Complex Analysis

Articles dedicated to Albert Pfluger on the occasion of his 80th birthday

Edited by
Joseph Hersch
and
Alfred Huber

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Editors:
Prof. Joseph Hersch
Prof. Alfred Huber
ETH Zürich
Mathematatik
ETH-Zentrum
CH-8092 Zürich

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Preface

The present volume contains articles pertaining to a wide variety of subjects such as conformal and quasiconformal mappings and related extremal problems, Riemann surfaces, meromorphic functions, subharmonic functions, approximation and interpolation, and other questions of complex analysis. These contributions by mathematicians from all over the world express consideration and friendship for Albert Pfluger. They reflect the wide range of his interests.

Albert Pfluger was born on 13 October 1907 in Oensingen (Kanton Solothurn) as the oldest son of a Swiss farmer. After a classical education he studied Mathematics at the ETH-Zurich. Among his teachers were Hopf, Plancherel, Pólya and Saxer. Pólya was his Ph.D. adviser. After some teaching at high schools (Gymnasien), he became professor at the University of Fribourg, and a few years later (1943) he was appointed as successor of Pólya at the ETH. He retired in 1978, but has always remained very active in research.

Pfluger's lectures were highly appreciated by the students. His vivid and clear teaching stimulated and challenged them to independent thinking. Many of his Ph.D. students are now themselves teaching in universities.

His main research relates to the following fields: entire functions, Riemann surfaces, quasiconformal mappings, schlicht functions. (See list of publications.) He collaborated with several mathematical colleagues, in particular with Rolf Nevanlinna, who taught parallel to him at the University of Zurich.

In 1973 Pfluger was nominated foreign member of the Finnish Academy of Sciences.

To Albert Pfluger, his wife Maria, their children and grandchildren we present our cordial wishes, and to the authors of this volume our sincere thanks.

Joseph Hersch, Alfred Huber

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Lars V. Ahlfors

Cross-ratios and Schwarzian Derivatives in R^n

This paper was written several years ago, but no part of it has been published previously. A preprint was distributed to selected experts and seems to have been favorably received. For some time I had hoped to improve on the results of the paper, but as years went by my research took a different direction, and it became implausible that I would add anything significant to the paper as it stands.

Meanwhile there has been considerable progress in this area, but my friends have insisted that the bulk of the paper still has at least some historical interest and should be made available to the mathematical public. It gives me great satisfaction that the paper will appear in its original form in this volume dedicated to Professor Albert Pfluger in appreciation of his lasting contributions to analysis

The research for the paper was supported by the National Science Foundation.

The cross-ratio is of fundamental importance in projective geometry and some aspects of complex function theory. In the latter connection the cross-ratio of four complex numbers a, b, c, d is defined as

$$(a,b,c,d) = \frac{a-c}{a-d} : \frac{b-c}{b-d},$$

another complex number. On the other hand, in geometry the cross-ratio occurs mainly as a double ratio $(AC/AD) : (BC)/(BD)$ of the lengths of four segments.

Recent developments in function theory, especially in connection with Kleinian groups. have made it even more essential than at the time of Poincaré and Klein to study the conformal structure of three-space as an extension of the conformal structure of the complex plane. Experience has shown that many methods which carry over effortlessly from two to three dimensions do not extend to arbitrary R^n. For this reason it seems to the author that the case of arbitrary dimension is not an idle generalization, but may serve to throw new light on the cases $n = 2$ and 3 as well. With some degree of justification it can be maintained that a method which does not generalize is not fully understood.

The theory of Möbius transformations does of course generalize almost automatically, but the lack of a natural generalization of the complex cross-ratio has been a considerable handicap. One of the purposes of this paper is to suggest a way to overcome this difficulty. Since the idea is quite simple it may have occured to others as well, but since I am not aware of any mention of it in the literature I have thought it worth while to give it some publicity.

The second part of the paper is devoted to a study of the Schwarzian derivative. It is commonly accepted that the Schwarzian derivative is an infinitesimal version of the cross-ratio, but this is seldom made explicit. Traditionally, the Schwarzian is considered only in connection with holomorphic functions of one variable, and in that case the relation to the cross-ratio is fairly obvious. It seems that the more general case of smooth mappings into R^n has hardly been explored at all. It turns out that the real and imaginary parts of the Schwarzian can both be generalized, albeit in somewhat different ways. At present these generalizations are more or less tentative, and there are no significant applications, but in view of the importance of the holomorphic Schwarzian it is not unreasonable to make at least a preliminary forage in this direction.

§1 Möbius transformations and cross-ratios.

1. When A.F. Möbius introduced the notion of what we call a Möbius transformation he did not connect it with the idea of a fractional linear transformation with complex coefficients, nor did he regard conformality as the main feature. His was a purely geometric theory of "Kreisverwandtschaften", a term that defies translation. In modern terminology a "Kreisverwandtschaft" is a homeomorphism of the extended complex plane $\bar{C}$ which maps circles on circles. This led him to the invariance of the cross-ratio and the angle, but he made only minimal use of the complex notation. It is interesting to note that Möbius was well aware that his definition works equally well in three dimensions. If it had not been considered esoteric at the time he would probably have used n dimensions.

2. The modern approach is much more direct. We begin with the complex case and define a Möbius transformation by the formula

$$\gamma z = \frac{az+b}{cz+d} \tag{1}$$

where $a, b, c, d \in C$ and $ad - bc = 1$. We regard (1) as a mapping of $\bar{C}$ on itself with the standard conventions for ∞.

From (1) one derives the difference formula

$$\gamma z - \gamma\zeta = \frac{z-\zeta}{(cz+d)(c\zeta+d)} \tag{2}$$

which implies the existence of the derivative

$$\gamma'(z) = (cz+d)^{-2}. \tag{3}$$

We rewrite (2) as

$$\gamma z - \gamma\zeta = \gamma'(z)^{1/2}\gamma'(\zeta)^{1/2}(z-\zeta). \tag{4}$$

We shall find (4) an extremely useful tool even, within certain limits, in the multidimensional case. For the moment we observe merely that it proves the invariance of the cross-ratio. Indeed, with the definition

$$(z, z', \zeta, \zeta') = (z-\zeta)(z-\zeta')^{-1}(z'-\zeta')(z'-\zeta)^{-1}$$

it follows that $(\gamma z, \gamma z', \gamma\zeta, \gamma\zeta') = (z, z', \zeta, \zeta')$, for the factors $\gamma'(z)^{1/2}$ etc. introduced by (4) cancel against each other.

We remark in passing that the cross-ratio is well defined, finite or infinite, as soon as no more than two of the z, z', ζ, ζ' are equal.

3. It is useful to recall that four numbers determine six cross-ratios, depending on the order. Each corresponds to four permutations: in simplified notation $(abcd) = (cdab) = (dabc) = (dcba)$.

There is a unique Möbius tranformation that carries three given distinct points b, c, d into $0, 1, \infty$ in this order. Therefore, there is a unique complex number z such that $(abcd) = (z, 1, 0, \infty) = z$, $(acbd) = (z, 0, 1, \infty) = 1 - z$, $(bacd) = z^{-1}$. The other three cross-ratios are $(1-z)^{-1}$, $1-z^{-1}$, $(1-z^{-1})^{-1}$. It is sufficient, however, to retain the basic relations

$$(abcd) + (acbd) = 1, \quad (abcd)(bacd) = 1. \tag{5}$$

They remain in force even in the event of one or two pairs of equal numbers.

In addition to $(z, 1, 0, \infty)$ there is another normal form for the cross-ratio, namely $(-1, -e^\tau, e^\tau, 1)$. The condition $z = (-1, -e^\tau, e^\tau, 1)$ translates to $z = ch^2(\tau/2)$. This means that τ is determined up to sign and additive multiples of $2\pi i$. It becomes unique if we require that $0 \le \operatorname{Im} \tau < \pi$, $\tau = 0$ if τ is real. In relation to $z = (abcd)$ the number τ, so normalized, is referred to as the *complex distance* between the ordered pairs (a, d) and (b, c). Its geometric meaning will be explained later. It seems to have been first introduced by F. Schilling in 1891.

4. We pass now to R^n, the complex plane being identified with R^2. We shall use the notations $x = (x_1, \ldots, x_n) \in R^n$, $|x|^2 = x_1^2 + \ldots + x_n^2$, and $(x, y) = x_1 y_1 + \ldots + x_n y_n$. As usual, R^n is compactified to $\bar{R}^n = R^n \cup \{\infty\}$. A *similarity* is a mapping $\bar{R}^n \to \bar{R}^n$ whose restriction to R^n is given by $x \to mx + b$, where $b \in R^n$ and m is a *conformal matrix*, i.e. a matrix λk, $\lambda > 0$, $k \in O(n)$. Also, ∞ is mapped on itself.

The *inversion*, or reflection in the unit sphere S^{n-1}, is defined by $x \to x^* = x/|x|^2$ when $x \neq 0, \infty$ and $0^* = \infty$, $\infty^* = 0$.

Definition 1. The group $M(\bar{R}^n)$ of Möbius transformations is the group generated by all similarities together with the inversion in the unit sphere.

If $\Omega \subset R^n$ is open the derivative of a mapping $f : \Omega \to R^n$ at $x \in \Omega$, if it exists, is the matrix $f'(x)$ or $Df(x)$ with elements $f'(x)_{ij} = \partial f_i / \partial x_j$. Clearly, $D(mx+b) = m$ and, by elementary calculation, $Dx^* = |x|^{-2}(\delta_{ij} - 2x_i x_j/|x|^2)$. In this paper we shall use the notation $Q(x)$ for the matrix with elements $Q(x)_{ij} = x_i x_j / |x|^2$ and I or I_n for the unit matrix. With this notation

$$Dx^* = |x|^{-2}(I - 2Q(x)). \tag{6}$$

One verifies that $Q(x)^2 = Q(x)$ and $(I - 2Q(x))^2 = I$, $I - 2Q(x) \in O(n)$. Matrices of the form $I - 2Q(a)$ will occur frequently. They have a simple geometric interpretation: $(I - 2Q(a))x$ is the mirror image of x with respect to the hyperplane through 0 perpendicular to a.

According to (6) Dx^* is a conformal matrix, and by the chain rule the derivative $\gamma'(x)$ of any $\gamma \in M(\bar{R}^n)$ is likewise a conformal matrix. In other words, the mapping by a Möbius transformation is conformal. For $n > 2$ the converse is a classical theorem due to Liouville.

As a conformal matrix $\gamma'(x)$ can be written in the form λk with $\lambda > 0$, $k \in O(n)$; unless γ is a similarity λ and k will depend on x. We shall denote λ by $|\gamma'(x)|$; because of the conformality λ is also the operator norm of the matrix $\gamma'(x)$ and the linear change of scale at x, the same in all directions.

The determinant of $k = \gamma'(x)/|\gamma'(x)|$ is constantly 1 if γ is sense-preserving, -1 if it is sense-reversing. It is possible to restrict attention to the sense-preserving subgroup, but this is not always an advantage. We prefer to stay with the original definition of $M(\bar{R}^n)$ as the group of all Möbius transformations.

As customary, we shall identify R^{n-1} with the set of $x \in R^n$ with $x_n = 0$. The points with $x_n > 0$ form the upper half-space H^n. We denote by $M(H^n)$ the subgroup which maps H^n on itself. Similarly, $M(B^n)$ will be the subgroup that preserves the unit ball. The groups $M(H^n)$, $M(B^n)$ and $M(R^{n-1})$ are isomorphic.

5. *Formula (4), restricted to absolute values, remains valid in* R^n.

Proposition 1. If $\gamma \in M(\bar{R}^n)$, then

$$|\gamma x - \gamma y| = |\gamma'(x)|^{1/2}|\gamma'(y)|^{1/2}|x - y| \tag{7}$$

for all $x, y \in R^n \setminus \gamma^{-1}\infty$.

The formula is trivial when γ is a similarity. For $\gamma x = x^*$ it reduces to $|x^* - y^*| = |x|^{-1}|y|^{-1}|x - y|$ which is easily verified. The general validity of (7) follows by the chain rule.

If $x, y, u, v \in R^n$ there is no immediate way of forming a cross-ratio since multiplication has no meaning. However, if we use only distances we can still form the *absolute cross-ratio*

$$|x, u, v, y| = |x - v|\,|x - y|^{-1}|u - y|\,|u - v|^{-1}. \tag{8}$$

It is again well defined as long as no three points coincide and we admit ∞ as a possible value.

Proposition 2. $|\gamma x, \gamma u, \gamma v, \gamma y| = |x, u, v, y|$ for every $\gamma \in M(\bar{R}^n)$.

This is a trivial consequence of (7).

6. We shall use Proposition 2 to prove:

Proposition 3. Every $\gamma \in M(\bar{R}^n)$ with $\gamma\infty = \infty$ is a similarity.

Since $\gamma - \gamma 0$ has fixed points at 0 and ∞ we may as well assume that $\gamma 0 = 0$, $\gamma\infty = \infty$ and show that $\gamma x = mx$ with a constant conformal matrix m. By Proposition 2, $|\gamma x, \gamma y, 0, \infty| = |x, y, 0, \infty|$ or $|\gamma x|/|\gamma y| = |x|/|y|$, and similarly $|\gamma x - \gamma y|/|\gamma y| = |x - y|/|y|$. From the first relation $|\gamma x| = \lambda|x|$ with constant λ. From the second $|\gamma x - \gamma y|^2 = \lambda^2|x - y|^2$ and hence $(\gamma x, \gamma y) = \lambda^2(x, y)$. On expanding the squares it follows that $|\gamma(x + y) - \gamma x - \gamma y|^2 = \lambda^2|(x + y) - x - y|^2 = 0$ and thus $\gamma(x + y) = \gamma x + \gamma y$, $\gamma'(x + y) = \gamma'(x)$, a constant.

7. Proposition 3 leads to a simple normal form for all Möbius transformations. For given γ we shall write $\gamma^{-1}0 = u$, $\gamma^{-1}\infty = v$ and assume that $v \neq \infty$. Then $\sigma x = (x-v)^* - (u-v)^*$ is a Möbius transformation with $\sigma u = 0$, $\sigma v = \infty$ so that $\sigma\gamma^{-1}$ has 0 and ∞ as fixed points. We conclude by Proposition 3 that

$$\gamma x = m[(x-v)^* - (u-v)^*] \tag{9}$$

where m is a constant conformal matrix.

Sometimes it is preferable to replace (9) by

$$\gamma x = m[(x^* - v^*)^* - (u^* - v^*)^*] \tag{10}$$

provided that u and v are different from 0. Since every mapping of the form (9) is also of the form (10) there exists a conformal m such that

$$(x^* - v^*)^* - (u^* - v^*)^* = m[(x-v)^* - (u-v)^*]. \tag{11}$$

To find m we shall first compare the absolute values. By (6) and (7)

$$\begin{aligned} |(x^* - v^*)^* - (u^* - v^*)^*| &= |x^* - u^*|/|x^* - v^*|\,|u^* - v^*| \\ &= |x-u|\,|v|^2/|x-v|\,|u-v| \end{aligned}$$

and $|(x-v)^* - (u-v)^*| = |x-u|/|x-v|\,|u-v|$ so that $m = |v|^2 k$, $k \in O(n)$. To determine k we differentiate (11). By (6) and the chain rule we obtain

$$(I - 2Q(x^* - v^*))\,(I - 2Q(x)) = k(I - 2Q(x-v)). \tag{12}$$

For $x = 2v$ the matrices $I - 2Q$ in this formula are all equal to $I - 2Q(v)$ so that (12) gives $k = I - 2Q(v)$. At the same time we have proved the identity $(I - 2Q(x^* - v^*))\,(I - 2Q(x)) = (I - 2Q(v))\,(I - 2Q(x-v))$ or, in different notation

$$I - 2Q(a^* - b^*) = (I - 2Q(a))\,(I - 2Q(a-b))\,(I - 2Q(b)). \tag{13}$$

We shall choose

$$\gamma_{uv} x = (x^* - v^*)^* - (u^* - v^*)^* \tag{14}$$

to be the standard mapping with $\gamma u = 0$, $\gamma v = \infty$. It is useful to display the alternative expression

$$\gamma_{uv} x = |v|^2 (I - 2Q(v))\,((x-v)^* - (u-v)^*) \tag{15}$$