\ldots)$.

Lemma 7. For each n

$$\langle y^E, c_n \rangle = \begin{cases} 1 & \text{if } E = (n,0,0,\ldots), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. True for $n = 1$. We have $\langle y_1^n, c_n \rangle = \langle y_1^{n-1} \otimes y_1, \psi(c_n) \rangle = \langle y_1^{n-1}, c_{n-1} \rangle \langle y_1, c_1 \rangle = 1$, and if $E \neq (n,0,0,\ldots)$, $y^E = y_i z$, $i > 1$, but then

$$\langle y^E, c_n \rangle = \langle y_i \otimes z, c_i \otimes c_{n-i} \rangle = 0,$$

since $\langle y_i, c_i \rangle = 0$ for $i > 1$.

Corollary 8. For each n, $\langle f(s_n), c_n \rangle = 1$.

Proof. See Corollary 6 and Lemma 7.

We are now ready to prove Theorem 1. First, f is a monomorphism. We do this by induction on the grading. Since $f(1) = 1$, f is a monomorphism in grading 0. Suppose $\operatorname{Ker} f|C_i = 0$ for $i \leq n$ and suppose $x \in \operatorname{Ker} f|C_{n+1}$. Since f is a map of Hopf algebras, $\psi(x) - x \otimes 1 - 1 \otimes x \in \operatorname{Ker} f \otimes f$, where

$$f \otimes f \colon \bigoplus_{i=1}^{n} C^i \otimes C^{n+1-i} \to C_* \otimes C_*$$

is a monomorphism, since C_* is A-free, so $\psi(x) = x \otimes 1 + 1 \otimes x$, or $x = \lambda s_k$ $(dk = n+1)$, but $\lambda = \lambda\langle f(s_k), c_k \rangle = \langle f(\lambda s_k), c_k \rangle = \langle f(x), c_k \rangle = 0$, so $x = 0$ and $f|C_{n+1}$ is a monomorphism.

To prove that f is an epimorphism, it is sufficient that $f: Q(C) \to Q(C_*)$, where $Q(B) = \overline{B}/\overline{B}\cdot\overline{B}$ denotes the indecomposable elements of a graded connected algebra B (here $\overline{B}$ = all $x \in B$ with grade $x > 0$). Since $P(C)$ is a direct summand of C, the exactness of

$$0 \longrightarrow P(C) \longrightarrow \overline{C} \xrightarrow{\overline{\psi}} \overline{C} \times \overline{C}$$

implies the exactness of

$$\overline{C}_* \otimes \overline{C}_* \xrightarrow{\overline{\psi}_*} \overline{C}_* \longrightarrow P(C)_* \longrightarrow 0 ,$$

so $P(C)_* = Q(C_*)$, and $\langle f(c_n), s_n\rangle = \langle y_n, s_n\rangle = 1$ by Corollary 6, so $f: C \to C_*$ is onto as well, completing the proof of Theorem 1.

Now to prove Theorem 3. Let $\Delta_i : C_* \to C_{* - di}$ be the map dual to multiplication by c_i. We have:

Lemma 9. The maps Δ_i are group homomorphisms and satisfy

$$\Delta_k(xy) = \sum_{i+j=k} \Delta_i(x)\Delta_j(y).$$

Proof. The coproduct of c_k is $\sum_{i+j=k} c_i \otimes c_j$.

We introduce a polynomial variable s and define $\Delta: C_* \to C_*[s]$ by setting $\Delta(x) = \sum_{i=0} (\Delta_i x)s^i$, then if we give the variable s the grading d (remember, $d = 1$ if char $\Lambda = 2$, $d = 2$ if char $\Lambda \neq 2$) we have:

Corollary 10. The map $\Delta: C_* \to C_*[s]$ is a homomorphism of graded algebras.

Since we now know that $C_* = A[y_1,\ldots,y_n,\ldots]$ we can introduce an additional bit of structure into C_*. If $E = (e_1,\ldots,e_r)$ is a sequence of natural numbers, let (as before) $y^E = y_1^{e_1}\ldots y_r^{e_r}$ and define deg $y^E = e_1 + \ldots + e_r$ and set $\deg(\sum a_E y^E)$ = maximum deg y^E where $a_E \neq 0$. The invariant deg is the algebraic degree in the polynomial generators y_i and is to be distinguished from the grading: recall that grade $y^E = d(e_1 + 2e_2 + \ldots + re_r)$. We also have the notion of degree in $C_*[s]$, namely $\deg(\sum a_i s^i)$ = maximum i such that $a_i \neq 0$.

Lemma 11. The map $\Delta: C_* \to C_*[s]$ preserves degree.

Proof. We have $\Delta y_k = y_k + y_{k-1}s$, so if $E = (e_1,\ldots,e_r)$, $y^E = y_1^{e_1}\ldots y_r^{e_r}$, then $\Delta y^E = (\Delta y_1)^{e_1}\ldots(\Delta y_r)^{e_r}$ has degree $e_1 + \ldots + e_r$

in s, and the top coefficient is $y_1^{e_2} \dots y_{r-1}^{e_r}$, so two monomials y^E, $y^{E'}$ of the same grade have the same top coefficient in $\Delta(y^E)$ and $\Delta(y^{E'})$ if and only if $(e_2, \dots, e_r) = (e'_2, \dots, e'_r)$, but this implies $e_1 = e'_1$ as well, so $y^E = y^{E'}$.

We define a filtration on C_* by setting $F_n C_* = \{x \in C_* | \deg \Delta x \leq n\}$, that is $x \in F_n C_*$ if and only if $\Delta_i = 0$ for $i > n$, we have:

Theorem 3. $F_n C$ is the free A-module on monomials y^E where $\deg y^E \leq n$.

Examples. 1. The standard map $BU(n) \to BU$ induces a monomorphism $H_*(BU(n);Z) \to H_*(BU;Z)$ and the image is precisely the free abelian group on y^E with $\deg y^E \leq n$.

2. The standard map $BO(n) \to BO$ induces a monomorphism $H_*(BO(n);Z_2) \to H_*(BO;Z_2)$ and the image is precisely the subspace over Z_2 with basis x^E, $\deg x^E \leq n$, where x_i is the element in $H_i(BO(1);Z_2)$ dual to w_1^i.

3. The operations Δ_i give us a quick way of determining the incidence matrices of C with C_* (VandeVelde [20] has them explicitly for $H_n(BU;Z)$ for $n \leq 24$). For example, $\langle c_2, y_1^2 \rangle = 1$, $\langle c_1^2, y_1^2 \rangle = \langle c_1, \Delta_1(y_1^2) \rangle = \langle c_1, 2y_1 \rangle = 2$ and we have the incidence matrix

	c_2	c_1^2
y_2	0	1
y_1^2	1	2

Indeed, VandeVelde [20] uses the triangularity of the incidence matrices under a clever ordering of the c^E, y^F bases to give a different proof of Theorem 1.

We now explain how this bears on the algebra of unoriented cobordism (see the original paper of Thom [19]). Let $\alpha : E(\alpha) \to X$ be a vector bundle with structure group $O(n)$ and $EB(\alpha)$ the total space of the unit ball bundle, $ES(\alpha)$ the total space of the unit sphere bundle, $M(\alpha) = EB(\alpha)/ES(\alpha)$ the Thom space of α. Notice that $M(\alpha \times \beta) = M(\alpha) \wedge M(\beta)$, $M(\varepsilon^n) = S^n$, where $\varepsilon^n : R^n \to$ point. The reduced Z_2-cohomology of $M(\alpha)$ is a free $H^*(X;Z_2)$-module on one generator

$U \in \tilde{H}^n(M(\alpha);Z_2)$ (dim $\alpha = n$) and the map

$$\phi : H^k(X;Z_2) \to \tilde{H}^{n+k}(M(\alpha);Z_2)$$

defined by $\phi(x) = x \cdot U$ is an isomorphism (U is called the Thom class of α and ϕ is called the Thom isomorphism). Of course, dually we have the Thom isomorphism in homology:

$$\phi_* : \tilde{H}_{n+k}(M(\alpha);Z_2) \to H_k(X;Z_2).$$

We write $MO(n) = M(\gamma^n)$, where γ^n is the classifying n-plane bundle over $BO(n)$, and we have maps

$$e_n : MO(n) \wedge S^1 \to MO(n+1)$$

$$\mu_{m,n} : MO(m) \wedge MO(n) \to MO(m+n)$$

induced by the standard inclusion $O(n) \to O(n+1)$ and the Whitney sum representation $O(m) \times O(n) \xrightarrow{w} O(m+n)$. Using the suspension homomorphisms and maps induced by e_n we define

$$\pi_m(MO) = \lim_{\substack{\to \\ n}} \pi_{m+n}(MO(n)) \quad ,$$

$$H_m(MO;Z_2) = \lim_{\substack{\to \\ n}} \tilde{H}_{m+n}(MO(n);Z_2).$$

Since the following diagram commutes

$$\begin{array}{ccc} \tilde{H}_{m+r}(MO(m);Z_2) \otimes \tilde{H}_{n+s}(MO(n);Z_2) & \xrightarrow{\mu_{m,n*}} & H_{m+n+r+s}(MO(m+n);Z_2) \\ \Big\downarrow \phi_* \otimes \phi_* & & \Big\downarrow \phi_* \\ H_r(BO(m);Z_2) \otimes H_s(BO(n);Z_2) & \xrightarrow{w_*} & H_{r+s}(BO(m+n);Z_2) \end{array}$$

(here ϕ_* is the Thom isomorphism in homology), the maps $\mu_{m,n}$ induce a ring structure in $\pi_*(MO)$, $H_*(MO;Z_2)$ and $\phi_* : H_*(MO;Z_2) \to H_*(BO;Z_2)$ is a ring isomorphism. We define $b_n \in H_n(MO;Z_2)$ by $\phi_*(b_n) = x_n$. Notice that b_n is born on $MO(1) \equiv BO(1)$ and there has the name x_{n+1}, the class dual to w_1^{n+1}. The Hurewicz homomorphisms over Z_2 $h : \pi_{m+k}(MO(k)) \to H_{m+k}(MO(k);Z_2)$ fit together to give a ring homomorphism $h : \pi_*(MO) \to H_*(MO;Z_2)$. Let A_* be the dual of the Steenrod algebra over Z_2 (see Milnor [17]), $\mu : H_*(MO;Z_2) \to A_* \otimes H_*(MO;Z_2)$

the coaction. Since μ is a homomorphism of algebras over Z_2 it is sufficient to specify $\mu(b_n)$. Now

$$\mu_*(b_n) = \sum_{s=0}^{s} \gamma_{n-s}^{(s+1)} \otimes b_s$$

where $\gamma_{n-s}^{(s+1)} \in A_{*n-s}$ satisfy the relations:

$$\gamma_s^{(1)} = \begin{cases} \xi_r & \text{if } s = 2^r - 1 , \\ 0 & \text{if } s \neq 2^r - 1 , \end{cases}$$

where ξ_r are the Milnor [17] generators, moreover $\gamma_0^{(n+1)} = 1$ and the Cartan relations are satisfied: for each pair of natural numbers i,j we have

$$\gamma_r^{(i+j)} = \sum_{r=s+t} \gamma_s^{(i)} \gamma_t^{(j)}$$

(see Liulevicius [15], for example).

Theorem 12 (Thom). The algebra $\pi_*(MO)$ is a polynomial algebra $Z_2[u_2,u_4,\ldots,u_n,\ldots]$, $n \neq 2^r - 1$ and $h:\pi_*(MO) \to H_*(MO;Z_2)$ is a monomorphism onto the elements x such that $\mu(x) = 1 \otimes x$ (the primitives under the coaction μ).

Proof. Let $N_* = Z_2[u_2,u_4,\ldots,u_n,\ldots]$, $n \neq 2^r - 1$ and define a homomorphism of algebras and comodules over A_*

$$f:H_*(MO;Z_2) \to A_* \otimes N_*$$

(the target being the extended A_*-comodule on N_*) by setting $\underline{f} = (\eta_* \otimes 1)f:H_*(MO;Z_2) \to N_*$ $\underline{f}(b_n) = u_n$ if $n \neq 2^r - 1$, $\underline{f}(b_n) = 0$ if $n = 2^r - 1$ for some r. Then since $f = (1 \otimes \underline{f})\mu$ we have $f(b_n) = 1 \otimes u_n$ modulo decomposables if $n \neq 2^r - 1$, $f(b_{2^r-1}) = \xi_r \otimes 1$ modulo decomposables, so f is onto, hence an isomorphism, since the dimensions of the domain and target are the same in each grading. Since $H_*(MO;Z_p) = 0$ for p an odd prime it follows that MO is equivalent to the Eilenberg-MacLane spectrum on the graded Z_2 vector space N_*, and the theorem follows.

Remark. This version of the proof appears in Liulevicius [14] (see also the correction in [15]). Of course, the image of the Hurewicz homomorphism is precisely $f^{-1}(1 \otimes N_*)$.

Let us simplify notation by identifying $H_*(MO;Z_2)$ with $A_* \otimes N_*$ under f, thus identifying $\pi_*(MO)$ with $1 \otimes N_*$. The following table gives the u_n in terms of b_i for $n \leq 10$ (for u_n, $n \leq 18$ see Table 1.1 in Liulevicius [16]).

Table

generator	algebraic degree	expression
ξ_1	1	b_1
u_2	1	b_2
ξ_2	1	b_3
	2	$+ b_1b_2$
u_4	1	b_4
	3	$+ b_1^2b_2$
u_5	1	b_5
	2	$+ b_1b_4 + b_2b_3$
	3	$+ b_1b_2^2$
u_6	1	b_6
ξ_3	1	b_7
	2	$+ b_3b_4 + b_1b_6$
	3	$+ b_1b_2b_4 + b_1^2b_5$
	4	$+ b_1^2b_2b_3 + b_1^3b_4$
	5	$+ b_1^3b_2^2$
u_8	1	b_8
	3	$+ b_2b_3^2 + b_1^2b_6$
u_8	1	b_8
	3	$+ b_2b_3^2 + b_1^2b_6$
	5	$+ b_1^2b_2^3 + b_1^4b_4$
	7	$+ b_1^6b_2$

Lecture 2. Immersions up to cobordism

There is a useful rule of thumb in differential topology for studying complicated geometric structures. Usually there is a wealth of structure so to understand it one tries to simplify the situation by deciding which parts of structure can be pruned off. The aim is to reduce the tangled, continuous picture of geometrical reality to the tangled but discrete domain of homotopy. Many geometers consider the problem solved at this point, by definition: if homotopy information is required, one just picks up the phone and calls Mahowald. Suppose, however, that you are stranded on a desert island -- you have to untangle the homotopy yourself. The technique of course is to reduce it to a problem of algebra. If the algebraic problem is still too complicated to handle it may be necessary to look if certain aspects of the homotopy situation can be simplified so that the algebra becomes manageable.

The work of Thom [19] on unoriented cobordism may be taken as an example (at the risk of making the scheme into a Procrustean bed, of course.) Suppose we want to get an overall view of the class of all closed smooth manifolds. The first problem is that there are too many of them even under diffeomorphism: there are lots and lots of those which are even topologically spheres, so the relation of diffeomorphism is too strong -- there are too many equivalence classes. We need a weaker, but hopefully still interesting, equivalence relation to cut down the number of equivalence classes. Cobordism, born in Poincaré's first attempt to define homology, is a reasonable candidate, especially if we are interested in homology of manifolds. Two closed manifolds M and M' of dimension m are said to be cobordant if there exists an $(m+1)$-dimensional manifold with boundary W such that ∂W is diffeomorphic to the disjoint sum of M with M'. It is immediately checked that cobordism is an equivalence relation (use the collaring theorem for transitivity). We denote the set of equivalence classes of m-dimensional closed manifolds under cobordism by N_m. Disjoint sum induces addition in N_m which makes it into a vector space over Z_2, and Cartesian product induces a product $N_m \otimes N_n \to N_{m+n}$ which makes $N_* = \{N_m\}_{m \in Z}$

into a graded algebra. The problem is now to determine the structure of this algebra, and the key to Thom's solution is a reduction of the geometric problem to a problem in homotopy, namely $\pi_*(MO)$. We exhibit the Thom-Pontrjagin map $\tau: \mathcal{N}_m \to \pi_m(MO)$. Let M be an m-dimensional manifold. Choose an embedding $e: M^m \to R^{m+k}$. Let T be a tubular neighborhood of the embedding e : T is diffeomorphic to $EB(\nu)$, the total space of the unit ball bundle associated with the normal bundle ν of the embedding e; let $d: T \to EB(\nu)$ be the diffeomorphism and notice that under d, ∂T corresponds to $ES(\nu)$, the total space of the unit sphere bundle associated with ν. Consider $S^{m+k} = R^{m+k} \cup \{\infty\}$ as the one-point compactification of R^{m+k}, let $t: S^{m+k} \to T/\partial T$ be the map induced by the identity on T which maps $S^{m+k} - T$ into the point of $T/\partial T$ represented by ∂T, let $\overline{d} : T/\partial T \to EB(\nu)/ES(\nu) = M(\nu)$ be the map induced by d, and $\hat{\nu}: M(\nu) \to M(\gamma^k) = MO(k)$ be the map induced by a classifying map of ν. If $x \in \mathcal{N}_m$ is the cobordism class defined by M, let $\tau(x) \in \pi_m(MO)$ be the class defined by the composition

$$S^{m+k} \xrightarrow{t} T/\partial T \xrightarrow{\overline{d}} M(\nu) \xrightarrow{\hat{\nu}} MO(k).$$

Of course, one has to verify that $\tau(x)$ is independent of all the choices made in its construction, but this is not difficult. It is almost immediate that τ is an algebra map, and one shows that τ is an isomorphism by explicitly constructing an inverse $\pi_m(MO) \to \mathcal{N}_m$ (here the transversality theorems of Thom are used).

This reduces the geometry to homotopy, and we consider the problem solved, since the algebra structure of $\pi_*(MO)$ has been determined: it is a polynomial algebra $Z_2[u_2,u_4,u_5,\ldots,u_n,\ldots]$, $n \neq 2^r - 1$. We would like to identify the monomorphism

$$\mathcal{N}_m \xrightarrow{\tau} \pi_m(MO) \xrightarrow{h} H_m(MO;Z_2) \xrightarrow{\phi_*} H_m(BO;Z_2)$$

and this is easily done: it is the <u>normal characteristic number map</u> $\nu_\#$ which is described as follows: let $u \in H^m(BO;Z_2)$, so we may consider u as a linear functional on $H_m(BO;Z_2)$, then

$$u\phi_*h\tau(\text{class of } M) = \langle \nu^*(u),[M]\rangle ,$$

where $\nu: M \to BO$ is the stable normal bundle of M, $[M]$ is the fundamental class of M, and $\langle,\rangle$ is the Kronecker pairing between cohomology and homology. For example, if $M = RP^2$ then $W(\nu) = 1 + x$, so