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Quantum Physics

A Functional Integral Point of View

Second Edition

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Preface

Twenty years after its inception [Jaffe, 1965b; Lanford, 1966; Glimm, 1967a], constructive quantum field theory is on the threshold of achieving its major goals. This level of success, while not unprecedented in contemporary mathematics, occurs with sufficient infrequency that it is worth commenting on (a) some of the factors which contributed to this success, and (b) what the implications of this success might be for mathematics and for science.

It is easier to address the second question. We see three consequences of a satisfactory mathematical foundation for the equations of quantum field theory. First, there is the question of principle as to whether the equations are correct and are correctly formulated. Having a mathematical foundation is a necessary but not a sufficient condition to answer this question positively. This concern was the original and primary motivation for the work from which this book is drawn. Second, the equations of quantum field theory are prototypes for other equations of independent interest, which arise in statistical mechanics, turbulence, and stochastic partial differential equations. The mathematical tools and concepts used to study quantum fields will likely find use in a variety of other problems having a similar mathematical structure. In fact, it is remarkable that these field-theoretic ideas have been instrumental through the work of Donaldson, Taubes, Uhlenbeck, and others in fundamental achievements in topology, where the mathematical structures appeared to be unrelated. Third, with the increasing power of computers, problems of the type mentioned above will be increasingly amenable to numerical solution. In this case, knowledge of the mathematical structure of the solution will be a considerable help in the discovery, understanding, and analysis of numerical algorithms and of numerical solutions. At the turn of the past century, Poincaré advanced essentially the same argument (citing the increased accuracy and quantity of astronomical observations) as a reason for the development of the qualitative theory of ordinary differential equations.

It is more difficult to determine the factors contributing to the success in this subject. It appears that a dedicated and talented group of workers, strong scientific leadership, sound scientific judgments, and constructive working arrangements each had a significant role to play.

In this second edition of *Quantum Physics*, we have added new chapters on correlation inequalities and the cluster expansion. Included is the remarkable proof that the ϕ^4 theories are trivial in high dimensions. Nonabelian gauge theories are required on both physical and mathematical grounds, and a new chapter is devoted to this topic. Also included in this chapter are phase cell expansions. This set of ideas has provided the basis for most of the estimates and proofs in constructive field theory. They were developed independently from renormalization group theory which implements similar ideas in problems with natural scaling behavior. An appendix on Hilbert space operators and function space integrals was added to make the book self-contained from a mathematical point of view. Certain proofs in Part I were simplified or expanded, to make them easier to follow.

Introduction

This book is addressed to one problem and to three audiences.

The problem is the mathematical structure of modern physics: statistical physics, quantum mechanics, and quantum fields. The unity of mathematical structure for problems of diverse origin in physics should be no surprise. For classical physics it is provided, for example, by a common mathematical formalism based on the wave equation and Laplace's equation. The unity transcends mathematical structure and encompasses basic phenomena as well. Thus particle physicists, nuclear physicists, and condensed matter physicists have considered similar scientific problems from complementary points of view.

The mathematical structure presented here can be described in various terms: partial differential equations in an infinite number of independent variables, linear operators on infinite dimensional spaces, or probability theory and analysis over function spaces. This mathematical structure of quantization is a generalization of the theory of partial differential equations, very much as the latter generalizes the theory of ordinary differential equations. Our central theme is the quantization of a nonlinear partial differential equation and the physics of systems with an infinite number of degrees of freedom.

Mathematicians, theoretical physicists, and specialists in mathematical physics are the three audiences to which the book is addressed.

Each of the three parts is written with a different scientific perspective. Part I is an introduction to modern physics. It is designed to make the treatment of physics self-contained for a mathematical audience; it covers quantum theory, statistical mechanics, and quantum fields. Since it is addressed primarily to mathematicians, it emphasizes conceptual structure—the definition and formulation of the problem and the meaning of the answer—rather than techniques of solution. Because the emphasis differs from that of conventional physics texts, physics students may find this part a useful supplement to their normal texts. In particular, the development of quantum mechanics through the Feynman–Kac formula and the use of function space integration may appeal to physicists who want an introduction to these methods.

Part II presents quantum fields. Boson fields with polynomial self-interaction in two space-time dimensions—$P(\phi)_2$ fields—are constructed. This treatment is mathematically complete and self-contained, assuming some knowledge of Hilbert space operators and of function space integrals. The original construction of the authors has been replaced by successive improvements and simplifications accumulated for more than a decade. This development is due to the efforts of a small and dedicated group of some thirty constructive field theorists including Fröhlich, Guerra, Nelson, Osterwalder, Rosen, Schrader, Simon, Spencer, and Symanzik, as well as the authors. Physicists may find Part II useful as a supplement to a conventional quantum field text, since the mathematical structure (normally omitted from such texts) is developed here.

Part II contains the resolution of a scientific controversy. For years physicists and mathematicians questioned whether nonlinear field theory is compatible with relativistic quantum mechanics. Could quantization defined by renormalized perturbation theory be implemented mathematically? The mathematically complete construction of $P(\phi)_2$ fields presented here and the construction of $\text{Yukawa}_{2,3}$, ϕ_3^4, sine-Gordon_2, Higgs_2, etc., fields in the literature provide the proof. Central among the issues resolved by this work is the meaning of renormalization outside perturbation theory. The mathematical framework for this analysis includes the theory of renormalization of function space integrals. From the viewpoint of mathematics the implementation of these ideas has involved essentially the creation of a new branch of mathematics.

Whether the equations are mathematically consistent in four space-time dimensions has not been resolved. There is speculation, for example, that the equations for coupled photons and electrons (in isolation from other particles) may be inconsistent, but that the inclusion of coupling to the quark field may give a consistent set of equations. A proper discussion of this issue is beyond the scope of this book, but is alluded to in Chapters 6 and 17.

Particle interaction, scattering, bound states, phase transitions, and critical point theory form the subject of Part III. Here we develop the consequences of the Part II existence theory and make contact with issues of broad concern to physics. This part of the book is written at a more advanced level, and is addressed mainly to theoretical and mathematical physicists. It is neither self-contained nor complete, but is intended to develop central ideas, explain main results of a mathematical nature, and provide an introduction to the literature.

Condensed matter physicists may find interesting the discussion of phase transitions and critical phenomena. The central matters are series expansions and correlation bounds. These methods find application in diverse areas. We give detailed justification of the connection (by analytic continuation) between quantum fields and classical statistical mechanics. Professional physicists could well start directly in Part III, returning to earlier material only as necessary.

Readers interested in the historical development of constructive quantum field theory are referred to the various survey articles of the authors and others. In this book the specific, detailed references are minimized, especially in the self-contained Parts I and II. A large bibliography has been included; we apologize for the inevitable omissions.

Numerous colleagues, students, and friends helped make this book possible. Of particular importance were R. D'Arcangelo, R. Brandenberger, B. Drauschke, J.-P. Eckmann, J. Gonzalez, W. Minty, K. Peterson, P. Petti, the staff at Springer-Verlag, and especially our wives Adele and Nora. We are also grateful to the ETH, the IHES, the University of Marseilles, and the CEN Saclay for hospitality as well as to the Guggenheim Foundation and the NSF for support.

For the Second Edition

In the preparation of this second edition, we highlight several definitive contributions to physics which have emerged from the program of mathematical analysis of fundamental physics, as explained in the first edition. We give three examples where mathematical theorems have given the cleanest and most definitive resolution of scientific controversies. These controversies concern properties of solutions of equations describing models of physical phenomena. They arose due to ambiguities in the analysis of these solutions through other methods, namely laboratory experiment, numerical simulation, or formal analysis according to the methods of theoretical physics.

The existence of a phase transition in a two-dimensional Coulomb gas (the Kosterlitz–Thouless transition) was proved by Fröhlich and Spencer; see Chapter 5. The appearance of disorder in the ground state of the three-dimensional random field Ising model (lower critical dimension) was proved by Imbrie; see Chapter 5. The nonexistence theory for pure ϕ^4 fields in dimensions $d > 4$ and the expectation of a similar result for $d = 4$ was obtained by Aizenman and Fröhlich; see Chapter 21. This result focuses attention on nonabelian gauge theory for which renormalization group methods suggest that the $d = 4$ theory exists; see Chapter 22.

In conclusion, we see that mathematical analysis must be included in the list of appropriate methods in the search for truth in theoretical physics. This conclusion is strengthened by experience in other areas of mathematical physics, including general relativity and attempts at a unified field theory based on string theory.

In this edition, we have expanded the proofs in Part I and added an appendix on Hilbert space operators and function space integrals to make the book self-contained. Three new chapters reflect recent developments.

We thank many people, including E. Nelson and R. Streater, for detailed comments on the first edition. Furthermore, we thank R. Cheng and B. Drauschke for assistance on the Bibliography and Index.

Conventions and Formulas

Fourier transforms

$$f(x) = (2\pi)^{-d/2} \int e^{ipx} f^{\sim}(p)\, dp,$$

$$f^{\sim}(p) = (2\pi)^{-d/2} \int e^{-ipx} f(x)\, dx,$$

$$f(\theta) = (2\pi)^{-d/2} \sum e^{in\theta} f^{\sim}(n),$$

$$f^{\sim}(n) = (2\pi)^{-d/2} \int_0^{2\pi} e^{-in\theta} f(\theta)\, d\theta.$$

Minkowski vectors

$$x = (x_0, \mathbf{x}) = (x_0, \ldots, x_{d-1}),$$

$$x^2 = x \cdot x = -x_0^2 + \mathbf{x}^2, \qquad p^2 = p \cdot p = -p_0^2 + \mathbf{p}^2,$$

$$x \cdot p = \sum x_i p^i = -x_0 p_0 + \mathbf{x} \cdot \mathbf{p},$$

$$\Box = -\partial_t^2 + \Delta = -\partial x_0^2 + \sum_{i=1}^{d-1} \partial x_i^2.$$

Euclidean vectors

$$x_d = ix_0,$$

$$x^2 = x \cdot x = \sum_{i=1}^{d} x_i^2,$$

$$\Delta = \sum_{i=1}^{d} \partial x_i^2.$$

Schrödinger's equation

$$\hbar = h/2\pi,$$

$$i\hbar\dot{\theta} = H\theta, \qquad \theta(t) = e^{-itH/\hbar}\theta(0),$$

$$p = -i\hbar\frac{\partial}{\partial q}, \qquad [p(x), q(y)] = -i\hbar\,\delta(x-y)$$

Covariance operators $C_m \in \mathscr{C}_m$ satisfy

$$(-\Delta + m^2)C_m = \delta.$$

σ and γ matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3,$$

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$\not{a} = \sum a_\mu \gamma_\mu,$$

$$\not{a}^2 = \sum a_\mu^2 = a^2.$$

Dirac equation (zero field)

$$(\hbar\not{\partial} - mc)\psi = 0.$$

Dirac equation in external field A

$$\left(\hbar\not{\partial} + i\frac{e}{c}\not{A} - mc\right)\psi = 0.$$

List of Symbols

a, a^*, A, A^*	annihilation and creation operators
a, A	free energy
A	antisymmetrization operator
$\mathscr{A}$	action
$\mathfrak{A}, \mathscr{A}$	algebra of operators
b	bond
B	observable; region in space-time
$\mathscr{B}$	set of bonds
c	diagonal values of C, $c(x) = C(x, x)$; critical (as a subscript); constant
cr	critical
cl	classical
C	covariance; (Chap. 7) complex numbers
$\mathscr{C}, \mathscr{C}_m$	a class of covariance operators (Sec. 7.9)
d	dimension of space-time
D	Dirichlet boundary conditions
$\mathscr{D}$	domain of an operator; C_0^∞ test function space
$\mathscr{D}'$	Schwartz distribution space
$\mathscr{D}^{(j)}$	domain for irreducible spin j representation of SU(2, C)
E	energy level; eigenvalue for H; Euclidean transformation; Euclidean group

$\mathscr{E}$	Euclidean group; Euclidean Hilbert space; time strip (Section 10.5)
f	test function; free energy
$\mathscr{F}$	Fock space
g	test function
$\mathscr{G}$	group
h, $\hbar$	Planck's constant
h	external field
H	Hamiltonian
HS	Hilbert–Schmidt
$\mathscr{H}(\mathbf{x})$	Hamiltonian density
$\mathscr{H}$	Hilbert space of quantum states
I	identity operator
J	interaction strength for Ising ferromagnet
j, $\mathbf{J}$	angular momentum
k	Boltzmann's constant
$\mathscr{K}$	kernel of semigroup
K	kernel of Bethe–Salpeter equation
L	angular momentum (Section 15.1)
L_s, L_i, L	lines in Feynman graphs (self-interacting, interacting)
$\mathscr{L}$	Lagrangian; lattice; multiple reflection norm (Section 10.5); Lorentz group
m, M	mass; magnetization; multiple reflection norm (Section 10.5)
n	number of field components; degree of polynomial P
nn	nearest neighbor
N	Neumann boundary condition; $N(f)$ = norm of f.
$\mathscr{N}$	null space for inner product
p	period boundary conditions; pressure; Lebesgue index; degree of polynomial P
p, P	momenta; momentum operator; momentum space
P	polynomial interaction; projection operator
P_n	Hermite polynomial
q, Q	configuration; configuration space; Lebesgue index
R	real numbers; multiple reflection norm (Section 10.5)
R^d	Euclidean d-space
s	time

s, S	entropy
S	generating function; Schwinger function; sphere; symmetrization operator
$\|S^n\|$	volume of n-sphere
$\mathscr{S}$	Schwartz space of rapidly decreasing test functions
$\mathscr{S}'$	Schwartz space of tempered distributions
$\mathfrak{S}_n$	symmetric group on n elements (permutation group)
t	Euclidean time ($= x_d$); Minkowski time ($= x_0$)
T	time ordering; truncation
U, V	unitary operator on Hilbert space
V	potential
W	Wightman function
dW	Wiener measure
$\mathscr{W}$	Wiener path space
$\mathscr{X}$	phase space
x	point in space time
$\mathbf{x}$	point in space
z	fugacity; activity
Z	partition function; field strength renormalization constant; integers
Z_+	nonnegative integers; partition function
β	$(1/kT)$ inverse temperature
γ	critical exponent
γ, Γ	boundary; phase boundary
Γ	Dirichlet boundary conditions on Γ; inverse to propagator or two point function
$\|\Gamma\|$	length or area of Γ
δ	Dirac δ function; Kronecker δ function; lattice spacing; critical exponent
Δ	Laplacian; special solution of wave or Laplace equation (propagator) also unit square
ε	$2\theta - 1$ (a type of Heaviside function); lattice spacing; reduced temperature $(T - T_c)/T_c$
ζ, η	critical exponents
θ	reflection operator; Heaviside function; state in $\mathscr{H}$
κ	momentum cutoff
λ	coupling constant
Λ	bounded region of space

$\|\Lambda\|$	area or volume of Λ
μ	$(-\Delta + m^2)^{1/2} = (p^2 + m^2)^{1/2}$; chemical potential; external field
$d\mu$	statistical weight or ensemble
ν	frequency; critical exponent
$d\nu$	statistical weight or ensemble
ξ	random variable
Ξ	partition function
π	3.14159; momentum conjugate to field ϕ
Π	projection operator; hyperplane
$\Pi_\pm$	half spaces of $R^d\backslash\Pi$
ρ	density
σ	mass^2; Ising spin variable; time
Σ	proper self-energy
ϕ, Φ	quantum field; configuration of classical field
$d\Phi_C$	Gaussian measure, covariance C
χ	susceptibility; random variable; state in $\mathscr{H}$; characteristic function
ψ	quantum field; state in $\mathscr{H}$
ω	frequency; Wiener path; angular integration variable
Ω	vacuum state; ground state; equilibrium state
∂	derivative; boundary operator
∇	gradient; divergence
$\|\cdot\|$	absolute value; area, volume or number of $\cdot$; norm
$\hat{}$	projection operator from the Euclidean path space to the Hilbert space of quantum states
$\tilde{}$	Fourier transform
$\langle\ ,\ \rangle$	inner product
$\langle\ \rangle$	expectation; integral with respect to $d\mu$
$[\ ,\ ]$	commutator: $[a, b] = ab - ba$
$\{\ ,\ \}$	anticommutator: $\{a, b\} = ab + ba$
$\varnothing$	free boundary conditions; empty set
$\times$	vector product
$\cdot$	time derivative; position for missing variable as in $f(\cdot) = f$ for a function f.
$\backslash$	set theoretic difference: $A\backslash B = \{x: x \in A, x \notin B\}$
$\bar{}$	complex conjugation; closure

Contents