Mathematical World • Volume 6

# Knots and Surfaces

## A Guide to Discovering Mathematics

David W. Farmer
and
Theodore B. Stanford

American Mathematical Society

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# Preface

This book is a guide to discovering mathematics.

Every mathematics textbook is filled with results and techniques which once were unknown. The results were discovered by mathematicians who experimented, conjectured, discussed their work with others, and then experimented some more. Many promising ideas turned out to be dead–ends, and lots of hard work resulted in little output. Often the first progress was the understanding of some special cases. Continued work led to greater understanding, and sometimes a complex picture began to be seen as simple and familiar. By the time the work reaches a textbook, it bears no resemblance to its early form, and the details of its birth and adolescence have been lost. The precise and methodical exposition of a typical textbook is often the first contact one has with the topic, and this leads many people to mistakenly think that mathematics is a dry, rigid, and unchanging subject.

We believe that the most exciting part of mathematics is the process of invention and discovery. The aim of this book is to introduce that process to you, the reader. By means of a wide variety of tasks, this book will lead you to discover some real mathematics. There are no formulas to memorize. There are no procedures to follow. By looking at examples, searching for patterns in those examples, and then searching for the reasons behind those patterns, you will develop your own mathematical ideas. The book is only a guide; its job is to start you in the right direction, and to bring you back if you stray too far. The discovery is left to you.

This book is suitable for a one semester course at the beginning undergraduate level. There are no prerequisites. Any college student interested in discovering the beauty of mathematics can enjoy a course taught from this book. An interested high school student will find this book to be a pleasant introduction to some modern areas of mathematics.

While preparing this book we were fortunate to have access to excellent notes taken by Hui–Chun Lee. We thank Klaus Peters and Gretchen Wright for helpful comments on an early version of this book.

*David W. Farmer*
*Theodore B. Stanford*
September, 1995

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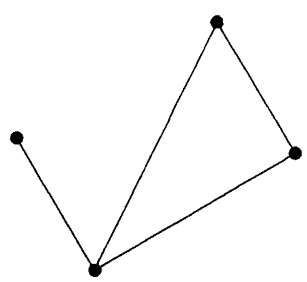
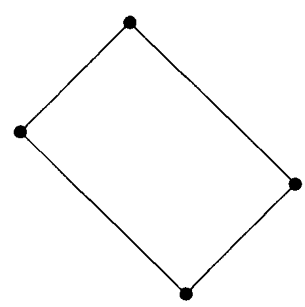
# 1

# Networks

**1.1 Countries of the insect world.** Imagine a world populated by semi–intelligent insects. The world of the insects is divided into small countries, each country consisting of a few cities connected by dark narrow tunnels. In the course of their work and leisure the insects slowly walk these tunnels, and by the time they reach adulthood all insects know how their country's cities are connected. If an insect needs to travel from one city to another, and those cities are directly connected, then the connecting tunnel is taken. Maybe the route could be shortened by taking two short tunnels through another city, but the insects are only semi–intelligent, so this possibility never occurs to them. And the insects are poor at measuring distances, so they probably couldn't identify a shorter route even it they looked for it. Life on the insect world is calm and uneventful, the citizens blissfully bumping along in the dark, performing their chores with calm inefficiency.

Let's take a closer look at the world of the insects. Here are two insect countries:





Our view from the 'outside' provides us with a complete picture of both countries. The insects are confined to the cities and tunnels, so they must expend more effort to get an accurate view of the layout. Suppose that communication between insect countries takes place by radio. Citizens from the above countries were talking, and they began to wonder if their two countries are the same. How can they determine that their countries have different layouts? First they observe that both countries have four cities and four main tunnels. So far, their countries appear similar. Then one says, "We have a city with just one tunnel leading to it." The other one says, "AHA! All our cities have two tunnels connected to them, so our countries are not set up the same way."

There are many other ways the insects could determine that their countries have different layouts. For example, each of these descriptions applies to exactly one of the countries above:

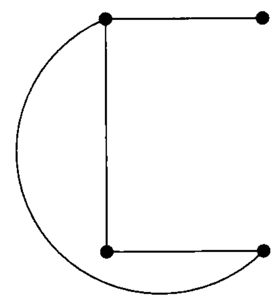
"My country has a city which connects directly to every other city."

"In my country, you can travel a route of four different tunnels and end up back where you started."

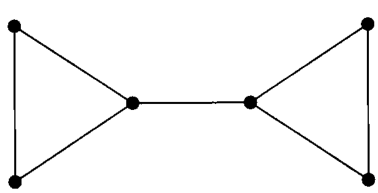
"In my country, you can travel a route of three different tunnels and end up back where you started."

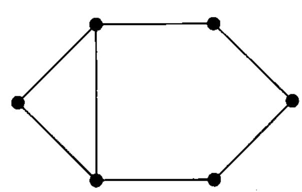
Since the insects are bad at measuring distances, they are not always able to distinguish between layouts which we would see as different.

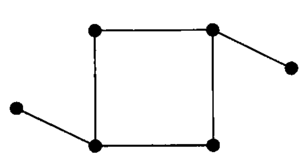
**Task 1.1.1:** Explain why the insects cannot distinguish between this country's layout and the first one shown previously.

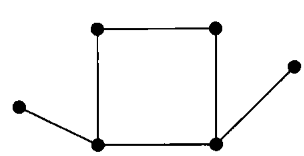


**Task 1.1.2:** For each pair of countries, determine whether the insects would view them as the same or different. For those that are different, describe how the insects can tell them apart. Note: for each pair, the number of cities is the same and the number of tunnels is the same. If this were not the case, then the insects could immediately tell that the two countries had a different layout.









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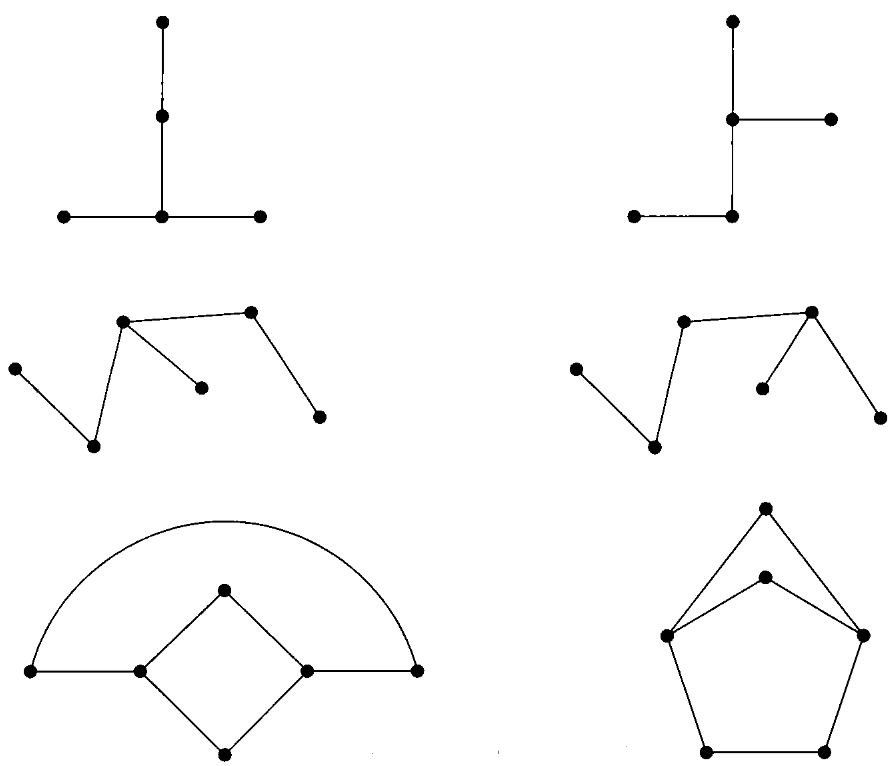


**Task 1.1.3:** Devise a precise description of what it means for two countries to be 'the same' as far as the insects are concerned.

**Task 1.1.4:** An insect says, "My country has seven cities and nine tunnels. One city has just one tunnel connected to it, one city has five tunnels connected to it, two cities have three tunnels connected to them, and the other three cities have two connecting tunnels." Draw two different countries which fit that description, and explain how the insects can tell them apart. How many different countries fit that description?

**Advice.** As you go through this book, you may find it helpful to keep a record of your thoughts and ideas. Set aside a notebook for this purpose. Put all of your work there, not just the final answers. It is important to keep a record of the entire process you went through as you worked on a problem, including work which didn't seem to lead to an answer. Your failed method on one problem could turn out to be the correct method for another problem. Having all your work in one place will help you see what you have done and will make it easy to find old work when you need it.

It is important that you spend sufficient time thinking about the Tasks as you encounter them. Some Tasks are easy and some are very difficult, so you should not expect to find a complete answer to every one. If a Task seems mysterious, it can help to discuss it with someone else. Occasionally you may skip a Task and come back to it later, but skipping a Task in the hope of finding the answers in the text will lead you nowhere. The only way for you to find an answer is to discover it yourself. Sometimes this will mean spending a long time on one Task. That is the nature of mathematical discovery. You will find that discovering your own mathematics is not at all like trying to learn mathematics which has already been discovered by someone else.

## 1.2 Notation, and a catalog

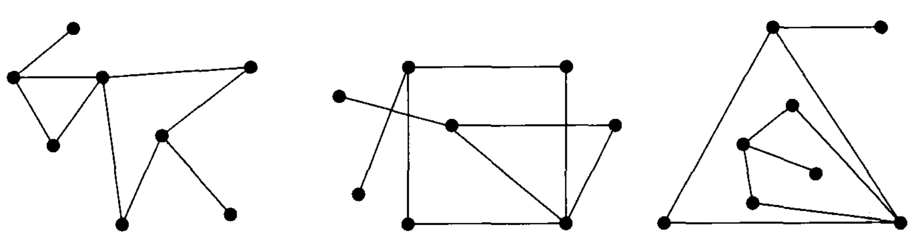
The ideas of the previous section fall under the mathematical topic of *graph theory*. The fanciful idea of insects crawling through dark tunnels will continue to be useful, but we will switch to using the mathematical terminology. Here is how to translate:

| Insect name: | Math name: |
|---|---|
| country | graph |
| city | point or vertex |
| tunnel | line or edge |

An example sentence is, "A graph is made up of points and lines." Note that 'vertices' is the plural of 'vertex,' so we can also say, "A graph consists of vertices connected by edges."

The actual picture we draw of a graph is called a **graph diagram**. Just as the insects could not distinguish between certain countries, the same graph can be represented by many different graph diagrams. The only important feature of the graph is how the various vertices are connected. Each graph diagram will have additional features, such as the lengths of the edges and the relative position of the vertices, but these aspects of the diagram have nothing to do with the graph itself.

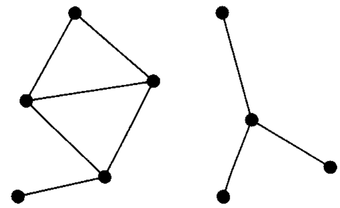
Here are three diagrams of the same graph:



A diagram may appear to show two edges crossing, but if there is not a vertex at the junction then the edges do not actually meet. Think of it as two insect tunnels which pass each other but do not intersect. The topic of drawing graphs without crossing edges will be explored in a later section.

A graph is called **connected** if we can get from any vertex to any other vertex by traveling along edges of the graph. The opposite of connected is **disconnected**.

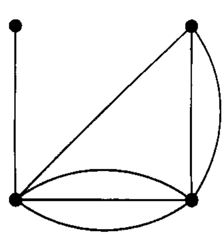
This can be thought of as a disconnected graph with 9 vertices, or as two separate connected graphs.



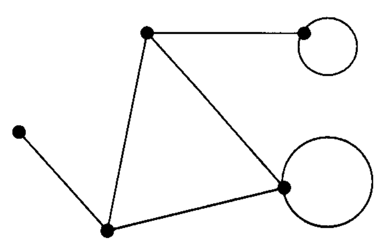
Any graph is just a collection of connected graphs; these are called the **components** of the graph.

The graphs we have been studying are presented as drawings on paper. It is easy to invent graphs which are described in other ways. For example, we can make a graph whose vertices are all of the tennis players in the world, and where an edge connects two players if they have played tennis together. We have defined a graph, although it would not be feasible to actually draw it. Another graph can be made by letting the vertices be the countries of the world, and having an edge connect two countries if those countries border each other. With the help of a map it would be possible to draw this graph. It is amusing to invent fanciful graphs and then try to determine what properties they have. For example, is the tennis player graph connected? If it is, that would mean each tennis player has played someone who has played someone who has . . . played Jimmy Connors. The play *Six Degrees of Separation* mentions, informally, the graph whose vertices are all of the people in the world, with edges connecting people who know each other. The title of the play comes from speculation that you can get from any one vertex to any other vertex by crossing at most 6 edges.

In order to make an organized study of graphs, we must impart a few more rules. Usually we do not allow our graphs to have more than one edge connecting two vertices, and we do not allow an edge to connect a vertex to itself.



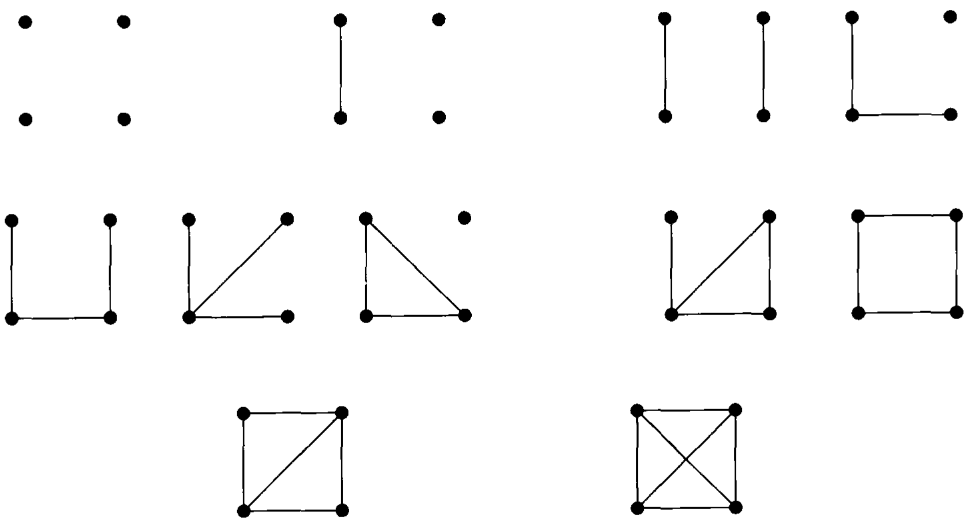
A graph with multiple edges



A graph with loops

Unless we state otherwise, a 'graph' is a 'graph without loops or multiple edges.'

We classify graphs according to how many vertices they have. Here is a catalog of all graphs with 4 vertices:



You should convince yourself that the list is complete.

**Task 1.2.1:** Make a catalog of all graphs with 5 vertices. Hint: there are between 30 and 40 of them. First find all the ones with no edges, then 1 edge, then 2 edges, and so on.

In the above Task it is difficult to be absolutely sure that you found all the graphs. Fortunately, there is something we can do to increase our confidence. For the graphs with 4 vertices we found a total of $1+1+2+3+2+1+1=11$ graphs, where we counted the graphs according to how many edges they have. Notice that the numbers form a symmetric pattern.

**Task 1.2.2:** Do your numbers from Task 1.2.1 form a symmetric pattern? If not, go back and fix your list. After your list is correct, explain why the symmetric pattern appears.

**Task 1.2.3:** Devise a code for describing a graph over the telephone. Note: your code only needs to describe a *graph*, not a *graph diagram*.

## 1.3 Trees

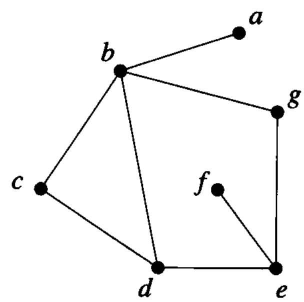
If we think of a graph as a roadmap then it is natural to look at the various routes we can take through the graph. A **path** in a graph is a sequence of edges, where successive edges share a vertex. To make things easier to read, we will describe a path by showing which vertices the path visits; this should not cause any confusion.

Example paths:

*(d, b, c, d)*

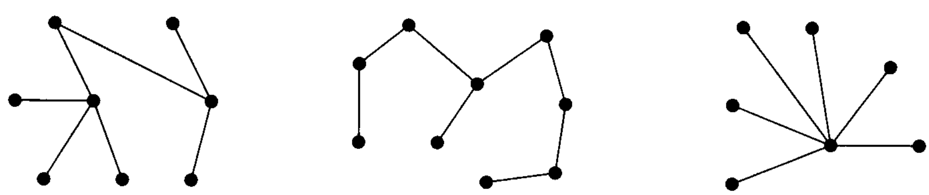
*(f, e, d, b, c)*

*(e, g, b, a, b, g)*



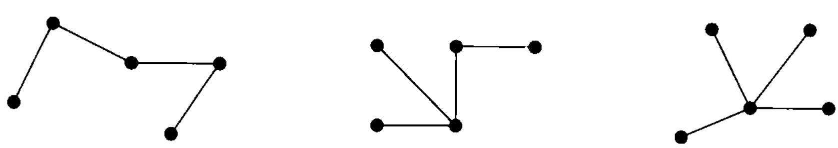

A path is **closed** if it ends at the same vertex as it began. The first path above is closed. A path is **simple** if it doesn't use the same edge more than once. The first two paths above are simple. A simple closed path is sometimes called a **circuit**. The first path above is a circuit, and so is *(b, c, d, e, g, b)*. A graph is **connected** if there is a path from any one vertex to any other vertex.

A graph is a **tree** if it is connected and it doesn't have any circuits. Here are three trees:

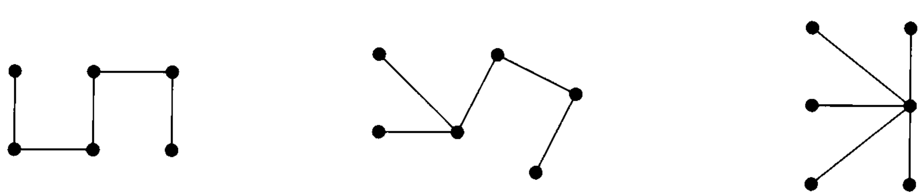


**Task 1.3.1:** What is the relationship between the number of vertices and the number of edges in a tree? Why does this relationship hold?

Trees are particularly simple kinds of graphs, so our plan is to study trees, and then to use trees to study other graphs. Here are all trees with 5 vertices:



Those trees should be in your catalog of graphs from Task 1.2.1. Here are all trees with 6 vertices:



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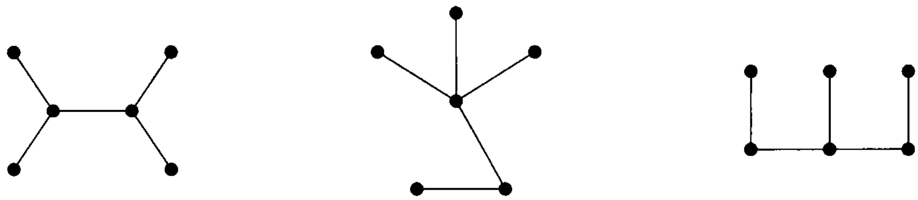


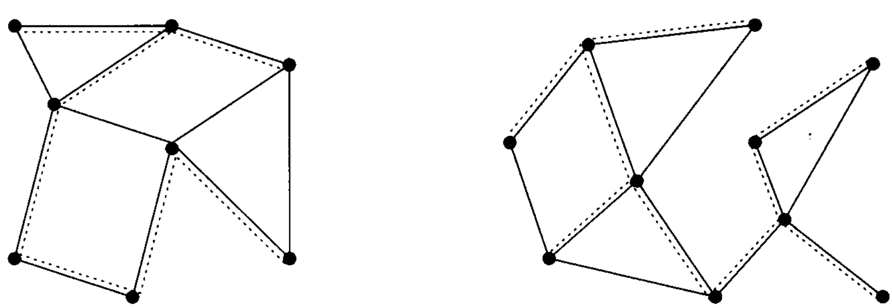
**Task 1.3.2:** Make a list of all trees with 7 vertices. If you feel ambitious, make a list of all trees with 8 vertices. Hint: there are between 20 and 30 of them.

**Task 1.3.3:** Suppose you had plenty of time and you wanted to make a list of all trees with a given large number of vertices; say, all trees with 12 vertices. Describe the method you would use. Is your method guaranteed to give a complete list with no repeats? Is your method practical?

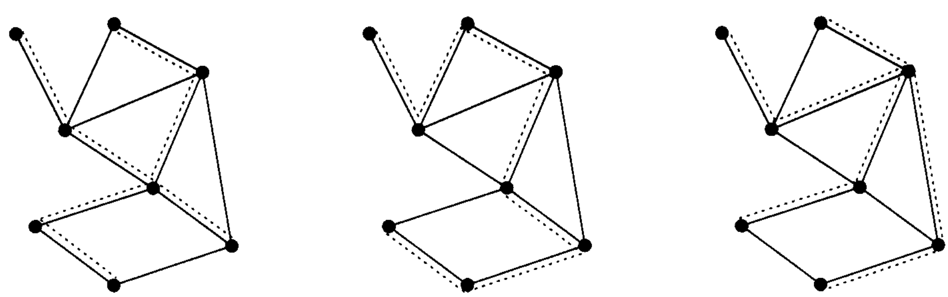
**Task 1.3.4:** In Task 1.2.3 you devised a code for describing a graph over the telephone. Suppose that you only needed the code for describing trees. Is it possible to devise a simpler code which still works in this case?

## 1.4 Trees in graphs

A tree inside a graph which hits every vertex of the graph is called a **spanning tree**. A spanning tree must use the edges in the graph, and it must hit every vertex. A useful way to show a spanning tree is to highlight the edges in the tree:



A graph can have many different spanning trees. Here are three different spanning trees for the same graph:



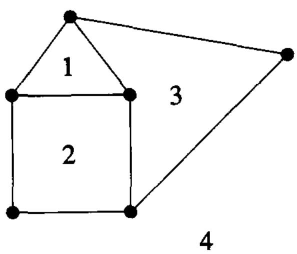
It is important to keep in mind that a graph can have several different spanning trees, so without a picture the term 'spanning tree' can be ambiguous.

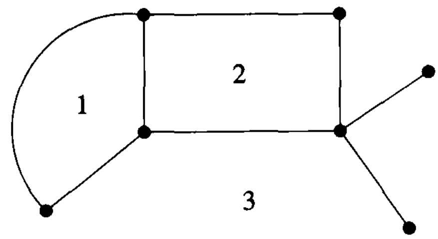
**Task 1.4.1:** Devise a way of counting the number of spanning trees of a graph.

In the next section we use spanning trees to study graphs.

## 1.5 Euler's formula

A graph diagram divides the plane into separate regions:



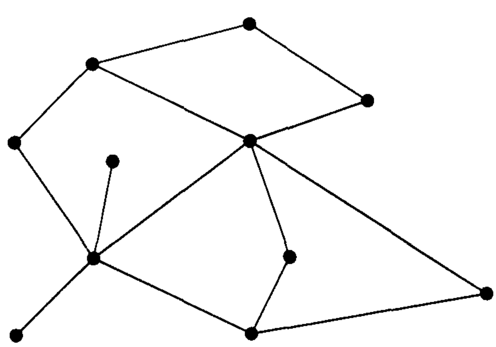




The first diagram divides the plane into 4 regions, and the second divides the plane into 3 regions. Note that the big outside area counts as a region.

**Task 1.5.1:** Draw several graphs and record the following information:

- the number of vertices in the graph (call it $v$)
- the number of edges in the graph (call it $e$)
- the number of separate regions of the graph (call it $f$)
- the number of vertices in a spanning tree (call it $A$)
- the number of edges in a spanning tree (call it $B$)
- the number of edges not in a spanning tree (call it $C$)

Here is an example. Find a spanning tree and check that the numbers are correct:



$v = 11$
$e = 14$
$f = 5$
$A = 11$
$B = 10$
$C = 4$

Note: for this Task you should only use connected graphs which you have drawn without crossing edges. A diagram drawn without crossing edges is called a

**planar diagram**. The importance of using planar diagrams in this Task is discussed in the next section.

**Task 1.5.2:** Look at the information you recorded and try to find patterns and relationships among the six quantities.

**Task 1.5.3:** Explain why the observations you made are correct. Note: one of your observations may have been $A = B + 1$. You already discussed this in Task 1.3.1.

**Task 1.5.4:** Explain why your observations can be used to show $v - e + f = 2$.

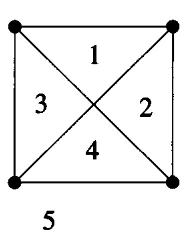
The equation $v - e + f = 2$ is known as **Euler's Formula**. It was first discovered by the Swiss mathematician Leonhard Euler in 1736. Note: Euler is pronounced 'Oiler.' Say it out loud a few times. This will keep you from looking foolish later.

**Task 1.5.5:** Suppose a graph has 7 vertices and 9 edges. Use Euler's formula to predict how many separate regions it would have if you drew the graph. Draw such a graph and check if your prediction is correct.

## 1.6 Planar graphs

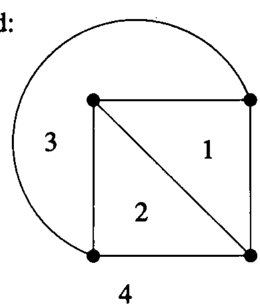
Euler's formula $v - e + f = 2$ is true for any connected graph which is drawn without crossing edges. For instance:

Bad:




$v = 4$
$e = 6$
$f = 5$

Good:



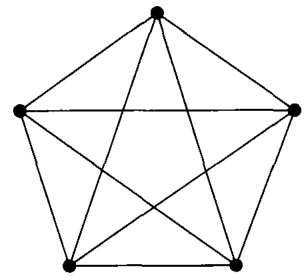

$v = 4$
$e = 6$
$f = 4$
$v - e + f = 2$

We say that a graph is **planar** if it has a diagram without crossing edges. Above are two diagrams of the same graph, but Euler's formula only works in the second case. This is usually expressed as "Euler's formula holds for connected planar graph diagrams."

**Task 1.6.1:** What can you say about $v - e + f$ if the graph is not connected?

The graph shown above is called 'the complete graph on 4 vertices,' and it is denoted $K_4$, pronounced "kay four." This means that it has 4 vertices and each vertex is connected to every other vertex. Similarly, $K_5$ is the graph with 5 vertices and each vertex is connected to every other vertex.

Here is a representation of $K_5$:



**Task 1.6.2:** How many edges does $K_5$ have? $K_6$? $K_7$?

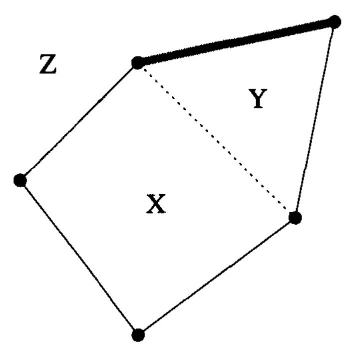
**Task 1.6.3:** Explain why $K_n$ has $1 + 2 + 3 + \cdots + (n - 1)$ edges. We will find another expression for this in Task 1.7.9.

**Task 1.6.4:** Try to draw $K_5$ without any crossing edges. Make at least four attempts.

After four attempts at Task 1.6.4, you should stop. Further attempts would be pointless because it is *impossible* to draw $K_5$ without any crossing edges. In other words, $K_5$ is not planar. We will use Euler's formula to show why the Task is impossible.

The reason we write $f$ for the number of separate regions of a graph is that those regions are usually called **faces**. A key fact we need is that each edge of a planar graph diagram is a border of two faces.

The dotted edge is a border of face X and face Y, and the fuzzy edge is a border of faces Y and Z.




The number of edges of a face is called the **order** of the face. In the diagram above, face X has order 4, face Y has order 3, and face Z has order 5.

An important relationship between the number of edges and the number of faces in a planar graph is:

$$3f \leq 2e.$$

We will use the concept of order, along with the observation that each edge borders two faces, to establish this inequality. Then we will use the inequality to show that $K_5$ does not have a planar diagram. But first, do this Task:

**Task 1.6.5:** Draw a few planar graph diagrams and check that $3f \leq 2e$ holds in each case. What graphs have $3f = 2e$?