Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1371

Sarah Glaz

Commutative Coherent Rings

Springer-Verlag

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1371

Sarah Glaz

Commutative Coherent Rings

Springer-Verlag

Berlin Heidelberg New York London Paris Tokyo

Author

Sarah Glaz
Department of Mathematics, Wesleyan University
Middletown, CT 06457, USA

Mathematics Subject Classification (1980): 13-02, 13B99, 13C11, 13C13, 13C15, 13D99, 13E99, 18G99, 20M14

ISBN 3-540-51115-6 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-51115-6 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

To my mother Amalia Dauer

and

In memory of my father Philip Dauer

INTRODUCTION

"Praise them! The Ring-bearers,
praise them with great praise!"

The Lord of the Rings
J. R. R. Tolkien

The history of coherent algebraic objects starts in 1944, when Cartan, without mentioning the term coherent, introduced and developed some properties of coherent sheaves. He gave the object its name in a paper in 1953. Serre, 1955-56, and Grothendieck and Dieudonne, 1961-1963, continued in Cartan's tradition and, through the concept of coherent sheaves, consolidated the foundation of modern algebraic geometry.

Coherent rings and modules first appear in the literature in 1960, in a paper by Chase, still without being mentioned by name. It is only in 1964 that coherent rings appear as such named, in Bourbaki.

From 1966 on, coherence in commutative rings became a vigorously active area of research. The body of research accumulated, beyond having an interest of its own, had significant impact on other areas of algebra. It is not my intention here to give a historical account of this research, since the interested reader can reconstruct it from the references given in the book; rather, I will briefly outline the interplay between the research done in coherent rings and the research done in other areas of commutative algebra.

Part of the research done in coherent rings was influenced by one of its most important examples, that of Noetherian rings. In this direction the questions investigated consisted of asking to what extent results known to hold in Noetherian rings are still valid for coherent rings. In some of these investigations the theory developed in order to obtain the answers has significance in the general theory of commutative rings. Two such examples are the investigation into the (nonexistent) analogue of Hilbert basis theorem for coherent rings

(Chapter 7), and the extension of the notion of regularity from Noetherian to coherent rings (Chapter 6) with the related questions pertaining to the weak dimension of a coherent ring.

In investigating the Hilbert basis theorem for coherent rings, cartesian squares were constructed (Chapter 5); this construction led to a complete structural description of rings of global dimension two (Chapter 6). The same question led to the most general definition of non-Noetherian grade and depth with applications to exactness of complexes for general commutative rings (Chapter 7).

The extension of the definition of regularity to coherent rings and its related questions prompted a renewed investigation into the relation between the minimal prime spectrum of a ring and its total ring of quotients (Chapter 4). This relation in its turn shed new light on the nature of flat epimorphic extensions (Chapter 4). Another influence of the definition of regularity was to start an investigation into projective dimensions of ideals in polynomial rings, group rings and symmetric algebras. This led to a better understanding of the homological properties of these rings (Chapter 7 and Chapter 8).

Not all research done in coherent rings was influenced by Noetherian rings results. All non-Noetherian classical type rings, like boolean algebras, absolutely flat rings, valuation and Prüfer domains and semihereditary rings are examples of coherent rings. The interest in coherence renewed an interest in these rings as a result of which our knowledge of these rings today is considerably more advanced than it used to be (Chapter 4 and Chapter 7). Moreover, new finite condition rings were defined and investigated, some as a direct result of coherence (Chapter 6), some probably influenced by the investigation into coherence.

Another aspect of research that was strongly influenced by the investigation into coherent rings is the development of certain ring constructions and rings extensions. What started as a (not-so-simple) question, When is D + M coherent?, evolved later to the generalization of the definition of rings of type D + M (Chapter 5). A serious investigation into new properties of trivial ring extensions was launched in a successful attempt to answer a conjecture arising from a coherent ring investigation (Chapter 4). The research into the nature of the integral closure of a one-dimensional coherent domain branched out into investigating, on one hand, general overrings (Chapter 5) and, on the other hand, the nature of prime ideals in polynomial rings (Chapter 7).

The notion of coherence touched other areas of algebra as well. There is an abundance of results in noncommutative ring theory in that direction. Coherent groups, coherent categories, and coherent functors were defined and investigated.

The whole subject of commutative coherent rings, in spite, and perhaps because, of all the research already done in the area, still has an unaccountable number of open problems, some of a very basic nature.

This book provides an extensive and systematic treatment of the theory of commutative coherent rings, blending and providing a link between the two sometimes disjoint approaches available in the literature, the ring theoretic approach and the homological algebra approach. The book covers most results in commutative coherent ring theory known to date, as well as a number of new results never published before.

The book assumes knowledge of basic commutative and homological algebra. Nevertheless, it is relatively self-contained, in the sense that, in Chapter 1 and in several later sections, all necessary basic results are summarized without proofs (references given). Chapter 1 also serves the purpose of setting a uniform notation and terminology for the book, as many of the notions used do not yet have standard notation.

Chapter 2, faithful to its title, introduces the reader to finitely presented modules and basic properties of coherent rings and modules.

Chapter 3 and section 1 in Chapter 7 develop many of the tools of modern research in commutative algebra such as several homological dimensions, Fitting invariants, Euler characteristic, Koszul complexes, and the general theory of grade.

Chapter 4 through Chapter 8 represent the main body of research in coherent ring theory. The presentation of most topics is as general as possible, with the results on coherent rings following from the general theory. An attempt has been made to provide a short historical overview with each of the main topics, to include many examples and counterexamples, and to expose the reader to open problems in the field, which are either explicitly stated or implied by the approach to the subject.

Chapters 4 and 5 present topics in ring extensions and ring constructions such as the total ring of quotients, flat epimorphisms, trivial ring extensions, cartesian squares, D + M constructions, and a general overring approach to the study of the integral closure.

Chapter 6 presents several particular rings which are either coherent or strongly related to coherence. The topic of uniform coherent rings is essentially a Noetherian ring theory topic with a strong coherence flavor; so is the syzygetic approach to some results in regular rings. The structure of rings of global dimension two is also presented in this chapter.

Chapter 7 presents the most studied question in coherent ring theory, namely the question of stable coherence. It also explores the relation between prime ideals in polynomial rings and the nature of the integral closure.

Chapter 8 investigates several universal algebras in the direction of coherence and regularity.

The book is suitable as a reference book for researchers and as a textbook for a second-year graduate course in algebra.

It is a pleasure to acknowledge my thanks to the numerous colleagues who have encouraged me by their continued interest in the progress of this book.

Particular thanks are due for helpful comments and for sending me reprints and preprints of papers, monographs, their own and their students' theses to B. Alfonsi, F. Barger, B. Greenberg, I. Papick, J. E. Roos, and W. V. Vasconcelos.

I must add a special word of thanks to W. V. Vasconcelos for introducing me to the subject of coherent rings and for his constant encouragement and interest during this work.

Many thanks are also due to my typist, Mrs. Janice Bittner, for the care, good judgment, and effort she put into producing this typescript.

I would also like to acknowledge my grateful thanks for the financial support of the National Science Foundation.

Last, but not least, I thank my husband Joe Glaz and my son Ron Glaz for their continuous and heartwarming support.

Table of Contents

CHAPTER 8 COHERENT ALGEBRAS

CHAPTER 1

PRELIMINARIES

SECTION 1. PROJECTIVE AND INJECTIVE MODULES

DEFINITION. Let R be a ring. An R module F is called <u>a free R module</u> if it is isomorphic to a direct sum of copies of R. If $Ra_\alpha \simeq R$ and $F = \bigoplus_{\alpha \in S} Ra_\alpha$ then the set $\{a_\alpha \,/\, \alpha \in S\}$ is called <u>a basis</u> of F over R.

Over a commutative ring R every two bases of a free R module have the same cardinality. Every R module is isomorphic to a quotient of a free R module.

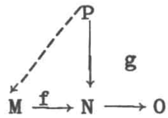
THEOREM 1.1.1 ([R2]). <u>Let R be a domain and let M be a finitely generated torsion free R module, then M can be embedded in a finitely generated free R module</u>.

THEOREM 1.1.2 ([B7]). <u>Let (R, m) be a local ring and let M and N be two finitely generated free R modules and $f: M \to N$ a homomorphism. The following conditions are equivalent</u>:

<u>(1) $1_{R/m} \otimes f: R/m \otimes_R M \to R/m \otimes_R N$ is injective</u>.

<u>(2) f is injective and coker f = N/M is a free R module</u>.

DEFINITION. Let R be a ring. An R module P is called <u>a projective R module</u> if the following diagram can be completed, for every R module M and N and every R homomorphism f and g:



Every free R module is projective. The converse is not necessarily true. For example, let $R = Z_2 \oplus Z_2$, where Z denotes the integers, then $M = Z_2$ is a projective, but not a free R module.

THEOREM 1.1.3 ([R2]). Let R be a ring and let P be an R module. The following conditions are equivalent:

(1) P is a projective R module.

(2) $Hom_R(P,-)$ is an exact functor, that is if $0 \to M \to N \to L \to 0$ is an exact sequence of R modules, then $0 \to Hom_R(P,M) \to Hom_R(P,N) \to Hom_R(P,L) \to 0$ is an exact sequence of R modules.

(3) P is a direct summand of a free R module.

(4) Every exact sequence $0 \to M \to N \to P \to 0$, where M and N are R modules, splits.

(5) $Ext^1_R(P,M) = 0$ for all R modules M.

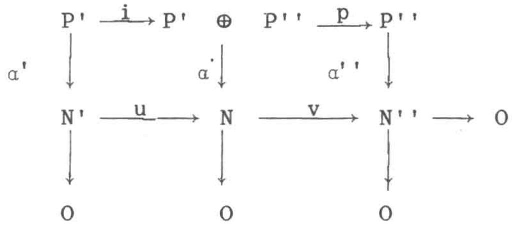
(6) $Ext^n_R(P,M) = 0$ for all R modules M and all integers $n > 0$.

An arbitrary direct sum of projective modules is a projective module. A projective module over a local ring is a free module.

DEFINITION. Let R be a ring and let M be an R module. An exact sequence $\ldots \to P_1 \to P_0 \to M \to 0$ with P_i projective R modules is called a projective resolution of M. If P_i are free R modules, this exact sequence is called a free resolution of M. If P_i are finitely generated then this exact sequence is called a finite projective (resp. free) resolution of M. If a module M admits a finite projective (resp. free) resolution of type $0 \to P_n \to \ldots \to P_0 \to M \to 0$, this resolution is called a finite resolution of length n, or a finite resolution of finite length if knowledge of n is not important.

Every module admits a projective (in fact free) resolution.

THEOREM 1.1.4 ([B7]). Let R be a ring, let $N' \xrightarrow{u} N \xrightarrow{v} N'' \to 0$ be an exact sequence of R modules and let $P' \xrightarrow{\alpha''} N' \to 0$ and $P'' \xrightarrow{\alpha''} N'' \to 0$ be two surjective maps. If P'' is a projective R module then there exists a surjective map $\alpha : P' \oplus P'' \to N \to 0$ such that the following diagram commutes:



where i and p are the corresponding inclusion and projection maps.

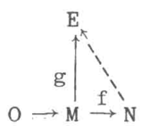
SCHANUEL'S LEMMA 1.1.5 ([R2]). Let R be a ring and let:

$$0 \to K \to P \to M \to 0$$

$$0 \to K' \to P' \to M \to 0$$

be two exact sequences of R modules with P and P' projective R modules. Then $K \oplus P' \simeq K' \oplus P$.

DEFINITION. Let R be a ring. An R module E is called an injective R module if the following diagram can be completed, for every R module M and N and every R homomorphism f and g:



An important example of an injective module is the Z module $M = Q/Z$ where Z denotes the integers and Q the rationals.

THEOREM 1.1.6 ([R2]). Let R be a ring and let E be an R module. The following conditions are equivalent:

(1) E is an injective R module.

(2) $Hom_R(-,E)$ is an exact functor, that is if $0 \to M \to N \to L \to 0$ is an exact sequence of R modules then $0 \to Hom_R(L,E) \to Hom_R(N,E) \to Hom_R(M,E) \to 0$ is an exact sequence of R modules.

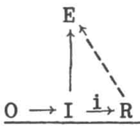
(3) E is a direct summand of every module of which it is a submodule.

(4) Every exact sequence $0 \to E \to N \to L \to 0$, where N and L are R modules, splits.

(5) $Ext^1_R(N,E) = 0$ for all R modules N.

(6) $Ext^n_R(N,E) = 0$ for all R modules N and all integers $n > 0$.

(7) For every ideal I of R, the following diagram can be completed:



where i is the inclusion map.

An arbitrary product of injective R modules is an injective R module. Every R module can be embedded in an injective R module.

DEFINITION. Let R be a ring and let M be an R module. An exact sequence $0 \to M \to E_0 \to E_1 \to \ldots$ with E_i injective R modules is called an injective resolution of M.

Every R module admits an injective resolution.

DEFINITION. Let R be a ring. An R module M is called a divisible R module if $rM = M$ for all $r \in R$.

THEOREM 1.1.7 ([R2]). Let R be a ring, then:

(1) Every injective R module is a divisible R module.

(2) Assume that R is a domain and M is a torsion free R module, then M is an injective R module iff M is a divisible R module.

(3) Assume that R is a domain with field of quotients K, and M is a torsion free R module, then M is a divisible R module iff M is a vector space over K.

THE DUALITY HOMOMORPHISMS ([C5]).

Let R and S be two rings, let M be an R module, E an S module and N an R and an S module, then there is a natural isomorphism: $Hom_R(M, Hom_S(N,E)) \simeq Hom_S(M \otimes {}_R N,E)$ which induces the first duality homomorphisms:

$$\rho_n: \ Ext^n_R(M, Hom_S(N,E)) \to Hom_S(Tor^n_R(M,N),E)$$

Next consider the homomorphism:

$\sigma: Hom_S(N,E) \otimes_R M \to Hom_S(Hom_R(M,N),E)$. If M is a finitely generated projective R module than σ is an isomorphism. In any event, σ induces the second duality homomorphisms:

$$\sigma_n: \ Tor^n_R(Hom_S(N,E),M) \to Hom_S(Ext^n_R(M,N),E)$$

THEOREM 1.1.8 ([C5]). With the above notation we have:

(1) If E is an injective S module then ρ_n are isomorphisms for all integers $n \geq 0$.

(2) If E is an injective S module and M has a projective resolution composed of finitely generated R modules then σ_n are isomorphisms for all integers $n \geq 0$.

DEFINITION. Let R be a ring and let M be an R module, an R module E is called an essential extension of M, if $M \subset E$ and for any nonzero submodule E' of E we have $E' \cap M \neq 0$.

Every R module M admits an essential injective extension E(M), which is unique up to isomorphism. This extension is called the injective envelope of M.

Let E(M) be the injective envelope of M, then there is no injective proper submodule between M and E(M). In fact, this is another characterization of an injective envelope.

If M and N are two R modules then $E(M \oplus N) = E(M) \oplus E(N)$.

DEFINITION. Let R be a ring. An R module E is called a universal injective R module if E is an injective R module and if for any R module M and any nonzero element $m \in M$, there exists a homomorphism $f: M \to E$ satisfying $f(m) \neq 0$.

THEOREM 1.1.9 ([M4]). Let R be a ring and let E be an R module. The following conditions are equivalent:

(1) E is a universal injective R module.

(2) E is an injective R module and the map $M \to Hom_R(Hom_R(M,E),E)$ is injective for all modules M.

(3) E is an injective R module and every R module can be embedded in a direct product of copies of E.

(4) E is an injective R module and contains a copy of every simple R module (an R module is simple if it has no proper submodules).

For any ring R, the module $E = Hom_Z(R,Q/Z)$ is a universal injective R module. It follows from Theorem 1.1.9(4) that if E is the injective envelope of the direct sum of one copy of each of the simple R modules, then E is a universal injective R module and it is isomorphic to a direct summand of every other universal injective R module.