Lecture Notes in Mathematics

Aníbal Moltó
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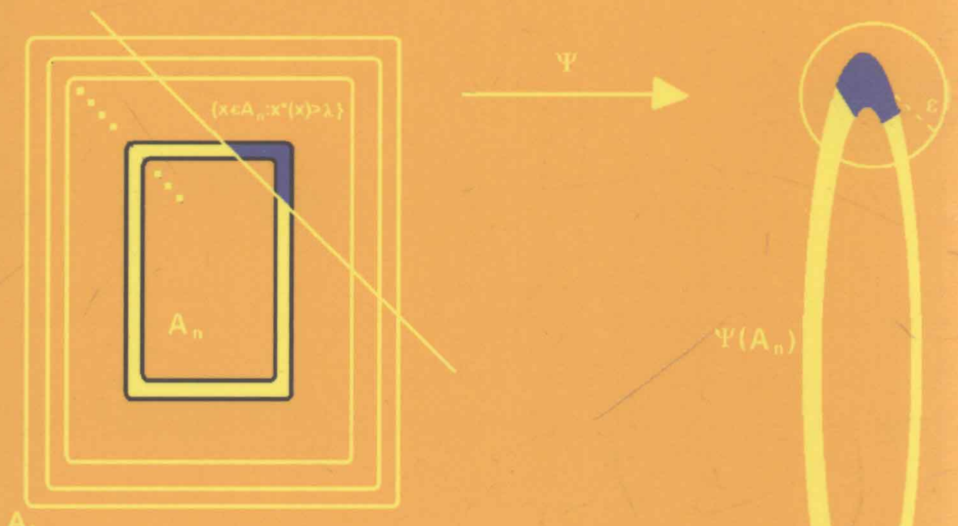
A Nonlinear Transfer Technique for Renorming

1951



Springer

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ISBN: 978-3-540-85030-4 e-ISBN: 978-3-540-85031-1
DOI: 10.1007/978-3-540-85031-1

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2008932180

Mathematics Subject Classification (2000): 46B03, 46B20, 46B26, 46T20, 54D20, 54E18, 54E25, 54E35

Cover design: SPi Publishing Services

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Preface

Banach spaces are objects with a linear structure so linear maps have been considered the natural tool for transferring *good* norms from one Banach space to another. It is well known that a Banach space X admits an equivalent strictly convex (rotund) norm if there is a bounded linear one-to-one operator $T : X \to Y$ where Y has such a norm. For example, J. Lindenstrauss proved that in any reflexive space X there is such an operator $T : X \to c_0(\Gamma)$ for some set Γ. F. Dashiell and J. Lindenstrauss gave an example of a strictly convex renormable space without such an operator into $c_0(\Gamma)$ for any Γ. For that reason we are searching for a non linear transfer technique. We consider here locally uniformly rotund (**LUR**) norms, a property adding to strict convexity the coincidence of the weak and the norm topologies on the unit sphere. For these norms a class of non linear maps was not only more powerful but even more natural for this purpose, as evinced by the solution of an old open problem due to Kadec using this class of non linear maps. The scope of this technique is not restricted to that particular case but, on the contrary, offers a unified method of obtaining this renorming, roughly speaking, in all cases in which this is known to be possible.

We have been lecturing on these new techniques throughout the courses given in the Spring School of Paseky nad Jizerou in 1998; in the Workshop in Banach spaces, Prague 2000; and in the 28th, 30th and 31st Winter Schools of Lhota nad Rohanovem on Abstract Analysis, in 2000, 2002 and 2003, places where these notes had their genesis. We would like to thank Professors J. Lukes, J. Kottas, V. Zizler, P. Holický, L. Zajíček, J. Tiser, M. Fabian and O. Kalenda for their invitations and their warm hospitality. Different parts of these notes have also been presented in seminars and conferences, such as the Choquet, Godefroy, Rogalski, Saint Raymond Analysis Seminar, University Pierre and Marie Curie, Paris VI, 1999 and 2001; Laboratoire de mathématiques pures de Bordeaux, University of Bordeaux, 1999; Functional Analysis Seminar and Analytic Topology Seminar, Mathematical Institute, Oxford University 2001 and 2002; VII Conference on Function Theory on

Infinite Dimensional Spaces, UCM, Madrid in 2001; Geometry of Banach spaces, Mathematisches Forschungsinstitut Oberwolfach, Germany, 2003; Interplay between Topology and Analysis at the International Congress Massee, Borovets, Bulgaria, 2003; Spring School on Non Separable Banach Spaces, Paseky nad Jizerou in 2004, and the Contemporary Ramifications of Banach space theory conference in honour of Joram Lindenstrauss and Lior Tzafriri, Institute of Advance Studies, Hebrew University of Jerusalem, 2005. We would like to thank G. Godefroy, R. Deville, C. J. K. Batty, P. Collins, J. L. González Llavona, D. Azagra, M. Jiménez, H. König, J. Lindenstrauss, N. Tomczak-Jaegermann, P. Kenderov, J. Lukes, M. Fabian, P. Hájek, V. Zizler, L. Tzafriri, T. Szankowski and M. Zippin for their excellent qualities as hosts and their grace and patience as audiences. J. Lindenstrauss deserves special gratitude for his insightful comments and encouragement with the topics presented here. Thanks are also due to I. Namioka for reading these notes, providing us with different points of view and excellent mathematical ideas. We wish to thank R. Haydon for many helpful suggestions and for our always interesting and stimulating conversations. Last, but certainly not least, we would like to express our debt to G. Godefroy, who was the first mathematician to suggest to us the idea of publishing all this material together, constantly encouraging us to finish our project.

Therefore despite the fact that the content of these notes is new and has not been published elsewhere, they have a self-contained and unified approach to the study of the existence of local uniformly rotund norms with a new point of view. As a result we hope they are accessible for readers with a basic knowledge of Functional Analysis and Set Theoretic Topology.

We study maps from a normed space X to a metric space Y which provide a **LUR** renorming in X. These maps are just those which satisfy two conditions that we call σ-slicely continuity and co-σ-continuity. Our main goal here is to characterize both properties, applying them as a new frame for **LUR** renormings. The characterization is an interplay between Functional Analysis, Optimization and Topology. We use ε-subdifferentials of Lipschitz functions and apply methods of metrization theory to the study of weak topologies. For example we find that any one-to-one operator T from X (reflexive, or even weakly countably determined) into $c_0(\Gamma)$ satisfies both conditions. Nevertheless our maps can be far away from the class of linear maps even when Y is a normed space. For instance the duality map from X into its dual is σ-slicely continuous if the norm of X is Fréchet differentiable. If in addition the dual norm is Gâteaux differentiable, then the duality map is co-σ-continuous and X is **LUR** renormable.

Murcia and Valencia,
July 2007

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Acknowledgement

The authors would like to thank all who supported the research for this work. Aníbal Moltó has been partially supported by BFM2003–07540/MATE and MTM2007-064521 del Ministerio de Ciencia y Tecnología MCIT y FEDER.
José Orihuela has been partially supported by PI–55/00872/FS/01, and 00690/PI/2004 Fundación Séneca CARM and BFM 2002–01719, MTM2005-08379 del Ministerio de Ciencia y Tecnología MCIT and Ministerio de Educación y Ciencia.
Stanimir Troyanski has been partially supported by Grant MM–1401/04 of the NSFR of Bulgaria, by PI–55/00872/FS/01, and 00690/PI/2004 Fundación Séneca CARM, by BFM 2002–01719, MTM2005-08379 del Ministerio de Ciencia y Tecnología MCIT and Ministerio de Educación y Ciencia and by a research project of the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences.
Manuel Valdivia has been partially supported by MEC and FEDER Project MTM2005-08210.

List of Symbols

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Introduction

Renorming in Banach space theory involves finding isomorphisms which improve the norm. That means making the geometrical and topological properties of the unit ball of a given Banach space as close as possible to those of the unit ball in a Hilbert space. Historically the first result in this direction is due to Clarkson [Clr36] who proved that every separable Banach space has an equivalent rotund norm. Indeed, if $\{f_i\}_1^\infty$ is a norm bounded sequence of linear functionals which separates the points of X then the equivalent norm given by

$$|x| = \|x\| + \left(\sum_1^\infty 2^{-i} f_i^2(x)\right)^{1/2}, \quad x \in X \tag{1.1}$$

is rotund. Let us recall that a norm $\|\cdot\|$ is rotund (strictly convex) if the unit sphere does not contain non-trivial segments, i.e. $x = y$ whenever $\|x\| = \|y\| = \|(x+y)/2\| = 1$.

An excellent monograph of renorming theory up to 1993 is [DGZ93]. In order to have an up-to-date account of the theory we should add [Hay99], [God01] and [Ziz03]. In [Hay99] the most important properties in renorming are characterized for $C(\Upsilon)$, where Υ is a tree, deducing a lot of counter-examples. In this way [Hay99] fixes the exact boundary of this theory. In the survey [God01] most of its results and proofs are devoted to separable and super-reflexive Banach spaces. The survey [Ziz03] gives an overview of the renorming theory of non-separable spaces with the classical approach.

In these notes we are focused mainly on locally uniformly rotund (locally uniformly convex) renorming. Let us recall that a norm $\|\cdot\|$ in a normed space is locally uniformly rotund (**LUR** for short) if $\lim_k \|x_k - x\| = 0$ whenever one of the two equivalent conditions holds $\lim_k \left(2\|x_k\|^2 + 2\|x\|^2 - \|x_k + x\|^2\right) = 0$ or $\lim_k \|(x_k + x)/2\| = \lim_k \|x_k\| = \|x\|$. Clearly every **LUR** norm is rotund. The converse is not true. If we construct in c_0 an equivalent norm using (1.1) we get a rotund norm which is not **LUR**.

A. Moltó et al., *A Nonlinear Transfer Technique for Renorming.*
Lecture Notes in Mathematics 1951,

The methods in these notes stem from the following result which gives a new starting point for **LUR** renorming.

Theorem 1.1. *Let X be a normed space and let F be a norming subspace of its dual. Then X admits an equivalent $\sigma(X,F)$-lower semicontinuous* **LUR** *norm if, and only if, for every $\varepsilon > 0$ we can write*

$$X = \bigcup_{n\in\mathbb{N}} X_{n,\varepsilon}$$

in such way that for every $x \in X_{n,\varepsilon}$ there exists a $\sigma(X,F)$-open half space H containing x with

$$diam\,(H \cap X_{n,\varepsilon}) < \varepsilon\,.$$

This linear topological concept is a particular case of a notion introduced in [JNR92] called countable cover by sets of small local diameter, which turns out to be equivalent for Banach spaces to the notion of descriptive spaces studied by R. W. Hansell in [Han01] (see Sect. 3.2).

The theorem above was proved in [MOT97] in the case where $F = X^*$. The proof was fully probabilistic and it was based on the following theorem.

For a set A in a normed space X we set

$$\gamma(A) = \sup_k \gamma_k(A), \quad \gamma_k(A) = \sup_m \inf \left(\mathbb{E}\,\|M_m\|^2\right)^{1/2}\,,$$

where the infimum is taken over all Walsh-Paley X-valued martingales $\{M_n\}_0^\infty$ such that

$$\#\left\{n \in \mathbb{N} :\ \int_{M_n^{-1}(A)} \|M_n - M_{n-1}\|^2 \geq 1\right\} \geq k\,.$$

The quantities $\gamma_k(A)$ measure how fast a dyadic tree must grow when it has many large branches ending at points of A.

Theorem 1.2. [Tro79] *A normed space X admits an equivalent* **LUR** *norm if, and only if, for every $\varepsilon > 0$ we can write $X = \bigcup_{n\in\mathbb{N}} X_{n,\varepsilon}$ in such a way that $X_{n,\varepsilon}$ are cones with*

$$\inf_n \gamma\,(X_{n,\varepsilon}) > \varepsilon^{-1}\,.$$

Historically the theorem above is the first characterization of **LUR** renormability in linear topological terms. The origin of this theorem goes back to Pisier's renorming [Pi75] of super-reflexive Banach spaces with power type modulus of rotundity.

The general case of Theorem 1.1 was proved in [Raja99], where instead of probabilistic arguments geometrical ones were applied, specially the Bourgain-Namioka superlemma (see, for example, [Die84, p 157]) which played an essential role there. In Sect. 4.2 we present another proof of this result where the Bourgain-Namioka superlemma is replaced by an optimization argument. An important contribution of M. Raja [Raja99] is an elegant proof to show that a rotund space in which the norm and the weak topologies coincide on the unit sphere admits a **LUR** renorming. Originally this result was proved in [Tro85] using Theorem 1.2. In turn, Raja's [Raja99] approach is a variation of a method of Lancien [Lan95] based on the dentability index which is defined through a modification of the "Cantor derivation". Namely, for a subset C of a normed space X and $\varepsilon > 0$

$$D_\varepsilon(C) = \{x \in C : \text{ diam } (C \cap H)\varepsilon \text{ for every open halfspace } H \text{ of } X \text{ with } x \in H\}.$$

Using this "derivation" Lancien got a new geometrical proof of the well-known renorming result of James-Enflo-Pisier for super-reflexive Banach spaces (see [God01, Sect. 3]).

It turns out that it is rather difficult to apply Theorem 1.2 and even Theorem 1.1 in a straightforward way. This motivates us to build up some technique to use Theorem 1.1. The most usual technique for renorming is the so-called transfer technique designed to transfer a good convexity property from a normed space to another. The easiest example illustrating this method is the following.

Theorem 1.3. *Let Y be a rotund space and T be a linear bounded one-to-one operator from X into Y, then the norm*

$$|x| = \|x\|_X + \|Tx\|_Y, \; x \in X$$

is rotund.

Actually (1.1) is a particular case of the above formula for the operator from X into ℓ_2 defined by $x \to \left(2^{-i} f_i(x)\right)_1^\infty \in \ell_2$. The simple geometrical interpretation of this fact is that the sum of convex functions is strictly convex whenever one of them, at least, is strictly convex. Unfortunately it is not possible to get **LUR** renormings by a direct application of this technique. Indeed let us consider the operator from ℓ_∞ to ℓ_2 defined by $x = (x_i)_1^\infty \to \left(2^{-i} x_i\right)_1^\infty$; it is one-to-one but ℓ_∞ is not **LUR** renormable. In [God82] (see also [GTWZ83] and [Fab91]) a transfer technique was developed to obtain rotund or **LUR** renormings by imposing compactness conditions on $T : X \to Y$. For example we have

Theorem 1.4. *Let X be a dual Banach space, let Y be a* **LUR** *Banach space and $T : Y \to X$ a bounded linear operator such that $\overline{TY}^{\|\cdot\|} = X$ and TB_Y is weak*-compact. Then X admits an equivalent dual* **LUR** *norm.*

In Sect. 4.1 we shall present Theorem 4.8, a reformulation of the former result in terms of our nonlinear approach to **LUR** renorming.

In order to be able to replace *rotund* by **LUR** in Theorem 1.3 we need the following.

Definition 1.5. Let Φ be a map from the metric space (X, d) into the metric space (Y, ϱ). Φ is said to be *co-σ-continuous* if for every $\varepsilon > 0$ we can write

$$X = \bigcup_n X_{n,\varepsilon}$$

and find $\delta_n(x) > 0$ for every $x \in X_{n,\varepsilon}$ in such a way that $d(x, y) < \varepsilon$ whenever $y \in X_{n,\varepsilon}$ and $\varrho(\Phi x, \Phi y) < \delta_n(x)$.

Now we can formulate the following (see [MOT97]).

Theorem 1.6. *Let Y be a* **LUR** *normed space and T be a bounded linear co-σ-continuous operator from the normed space X into Y, then X admits an equivalent* **LUR** *norm.*

In order to apply the former theorem we need the following characterization of co-σ-continuous maps.

Theorem 1.7. *A map Φ from a metric space (X, d) into a metric space (Y, ϱ) is co-σ-continuous if, and only if, for every $x \in X$ there exists a separable subset Z_x of X such that*

$$x \in \overline{\bigcup \{Z_{x_n} : n \in \mathbb{N}\}}^{d} \tag{1.2}$$

whenever $\lim_n \Phi x_n = \Phi x$.

If X is a normed space then the condition (1.2) can be replaced by

$$x \in \overline{\operatorname{span} \bigcup \{Z_{x_n} : n \in \mathbb{N}\}}^{\|\cdot\|}.$$

The proof of the former theorem can be found in Sect. 2.2 where co-σ-continuous maps are fully studied (see Theorem 2.32 and Proposition 2.33).

Example 1.8. Let us recall that the class of Baire maps between two metric spaces is the smallest family of functions which contains all continuous functions and the pointwise limit of sequences in it. So for any Baire map Ψ between metric spaces (Y, ϱ) and (X, d) there exists a countable family $\{\Psi_n : n \in \mathbb{N}\}$ of continuous functions such that $\Psi y \in \overline{\{\Psi_n y : n \in \mathbb{N}\}}$ for all $y \in Y$. A straightforward consequence of Theorem 1.7 is that *when Φ is a one-to-one map from (X, d) into (Y, ϱ) and Φ^{-1} is a Baire map then Φ is co-σ-continuous.*

From the last two theorems we now have corollaries that are easier to apply.

Corollary 1.9. *Let Y be a* **LUR** *normed space, let T be a bounded linear operator from the normed space X into Y such that for every $x \in X$ there exists a separable subspace Z_x of X with*

$$x \in \overline{\operatorname{span} \bigcup \{Z_{x_n} : n \in \mathbb{N}\}}^{\|\cdot\|}$$

whenever $\lim \|Tx_n - Tx\| = 0$.

Then X admits an equivalent **LUR** *norm.*

Actually in many cases we can require less than **LUR** renormability of Y in Corollary 1.9 and this fact will be a contribution developed in Sect. 3.4. To explain it let us firstly extend the notion of **LUR** norm.

Definition 1.10. Let $(X, \|\cdot\|)$ be a normed space and $\mathcal{T}$ a topology on it. We say that the norm $\|\cdot\|$ is $\mathcal{T}$ **LUR** if

$$\mathcal{T} - \lim_k x_k = x \ ,$$

whenever

$$\lim_{k\to\infty} \left\| \frac{x_k + x}{2} \right\| = \lim_{k\to\infty} \|x_k\| = \|x\| \ . \tag{1.3}$$

In this way we define weak **LUR**, weak* **LUR** and more general $\sigma(X, F)$ **LUR** norms if F is a subspace of X^*. In the case when X is a subspace of $\ell_\infty(\Gamma)$ we define pointwise **LUR** norm requiring that for all $\gamma \in \Gamma$

$$\lim_k x_k(\gamma) = x(\gamma)$$

whenever (1.3) holds.

Clearly $\sigma(X, F)$ **LUR** does not imply **LUR** renorming in general, for example ℓ_∞ has a pointwise **LUR** norm which is weak* **LUR** but fails to be weakly **LUR** renormable [Lin72] and therefore **LUR** renormable. However we have (see Corollary 3.23, 3.24 and [MOTV99]) the following.

Theorem 1.11. *Every weakly* **LUR** *normed space is* **LUR** *renormable. Every weak* **LUR** *dual norm in a dual Banach space with the Radon-Nikodym property has an equivalent dual* **LUR** *norm.*

By $\ell_c^\infty(\Gamma)$ we denote the subspace of $\ell^\infty(\Gamma)$ containing only those $x \in \ell^\infty(\Gamma)$ for which $\#\operatorname{supp} x \le \aleph_0$ and δ_γ is the projection on the γ-coordinate for $\gamma \in \Gamma$, i.e. $\delta_\gamma(x) := x(\gamma)$. Now we state the following.

Theorem 1.12. *Let Y be a subspace of $\ell_c^\infty(\Gamma)$ with a pointwise* **LUR** *norm which is pointwise lower semicontinuous, let T be a bounded linear operator from the normed space X into Y and $\{X_\gamma\}_{\gamma\in\Gamma}$ be a family of separable subspaces of X such that for every $x \in X$ we have*

$$x \in \overline{\operatorname{span} \bigcup \{X_\gamma : \gamma \in \operatorname{supp} Tx\}}^{\|\cdot\|} \ .$$

Then X admits an equivalent **LUR** *norm.*

This theorem is a generalization of a result in [FT90] where Y is the Mercourakis space $c_1(Z \times K)$ [Mer87], which is not **LUR** renormable since it contains a subspace isomorphic to ℓ^∞ whenever Z is an infinite set. It seems surprising that it is not necessary to assume that Y is **LUR** renormable but it is enough that Y is pointwise **LUR**. In Theorem 3.46 we shall present a nonlinear version of it. For the moment let us apply Theorem 1.12 to a large class of Banach spaces X which admits some suitable linear bounded operator with range in $c_0(\Gamma)$.

Indeed J. Lindenstrauss [Lin65] and [Lin66] introduced the projectional resolution of the identity (**PRI** for short) and using it constructed in every reflexive Banach space a linear bounded one-to-one map $T : X \to c_0(\Gamma)$ for some Γ. Later this technique was extended to weakly compactly generated Banach spaces by D. Amir and J. Lindenstrauss [AL68], to weakly compactly determined spaces by L. Vašak [Vas81], to weakly Lindelöf determined spaces by S. Argyros and S. Mercourakis [AM93, Val90, Val91, Vald90], and to duals of Asplund spaces by M. Fabian and G. Godefroy [FG88]. There exists a **PRI** in $C(K)$ when K is a Corson compact and its generalization (see [AMN88] and [Val90] respectively), when K is a compact of ordinals [Alex80] and a compact topological group [Alex82]. Quite recently M. Fabian, G. Godefroy and V. Zizler [FGZ01] have obtained a **PRI** for Banach spaces with a uniformly Gâteaux diffentiable norm. All these classes of Banach spaces are included in the so-called class $\mathcal{P}$ and, as is shown in [DGZ93, p. 236], using **PRI** and some hereditary properties of some complemented subspaces it is possible to construct a transfinite sequence of projections $\{Q_\alpha : 0 \le \alpha \le \mu\}$ such that if we set $R_\alpha = (Q_{\alpha+1} - Q_\alpha) / (\|Q_{\alpha+1}\| + \|Q_\alpha\|)$ we have

i) $Q_0 = 0$, $Q_\alpha \neq 0$ for $\alpha > 0$, $Q_\mu = Id$;
ii) $Q_\alpha Q_\beta = Q_\beta Q_\alpha = Q_{\min(\alpha,\beta)}$;
iii) $(Q_{\alpha+1} - Q_\alpha) X$ is separable for all $\alpha \in [0, \mu)$;
iv) $\{\|R_\alpha x\|\}_{0 \le \alpha < \mu} \in c_0([0, \mu))$ for all $x \in X$;
v) $Q_\beta x \in \overline{\operatorname{span}\{R_\alpha x : 0 \le \alpha < \beta\}}^{\|\cdot\|}$ for all $x \in X$.

If a Banach space has such a transfinite sequence of projections it is easy to construct a bounded linear operator $T : X \to c_0([0, \mu) \times \mathbb{N})$ and to find a separable subspace $X_{\alpha,n}$ satisfying the conditions of the last theorem. Indeed we can find for every $\alpha < \mu$ a sequence $f_{\alpha,n} \in X^*$, $\|f_{\alpha,n}\| \le 1$, $n \in \mathbb{N}$, which separates the points of $R_\alpha X$. We set $X_{\alpha,n} = R_\alpha X$ and define a bounded linear operator $T : X \to c_0([0, \mu) \times \mathbb{N})$ by the formula

$$Tx(\alpha, n) = \frac{f_{\alpha,n}(R_\alpha x)}{n} .$$

having in mind that $\{Q_\alpha : 0 \le \alpha \le \mu\}$ satisfies conditions i)–v) it is easy to see that T and $\{X_{\alpha,n} : (\alpha, n) \in [0, \mu) \times \mathbb{N}\}$ fulfill the conditions of Theorem 1.12, and therefore X is **LUR** renormable.

First J. Lindenstrauss [Lin72] asked whether every strictly convex Banach space X admits a one-to-one bounded linear operator to $c_0(\Gamma)$ for some Γ. Later in a joint paper with Dashiell [DL73] they constructed a strictly convex Banach space without a one-to-one bounded linear operator into $c_0(\Gamma)$ for any Γ. The first example of a **LUR** Banach space without a one-to-one bounded linear operator in $c_0(\Gamma)$ was found by R. Deville [Dev86].

Throughout these notes some applications of the above theorems will be shown. However, in many cases the linearity of T is rather restrictive. At first glance it seems that the linearity of T is necessary to transfer slices from Y to X. It is clear that if a map sends zero into zero and transfers slices into slices then it must be linear. Nevertheless the linearity of T can be avoided as it is shown by a comparison of Theorems 3.46 and 1.12. Taking advantage of the possibility to take additional countable splittings we can replace the linearity of T by something less restrictive. Our Theorem 1.1 motivates the following.

Definition 1.13. Let A be a subset of a linear topological space X, let Φ be a map from A into a metric space (Y, ϱ). We say that Φ is *slicely continuous at* $x \in A$ if for every $\varepsilon > 0$ there exists an open half space H of X containing x with osc $(\Phi \lceil_{H\cap A}) = \text{diam } \Phi(H \cap A) < \varepsilon$. We say that Φ is *σ-slicely continuous on* A if for every $\varepsilon > 0$ we can write

$$A = \bigcup_{n\in\mathbb{N}} A_{n,\varepsilon} \tag{1.4}$$

in such a way that for every $x \in A_{n,\varepsilon}$ there exists an open half space H of X containing x with osc $\left(\Phi \lceil_{H\cap A_{n,\varepsilon}}\right) = diam\ \Phi(H \cap A_{n,\varepsilon}) < \varepsilon$.

Maps of this kind can be very far from linear. For example see the oscillation map defined in Sect. 2.7, the maps defined in Propositions 4.1–4.5 and Theorems 5.1, 5.13 and 5.15.

Now we can reformulate Theorem 1.1 in the following way: *A normed space X admits an equivalent $\sigma(X, F)$-lower semicontinuous* **LUR** *norm if and only if the identity map $Id : (X, \sigma(X, F)) \to (X, \|\cdot\|)$ is σ-slicely continuous.* And consequently we have for any bounded linear operator the following.

Proposition 1.14. *Let T be a bounded linear operator from the normed space X into the normed space Y. Then T is σ-slicely continuous provided one of the spaces X or Y is* **LUR** *renormable.*

Moreover taking advantage of the nonlinear structure of the sets which satisfy (1.4) for the identity map, we can formulate our transfer result as follows.

Theorem 1.15. *Let X be a normed space and let F be a norming subspace of its dual. Then X admits an equivalent $\sigma(X, F)$-lower semicontinuous* **LUR** *equivalent norm if, and only if, there exists a metric space (Y, ϱ) and a map $\Phi : X \to Y$ which is σ-slicely continuous for $\sigma(X, F)$ and co-σ-continuous for the norm topology.*

Proof. If X admits an equivalent $\sigma(X,F)$-lower semicontinuous **LUR** equivalent norm we can take $Y = X$ and the identity map as Φ, which according to Theorem 1.1 is σ-slicely continuous for $\sigma(X,F)$. Conversely, let $\Phi : X \to (Y, \varrho)$ be co-σ-continuous and σ-slicely continuous for $\sigma(X,F)$.

Let us fix $\varepsilon > 0$, by co-σ-continuity we have that $X = \bigcup_{n=1}^{\infty} X_{n,\varepsilon}$ where for every $x \in X_{n,\varepsilon}$ there is $\delta(x,n,\varepsilon) > 0$ so that $\|x-y\| < \varepsilon$ whenever $y \in X_{n,\varepsilon}$ and $\varrho(\Phi x, \Phi y) < \delta(x,n,\varepsilon)$. Let us make another decomposition defining

$$X_{n,p,\varepsilon} := \left\{ x \in X_{n,\varepsilon} : \ \delta(x,n,\varepsilon) > \frac{1}{p} \right\}$$

then we have $X_{n,\varepsilon} = \bigcup_{p=1}^{\infty} X_{n,p,\varepsilon}$. Now we apply the σ-slicely continuity of the map Φ for fixed p and we get another splitting of X as $X = \bigcup_{m=1}^{\infty} X_p^m$ in such a way that for every m and every $x \in X_p^m$ we have a $\sigma(X,F)$-open half space H_x with $x \in H_x$ and $\operatorname{osc}\left(\Phi_{\restriction H \cap X_p^m}\right) < 1/p$. Fix m, n, p and then if $x \in X_{n,p,\varepsilon} \cap X_p^m$ we have $\|y - x\| < \varepsilon$ whenever $y \in H_x \cap X_{n,p,\varepsilon} \cap X_p^m$. Indeed since $y \in H_x \cap X_p^m$ we have $\varrho(\Phi x, \Phi y) < 1/p$ and consequently $\|y-x\| < \varepsilon$ since $y \in X_{n,\varepsilon}$ and $\varrho(\Phi x, \Phi y) < 1/p < \delta(x,n,\varepsilon)$. From the construction it is clear that $X = \bigcup \left\{ X_{n,p,\varepsilon} \cap X_p^m : \ m,n,p \in \mathbb{N} \right\}$, and the argument holds for every $\varepsilon > 0$ so the identity map from $(X, \sigma(X,F))$ into X is σ-slicely continuous, and to finish the proof it is enough to apply Theorem 1.1. □

From Theorem 1.15 and Proposition 1.14 the proof of the linear transfer technique (Theorem 1.6) immediately follows. In particular, we see that if $T : X \to Y$ is a bounded linear one-to-one map with Y **LUR** renormable and T^{-1} a Baire map for the norms, then X is **LUR** renormable.

In these notes we characterize co-σ-continuous and σ-slicely continuous maps and using Theorem 1.15 we obtain almost all known **LUR** renorming results as well as some new ones. Until now, **LUR** equivalent norms have been constructed ad hoc for each particular situation (see, for example, [Tro71], [GTWZ83], [GTWZ85], [HR90], [Fab91], [Hay99], [HJNR00] and others). Mainly they were based on the Deville Master Lemma [DGZ93, Chap. VII, Lemma 1.1.] (whose origin is in [Tro71]), distance to the unit sphere of **LUR** spaces, convolutions with **LUR** norms, and the three space problem for **LUR** renorming.

Theorems 1.7 and 1.15 together assert that a normed space X has an equivalent **LUR** norm if, and only if, there is a metric d on X generating a topology finer than the weak topology and such that the identity map from (X, weak) into (X,d) is σ-slicely continuous. For that reason it cannot be a surprise that the method of covers which has had such a strong influence in the problem of metrization [Fro95] of topological spaces turns out to be an important tool in **LUR** renorming. Let us recall some definitions to be precise on the relationships.