Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Degrees of Unsolvability: Structure and Theory

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Dedicated

to

URSULA BRODBECK

"Like one that stands upon a promontory,
And spies a far-off shore where he would tread,
Wishing his foot were equal with his eye."

Shakespeare, *Henry VI, Part 3*

ABSTRACT

This book presents the theory of degrees of unsolvability in textbook form. It is accessible to any student with a slight background in logic and recursive function theory. Degrees are defined and their basic properties established, accompanied by a number of exercises.

The structure of the degrees is studied and a new proof is given that every countable distributive lattice is isomorphic to an initial segment of degrees. The relationship between these initial segments and the jump operator is studied. The significance of this work for the first-order theory of degrees is analyzed: it is shown that degree theory is equivalent to second-order arithmetic. Sufficient conditions are established for the degrees above a given degree to be not isomorphic to and have different first-order theory than the degrees, with or without jump.

The degrees below the halting problem are introduced and surveyed. Priority arguments are presented. The theory of these degrees is shown to be undecidable.

The history of the subject is traced in the notes and annotated bibliography.

PREFACE AND ACKNOWLEDGEMENTS

January 1973 I went to Barry Cooper to ask him if anyone had shown that the degrees $\leq \underline{a}$ is undecidable for every non-zero r.e. $\underline{a}$. What a good idea for a Ph.D. thesis! Well, it turned out to be harder than I imagined.

By November of that year I'd written up the background work on the degrees $\leq \underline{0}'$ which was necessary for it (that appeared as [24]). And I had a "plausible" proof that the three element chain is isomorphic to an initial segment of the degrees $\leq \underline{a}$ for such $\underline{a}$ - the essential first step towards undecidability. I'm grateful to Barry Cooper for suffering through this early (alas, wrong) proof.

I returned to this project in early 1975 determined to give a proper proof of that embedding. Later in the year I worked with Dave Posner at Berkeley where we at least convinced ourselves that we weren't missing an easy proof. His interest kept me going and I'm thankful for it.

In 1976, finally grasping the proper definitions of uniformity below $\underline{0}'$ I wrote what's now Chapter 2 of Epstein [55] - the proper proof. Then I began to write the rest of this book. I was very fortunate that during 1976 and 1977 Victoria University of Wellington sponsored this work with a Postdoctoral Research Fellowship, and I'm particularly grateful to Wilf Malcolm for arranging that. He and all the Logic Group there gave me the emotional support so necessary in a project of this size.

I finished the first draft in November of 1977. Funds were provided for the typing of it (by Shelly Carlyle) by the Victoria University Research Council. I discussed that version with R. W. Robinson in December 1977 at Newcastle, Australia. His comments, especially on Chapter X, improved the exposition. Throughout his encouragement has been valuable.

Following our talks I was able to show that there is a model of arithmetic in the degrees $\leq \underline{0}'$. The main results of Chapter XII then date from January 1978 in Wellington.

In February 1979 at Iowa State University I began the final rewrite, getting out all the old mistakes and putting in the new ones. The comments and criticisms of my students, Steve Wegmann and Richard Kramer, helped shape this into a textbook rather than a collection of results. Richard Kramer provided a number of simplifications. But the mistakes are all mine.

Working in isolation from other recursion theorists could have killed much of this book. But I was lucky to establish correspondence with M. Lerman and Carl Jockusch, Jr. Their many comments, explanations, and encouragement helped enormously. I'm also grateful to Richard Shore who in this last year has patiently discussed his and A. Nerode's new work with me.

Funds for the diagrams and proofreading were provided by Iowa State University Research Foundation. I much appreciate Beverly Hickey's work in typing this final version.

As you can see, I never did get that undecidability result. Maybe that would make a good Ph.D. project for someone ... .

Victoria University
Wellington, New Zealand 1975-1977

Iowa State University 1979

INTRODUCTION

"It is my conviction that to withstand and counteract the deadening impact of mass society, a man's work must be permeated by his personality."

Bruno Bettleheim.

This is a textbook, either for a class or for your own study. If you've had an introductory course on logic where the recursive functions have been defined you'll find this virtually self-contained. If not, well, it's one way to pick that up.

Beginning each chapter we tell you what previous chapters you need to have read. Then we present a motivation section in the hope that you can get the ideas clear before you plunge into the formal work. The motivation may even enable you to avoid the proofs. The exercises and examples are there for your benefit; only in Chapter I are they really essential.

In Part 1 we first introduce you to the degrees of unsolvability and establish some basic facts in Chapter I. This is the longest and hardest chapter for someone new to the subject: persevere. In Chapter II we survey the first-order theory of degrees and establish the undecidability of that theory. All the facts about lattices which we use here and elsewhere can be found in Appendix 1.

Part 2 is devoted to proving that every countable distributive lattice with least element is isomorphic to an initial segment of the degrees of unsolvability (notated $L \overset{*}{\Rightarrow} \mathbb{D}$). This is the theorem needed for the proofs in Chapter II. We proceed leisurely with lots of examples. If you want to speed through just the formal sections of this part you only need to read the definition of uniform tree in Chapter III, Chapter IV §C for the general definitions, and the constructions of Chapters V and VII. In Chapter VII we also show which of the constructions we do in this part can, in some sense, be made effective. Chapter VIII first demonstrates that there are uncountably many minimal degrees, and then connects our constructions with the jump operator.

Part 3 looks more closely at the first-order theory of degrees. In Chapter IX we refute the homogeneity conjectures and look at automorphisms of the degrees. In Chapter X we prove that the first-order theory of degrees is equivalent to second-order number theory.

Part 4 is devoted to the degrees below $\underline{0}'$, the degree of the halting problem. Chapter XI is a survey and overview, including an introduction to priority arguments. Appendix 2 supplements our discussion of classifications of degrees below $\underline{0}'$. And Appendix 3 makes some observations on how the nature of tree constructions affects what degrees we can build. In Chapter XII we show how to translate fragments of arithmetic into the theory of degrees $\leq \underline{0}'$. This allows us to prove that the theory of degrees $\leq \underline{0}'$ is undecidable. Chapter XII assumes a considerable background in logic as well as several technical theorems on initial segments below $\underline{0}'$ which we prove elsewhere.

We trace the history of the subject and suggest further readings in the Notes and the Annotated Bibliography. In the Notes we also present a number of open questions and conjectures.

* * *

Certain areas of degree theory are not included here: non-distributive initial segments (see the Bibliography for references), r.e. degrees (discussed briefly in Chapter XI where references are provided), and the relation of the arithmetical degrees to first-order logic (see Chapter I §E for references). We can't be all things to all students.

* * *

Within each chapter we number the theorems consecutively (if there's more than one), and number the corollaries to each theorem. Lemmas are numbered separately as are the exercises. The end of a proof is marked ■, and the end of a subproof is marked □.

* * *

ENJOY!

Background Requirements for the Chapters

A flow chart would be more difficult to read than most of the proofs in this book. We simply list the prerequisites.

Chapter I: Some familiarity with recursive functions or a lot of tenacity.

Chapter II: Some lattice theory (Appendix 1 §A, B), Chapter I §A-C.

Chapter III: Chapter I §A-H.

Chapter IV: Chapter III and prerequisites for that. Appendix 1 §A, B.

Chapter V: Chapter I §A-H. Chapter III: definition of fully uniform tree. Chapter IV §C. Appendix 1. If you skip Chapter IV the notation will seem pretty hairy.

Chapter VI: Chapter III, Chapter IV §B, C.

Chapter VII: Chapter V, Chapter VI motivation. For §C: if you don't know about priority arguments Chapter XI §A will help.

Chapter VIII: Chapter VII.

Chapter IX: Chapter II, statement of Theorem 1 Chapter VIII, definition of degree of a presentation of a lattice in Chapter VII.

Chapter X: Chapter I, Chapter II, statement of Theorem on Jumps and Chains, Chapter VIII. Some knowledge of the undecidability of arithmetic.

Chapter XI: Chapter I except §J.

Chapter XII: Chapters I and II. Chapter X §A and Lemma 1 of §B. Definition of high degree from Chapter XI. Some knowledge of the undecidability of arithmetic and the representability of recursive functions in arithmetic. Chapters IX and X are extremely useful motivation but are not essential.

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CHAPTER I: AN INTRODUCTION TO DEGREES OF UNSOLVABILITY

"Take but degree away, ...
And hark! what discord follows..."
Shakespeare, *Troilus and Cressida*.

In this chapter we define the degrees of unsolvability, prove some basic facts about them, and establish notation. You may already be familiar and happy with a definition of degrees of unsolvability. We present our own partly for completeness and partly because it is useful to have a concrete version available for reference. However, the technical details of it are generally absent from our constructions - any other definition would work equally well. You may therefore wish to simply skip to the facts which we list below in the exercises of §C.

In §D we develop the basic properties of 0' and of r.e. sets. Then in §E we establish a characterization that allows us to calculate bounds for how effective our constructions are.

Sections F and G set up the machinery which is fundamental to all our constructions.

In §H we use this machinery to show that the degrees are not dense and in §J that they are not linearly ordered.

*   *   *

A. The Recursive Functions

Degrees of unsolvability were originally designed as measures of complexity of unsolvable problems. As such they are concerned solely with functions on and subsets of the natural numbers, $\mathbb{N}$.

We usually confuse a set A with its characteristic function

$$A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Small Roman letters will usually denote natural numbers.

The recursive functions, which we define next, are generally identified as the class of computable functions, or solvable problems. This is Church's Thesis. This thesis has considerable intuitive and philosophical importance, but the mathematical structures we investigate and theorems we prove do not depend on it.

An algorithmic class of functions is one which

	contains:	the zero function
		the projections
		successor
and is	closed under:	composition
		primitive recursion
		μ-operator.

The zero function assigns 0 to every input.

The successor is $s(x) = x+1$

The projections are $p_n^i(\vec{x}) = x_i$ where $\vec{x} = (x_0,\ldots,x_n)$.

Composition of g and h is: $f(\vec{x},\vec{y}) = g(\vec{x},h(\vec{y}))$

Primitive recursion on g and h is: $f(0,\vec{x}) = g(\vec{x})$ and $f(n+1,\vec{x}) = h(f(n,\vec{x}),n,\vec{x})$.

The μ-operator applied to g is $f(\vec{x}) = \mu y(g(\vec{x},y) = 0)$ defined by: