Lecture Notes in Mathematics
1515

E. Ballico F. Catanese C. Ciliberto (Eds.)

Classification of Irregular Varieties

Minimal Models and Abelian Varieties

Proceedings, Trento 1990

Springer-Verlag

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Lecture Notes in Mathematics 1515

PREFACE

The conference on "Classification of Irregular Varieties, Minimal Models and Abelian Varieties" was held in Villa Madruzzo, Cognola (Trento) from December 17 to 21,1990. The meeting has been sponsored and supported by C.I.R.M. (Centro Internazionale per la Ricerca Matematica, Trento), the Mathematical Department of the University of Trento and Centro Matematico Vito Volterra. This volume contains most of the works reported in the formal and informal lectures at the conference. The topics of the volume are:

Abelian varieties and related varieties (papers by Bardelli, Birkenhake - Lange, Ciliberto - Harris - Teixidor, Ciliberto - Van der Geer, Salvati Manni,Van Geemen);

Minimal models and classification of algebraic varieties (the papers by Andreatta - Ballico - Wisniewski and by Kollar - Miyaoka - Mori);

K-theory (the paper by Vistoli).

During the conference some "examples" were worked out by the participants. They are collected here under the heading " Trento examples". They are listed with the names of the discoverers, a discussion of the problems considered and, of course, the proofs. The two of us not connected with the "Trento examples" think that they are very interesting.

We liked the idea of inserting at the end of this volume a list of problems and questions. We collected this list mentioning the proposers of the questions and a few related references. We believe that publishing such lists may be a very useful contribution to the mathematical life, and hope this will be done more often. At the beginning of the list we described the fate of some of the problems in the list published in the Springer Lecture Notes in Mathematics 1389 (Proceedings of the conference on "Algebraic curves and Projective Geometry", Trento 1988, edited by two of us).

We are very grateful to all participants for their enthusiasm (and the "Trento examples" are a fruit of their enthusiasm), to the contributors of this volume, to the referees for their precious help, to the three organizations which supported and sponsored the meeting, to C.I.R.M. for his help and assistance in running the meeting and in editing this volume.

All the papers were refereed. They are in final form and will not be published elsewhere.

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CONTENTS

Projective manifolds containing large linear subspaces

M. Andreatta 1), E. Ballico 1) and J. Wiśniewski 2)

Let $X \subset \mathbf{P}^N$ be a complex manifold of dimension n containing a linear subspace Π isomorphic to $\mathbf{P}^r$. By $N_{\Pi/X}$ we will denote the normal bundle to Π in X; let c be the degree of $N_{\Pi/X}$. The manifold X can be studied in terms of adjunction theory. If L denotes the restriction of $\mathcal{O}(1)$ to X then, for some positive rational number τ (which is called the nef value of L), the divisor $K_X + \tau L$ is semiample (but not ample) and its large multiple defines an adjunction mapping of X. Note that, for X as above, $\tau \geq r+c+1$ and the equality holds if and only if the adjunction map contracts Π to a point.

The existence of a linear subspace $\mathbf{P}^r$ in X makes X rather special. If the normal bundle of Π in X is numerically effective then X is covered by lines and there is the following:

Theorem. ([B-S-W], thm (2.3))Let X, Π and L be as above. Assume moreover that the normal bundle of Π in X is numerically effective. If $r+c \geq n/2$ then the map associated to the adjoint divisor $K_X + \tau L$, τ being the nef value of L, contracts Π to a point. Moreover the map is an extremal ray contraction, unless $X \cong \mathbf{P}^{n/2} \mathrm{x} \mathbf{P}^{n/2}$.

Similarly one can reformulate the theorem (2.5) of [B-S-W] to describe the case when $(r+c) \geq (n-1)/2$. If r itself is large enough then the adjunction map is expected to be a projective bundle. In particular we have

Theorem ([Ei], thm (1.7), [Wi2] thm (2.4)) If the normal bundle to Π in X is trivial and $r \geq n/2$ then X has a projective bundle structure, Π being one of the fibers of the bundle.

It turns out that the assumption on the normal bundle being trivial could be replaced by "numerically effective" to obtain a theorem similar to the one above, see the theorem (0.7).

In the present paper we deal with the case when the normal bundle is not nef but still not too negative so that X contains a subvariety having projective bundle structure, see the theorem (1.1). As an application, we describe special adjunction morphisms, which then turn out to have very nice structure; namely they are either blow-downs or can be flipped, see the theorem (1.2) and the theorem (1.3). In the remainder of the paper we discuss the question of projectivity of some manifolds obtained by contracting subvarieties having projective bundle structure, see the theorem (1.4) and the remark (1.4.1).

The present paper was prepared when the third author was visiting The University of Trento in the Fall of 1990. He would like to express his thanks to the University for the financial support as well as to the members of the Mathematical Department for their help and warm welcome. The first two authors were partially supported by MURST and GNSAGA.

§0. Notations and Preliminaries.

In this paper we work over the complex field **C**. We are going to use some notations which were developed in the context of the Minimal Model Program by Mori, Kawamata and others. For these we fully refer to the paper [K-M-M], but for convenience of the reader we just recall the following.

Let X be a smooth connected projective variety of dimension $n \geq 2$.
(0.1) **Definition**. Let $R = \mathbf{R}_+[C]$ be an extremal ray on X. We define

a) The *length* of R as $\ell(R) = \min \{ -K_X \cdot C$, C rational curve and $[C] \in R\}$.

b) The *locus* of R, E(R), as the locus of curves whose numerical classes are in R.

(0.2) **Definition.** Let $\varphi = \mathrm{cont}_R$ be an elementary contraction, i.e. the contraction of an extremal ray R and let $\delta = \dim(E(R))$, where "dim" denotes, as usual, the maximum of the dimensions of the irreducible components.
The contraction φ is said to be of flipping type or *a small contraction* if $\delta < n-1$ (resp. of *fiber type* if $\delta = n$, resp. of *divisorial type* if $\delta = n-1$).

(0.3) **Definition**. Let $\varphi: X \longrightarrow Y$ be an elementary small contraction, the *flip* of φ is a birational morphism $\varphi': X' \longrightarrow Y$ from a normal projective variety X' with only terminal singularities such that the canonical divisor $K_{X'}$ is φ'-ample as a **Q**-divisor.

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi^+} & \\ & Y & \end{array}$$

The following inequality was proved in [Wi1].
(0.4) **Proposition.** Let $\varphi := \mathrm{cont}_R$ the contraction of an extremal ray R, E'(R) be any irreducible component of the exceptional locus and d the dimension of a general fiber of the contraction restricted to E'(R). Then

$$\dim E(R) + d \geq n + \ell(R) - 1.$$

(0.5) **Proposition**. Let $Z \subset X$ be a closed subvariety of X such that the map Pic(X) —> Pic(Z) has 1-dimensional image.
Let R be an extremal ray of X such that

$$\ell(R) + \dim(Z) > \dim X + 1.$$

Then either locus(R) (= E(R)) and Z are disjoint or Z is contracted to a point by contr_R.

Proof Suppose that locus(R) and Z are not disjoint and therefore let F be a fiber of contr_R such that F and Z are not disjoint. By the inequality in (0.4) we have

$$\dim(F) \geq n + \ell(R) - 1,$$

and then, by our assumption, we have $\dim(F \cap Z) \geq 1$. Therefore at least a curve of Z is contracted to a point by contr_R, therefore all Z by the assumption on the Pic.

In fact we have the following result announced to us by Beltrametti and Sommese in [Be-So]:

(0.5.1) **Corollary**. Let R_1 and R_2 be distinct extremal ray with $\ell(R_1) = a$ and $\ell(R_2) = b$ and assume that they are not nef. Then $E(R_1)$ and $E(R_2)$ are disjoint if $a+b > \dim X$.

(0.6) **Definition**. Let E be a vector bundle on a smooth projective variety X. E is called nef (resp. semiample or ample) if the relative hyperplane-section divisor ξ_E on **P**(E) ($\mathcal{O}(\xi_E) = \mathcal{O}_{\mathbf{P}(E)}(1)$) is nef (resp. semiample or ample).

The following result is a slight generalization of the theorem (1.7) in [Ei] (see also [Wi2], thm (2.4)).
(0.7) **Theorem**. Let $\Pi \subset X \subset \mathbf{P}^N$ be as in the introduction. Assume moreover that the normal bundle to Π in X is numerically effective. If $r > n/2$ then X has a projective bundle structure and Π is contained in one of the fibers of the bundle.

(0.8) **Remark**. The examples of even-dimensional quadrics and Grassmanians of lines show that the bound on r is sharp (see the point in the proof when we discuss decomposability of the normal bundle).

Proof of (0.7). First we claim that the normal bundle to Π in X is decomposable and isomorphic to $\mathcal{O}(1)^{\oplus\alpha} \oplus \mathcal{O}^{\oplus\beta}$, $\alpha+\beta = n-r$. Indeed, the normal bundle of Π in X is a sub-bundle of the normal bundle of Π in $\mathbf{P}^N$ isomorphic to $\mathcal{O}(1)^{\oplus N-r}$. Therefore, being nef, the normal bundle has the same splitting type, precisely (0,...,0,1,...,1), on any line contained in Π. Since $r > \mathrm{rank}(N_{\Pi/X})$ the decomposability of the bundle follows (see [O-S-S], thm (3.2.3)).

Consider the injective morphism of normal bundles

$$0 \longrightarrow N_{\Pi/X} \longrightarrow N_{\Pi/\mathbf{P}^N}$$

and the above splitting of $N_{\Pi/X}$ to obtain the following injective morphism of vector bundles

$$0 \longrightarrow \mathcal{O}_{\Pi}(1) \longrightarrow N_{\Pi/\mathbf{P}^N} = \mathcal{O}(1)^{\oplus k}.$$

This inclusion single out a linear form $z \in H^0(\mathbf{P}^N, \mathcal{I}_{\Pi}(1))$, $\mathcal{I}_{\Pi}$ the ideal sheaf of Π in $\mathbf{P}^N$.
Set $H = \{z = 0\}$; we have the following:

Claim: for every $x \in \Pi$, H does not contain $(TX)_x$, the tangent space of X at x.

Proof. In fact for every x we have the surjective map

$$((TX)_{|\Pi})_x \longrightarrow (N_{\Pi/X})_x$$

and by construction z induces a non zero linear form on $(N_{\Pi/X})_x$.

Therefore the hyperplane section $H \cap X$ is smooth along Π and thus, by Bertini, a general hyperplane section of X containing Π, call it X', is smooth everywhere.
Moreover, using an exact sequence of normal bundles it can be seen that

$$N_{\Pi/X'} = \mathcal{O}(1)^{\oplus(\alpha-1)} \oplus \mathcal{O}^{\oplus\beta}.$$

Therefore we can inductively produce a smooth subvariety $Y \subset X$ containing Π and such that Π has trivial normal bundle in Y. Thus, by [Ei], thm (1.7), Y has a projective bundle structure, Π being a fiber of such a bundle. Using the Lefschetz hyperplane section theorem it can be easily seen that the projective bundle map commutes with the adjunction map which we have from [B-S-W], thm (2.3), quoted in the introduction. Therefore the fibers of the projective bundle map are obtained by hyperplane slicing of the adjunction map fibers, so the latter map must be a projective bundle.

§1. Projective manifolds containing large linear subspaces.

The following theorem concerns a variation of the theorem (0.7); the proof follows the same lines as the one of Ein's (see the theorem (1.7) in [Ei]).

(1.1) **Theorem**. Let $X \subset \mathbf{P}^N$ be a projective n-fold, $n > 2$, containing a r-dimensional linear projective space, $\Pi_0 = \mathbf{P}^r$, such that either

(a) $N_{\Pi_0/X} = \mathcal{O}^{\oplus n-r-1} \oplus \mathcal{O}(-1)$ and $r > (n/2)$

or, respectively,

(b) $N_{\Pi_0/X} = \mathcal{O}^{\oplus n-r-2} \oplus \mathcal{O}(-1)^{\oplus 2}$ and $r > (n+1/2)$.

Then there exists a smooth subvariety E in X of codimension 1, or 2, respectively, which is a $\mathbf{P}^r$ - bundle over a smooth projective manifold T, such that Π_0 is one of its fiber and the normal bundle of E restricted to every fibers Π_t of the projective bundle is isomorphic to $O(-1)$, $O(-1)^{\oplus 2}$, respectively.

Proof. We have that $h^1(N_{\Pi_0/X}) = 0$ and therefore the Hilbert scheme of r-planes in X is smooth at the point t_o corresponding to Π_0. Let T be the unique irreducible component of the Hilbert scheme containing t_0. Call m = n-r. Since $h^0(N_{\Pi_0/X}) = m-1$, respectively m-2, the dimension of T is m -1 in case (a), or m-2 in case (b) respectively.

Suppose Π_t is an arbitrary r-plane in the family T; then we claim that the normal bundle $N_{\Pi_t/X}$ is of type (a) or (b), respectively. To prove this note first that, since a small deformation of the decomposable bundle is trivial, the assertion holds for a general t in T. In particular, for a general t, $(N_{\Pi_t/X})^*$ is a numerically effective vector bundle. On the other hand, applying the sequence of conormal bundles

$$0 \longrightarrow (N_{X/\mathbf{P}^N})^* \longrightarrow (N_{\Pi_t/\mathbf{P}^N})^* = \oplus O(-1) \longrightarrow (N_{\Pi_t/X})^* \longrightarrow 0$$

we see that $(N_{\Pi_t/X})^*(1)$ is spanned, hence nef for any t. Moreover, by our hypothesis on r the line bundle $-c_1(N_{\Pi_t/X})^*(1) - K_{\Pi_t}$ is ample, so we can apply the following consequence of a result from [Wi2] and conclude that $(N_{\Pi_t/X})^*$ is numerically effective for all t.

(1.1.1) **Lemma**. Let E_0 be a vector bundle on $\mathbf{P}^r$, such that $c_1(E_0(1)) < r+1$ and E(1) is nef. If E_0 is a specialization of a nef vector bundle, then it is also nef.

Proof of lemma. Apply the theorem on rigidity of nef values (1.7) of [Wi2] to the deformation of E; the proof is then similar to the proof of (2.1) in [Wi2].

Coming back to the proof of the theorem, let us note that the results of [P-S-W] imply then that indeed the normal bundle $N_{\Pi_t/X}$ is either of type (a) or (b), respectively.

Now, by the property of Hilbert scheme, T is smooth and, if Π is the universal r -plane over T, we have that Π is a $\mathbf{P}^r$-bundle over T, $\Pi \longrightarrow T$. There is moreover a natural "evaluation" map h: $\Pi \longrightarrow X$. The map h is an immersion at each point: this can be proved exactly as done in the proof of 1.7 in [Ei], p. 901, using the fact that no non zero section of $(N_{\Pi/X})$ has zeros and thus a part of the differential of h, being the evaluation of the normal bundle, is of highest rank everywhere.

To prove that h is one to one, a modification of Ein's argument has to be used. What we need is that $\Pi_t \cap \Pi_{t'} = \emptyset$ for every $t \neq t'$. Assume on the contrary that $\Pi_t \cap \Pi_{t'} \neq \emptyset$, and let Δ denote the linear space being this intersection. Then $\dim(\Delta) \geq 2r - n \geq 1$, with equality if and only if $n = 2r-1$.

Assume first that Δ is a line and $n = 2r - 1$. In this case we have that $N_{\Delta/X} = N_{\Delta/\Pi_t} \oplus N_{\Pi_t/X|\Delta}$, that is $N_{\Delta/X} = \mathcal{O}(-1) \oplus \mathcal{O}(1)^{\oplus r-1} \oplus \mathcal{O}^{\oplus m-1}$. Therefore the Hilbert scheme of lines in X is smooth at d, the point corresponding to Δ, and therefore there is a unique component of it containing d, $T(\Delta,X)$ and $\dim(T(\Delta,X)) = (m-1) + 2(r-1)$. Analogously if we consider the Hilbert scheme of lines respectively in Π_t and in $\Pi_{t'}$ we found that they are smooth at the point corresponding to Δ and their components through this point, $T(\Delta,\Pi_t)$ resp. $T(\Delta,\Pi_{t'})$, are of dimension $2(r-1)$. Therefore, counting the dimension of intersection of them inside $T(\Delta,X)$ we get

$$\dim (T(\Delta,\Pi_t) \cap T(\Delta,\Pi_{t'})) \geq 4(r-1) - [2(r-1) + (n-r-1)] > 0,$$

since by our assumption $r > (n/2)$. This is a contradiction as we assumed that their intersection is just a point d.

Assume then $\dim(\Delta) \geq 2$. Let l be a line in Π_t, for some t, and $\mathbf{P}_0$ a projective plane through l contained in Π_t. Let $\mathcal{H}$ be the subscheme of the Hilbert scheme of planes in X defined by

$$\mathcal{H} = \{\mathbf{P}: l \subset \mathbf{P} \text{ and } \mathbf{P} \text{ is a plane in } X\}.$$

Since $h^1(N_{\mathbf{P}_0/X} \otimes J_{l/\mathbf{P}_0}) = 0$, $\mathcal{H}$ is smooth at the point p_o corresponding at $\mathbf{P}_0$. Hence there is a unique component $\mathcal{H}_0$ of $\mathcal{H}$ containing p_0 and $\dim(\mathcal{H}_0) = r - 2$.
Let Σ_l be the set swept out in X by the planes from $\mathcal{H}$. Then $\dim \Sigma_l = r = \dim\Pi_t$ therefore $\Sigma_l = \Pi_t$. In particular we have that if l is a line in $\mathbf{P}_0$ contained in Δ then $\Sigma_l = \Pi_t = \Pi_{t'}$, giving the absurd.

Therefore h is an embedding, let $E = h(\Pi)$; the statement on the normal bundle of E is then clear and the theorem is proved.

The following two theorems come from an application of the above result. We will give a proof of the second one, which uses the case (b) of the theorem (2.1); a proof of the first one can be obtained similarly using the case (a).

(1.2) **Theorem**. Let X be a smooth projective variety of dimension $n \geq 3$ and L a very ample line bundle on X. Let $H = K_X + rL$ for $r > n/2$: assume that H is nef and big but not ample.
Let $\varphi: X \longrightarrow Y$ be the morphism associated to some high multiple of H (as usually the variety Y is normal and the fibers of φ are connected); let $E = \cup E_i$ be the decomposition into irreducible components of the exceptional set.
Assume that every component E_i of E is contracted to a set of dimension not smaller than n-r-1. Then the components of E are pairwise disjoint and each $\varphi_{|E_i} : E_i \longrightarrow Z_i := \varphi(E_i)$ is a $\mathbf{P}^r$-bundle

over a smooth variety Z_i of dimension n-r-1. That is Y is a smooth n-fold and, by the Nakano contraction theorem (see [Na]), the map φ is a blow-down of divisors E_i's to varieties Z_i's.

(1.3) **Theorem.** Let X be a smooth projective variety of dimension $n \geq 4$ and L a very ample line bundle on X. Let $H = K_X + rL$ for $r > (n-1)/2$: assume that H is nef and big but not ample.
Let $\varphi: X \longrightarrow Y$ be the morphism associated to some high multiple of H; let $E = \cup E_i$ be the decomposition into irreducible components of the exceptional set.
Assume that every component E_i of E has codimension at least 2 and that it is contracted to a set of dimension not smaller than n-r-3. Then the components of E are pairwise disjoint and each $\varphi_{|E_i}$: $E_i \longrightarrow Z_i := \varphi(E_i)$ is a $\mathbf{P}^{r+1}$-bundle over a smooth variety Z_i of dimension n-r-3. Moreover there exists a flip

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \varphi \searrow \quad \swarrow \varphi^+ & \\ & Y & \end{array}$$

to a smooth projective variety X^+, which is an isomorphism outside E, such that the canonical divisor K_{X^+} is φ^+-ample. The assumption $r > (n-1)/2$ is not needed if r = n-3, i.e. for (n,r) = (4,1) or (5,2).

Proof of the theorem (1.3). We first prove the following

(1.3.1) **Lemma.** In the hypothesis of the theorem we have that

$$\dim \varphi(E_i) = n-r-3.$$

Proof of the lemma. We will give two different proofs of this lemma:
Suppose for absurd that $\dim \varphi(E_i) > n-r-3$: then we can take n-r-2 divisors $H_i \in |mH|$ for $m >> 0$ such that $X' = H_1 \cap \ldots \cap H_{n-r-2}$ is smooth and $(X' \cap E_i) \neq \emptyset$. Take now r-1 divisor $L_j \in |L|$ such that $X'' = X' \cap L_1 \cap \ldots \cap L_{r-1}$ is smooth and $\dim (X'' \cap E_i) > 0$.
Therefore $\varphi_{|X''}$ would be a small contraction on a smooth 3-fold, which is absurd.

(sketch of proof) Let F be a fiber of φ contained in E_i and let C be a rational curve on F such that $-n-1 \leq K_X \cdot C < 0$. By hypothesis we have that $-K_X \cdot C = r$ (i.e. C is a line relative to L). Therefore we can construct a non breaking family of rational curves whose dimension at every point is at least r-2 (see [Mo] or [Io] or [Wi1]). Arguing as in the proof of the inequality in (0.4) (see [Wi1]) we get

$$\dim(E_i) + \dim(F) \geq n + r - 1.$$

With this the proof of the lemma is immediate.

Going back to the proof of the theorem we claim that each E_i contains a linear space $\Pi_i = \mathbf{P}^{r+1}$ such that $N_{\Pi_i/X} = \mathcal{O}^{\oplus n-r-2} \oplus \mathcal{O}(-1)^{\oplus 2}$.

We choose m >> 0 such that the linear system $|mH|$ is base-point free and take n-r-3 general divisors from this system such that the intersection of them is a smooth variety X' of dimension r+3. For a general choice of these divisors the variety X' will contain only a finite number of positive-dimensional (of dimension r+1) fibers of φ (but at least one from each E_i) each of them contracted to a point.

Now take r-1 general divisors from the very ample linear system $|L|$ and intersect them with X' to obtain a smooth 4-fold which we denote by X" . Let Y" denote the normalization of the image of the map $\varphi'' = \varphi_{|X''}$ restricted to X"; since Y" has isolated singularities, the exceptional locus of φ'' is of dimension 2. By adjunction we find out that the divisor $-K_{X''}$ is φ''-ample so that, locally, we are in the situation of [Ka], (2.1). In particular, the exceptional locus of φ'' consists of a number of disjoint projective planes with normal bundle $\mathcal{O}(-1)^{\oplus 2}$. Therefore the exceptional locus of $\varphi_{X'}$ consists of a number of disjoint linear $\mathbf{P}^{r+1}$'s which proves the first part of our claim. The statement on the normal bundle of Π_i is then clear: we check it first on the projective planes (c.f.[ibid] then on Π_i it follows from the well known fact that the extension of a decomposable bundle on $\mathbf{P}^2$ to $\mathbf{P}^{r+1}$ must be decomposable.

Now, from the previous result it follows that each E_i has a structure of a projective bundle, and by an argument as in the proof of (1.1) it follows that they are pairwise disjoint. Moreover from (1.1) it follows that the normal bundle to E_i restricted to any fiber of φ is isomorphic to $O(-1)^{\oplus 2}$.

Now the construction of the flip is standard. We blow-up X along E_i's , $Bl_X(\cup E_i)$, the exceptional divisors being the fiber product of $\mathbf{P}^{r+1}$- and $\mathbf{P}^1$-bundle over Z_i's; we can contract $Bl_X(\cup E_i)$ to a smooth variety X^+ contracting the exceptional divisor to the $\mathbf{P}^1$-bundle over Z_i's. The divisor K_{X^+} is then φ^+-ample so X^+ is projective.

(1.3.2) **Remark**. Let L^+ denote the strict transform of L to X^+. Then the divisor $H^+ := (K_{X^+}) + ((r-\varepsilon)L^+)$, for $0 < \varepsilon << 1$, can be proved to be ample; it may be a good candidate in order to consider the pair (X^+,L^+) and proceed further with the adjunction program.

(1.3.3) **Remark**. Note that the first part of the above proof (i.e. concerning Π_i's) works for L merely ample and spanned, so that the theorem is true for such L if r = n-3.

The first part of the next theorem is exactly the theorem (1.1) case (a); the second one follows from the contraction theorem of Nakano (see[Na]).

(1.4) **Theorem**. Let $X \subset \mathbf{P}^N$ be a projective n-fold containing a r-dimensional projective space, $\Pi_0 = \mathbf{P}^r$, such that $N_{\mathbf{P}^r/X} = O^{\oplus n-r-1} \oplus O(-1)$ and $r > (n/2)$. Then there exists a divisor D in X which is a $\mathbf{P}^r$ bundle over a smooth projective manifold T, such that Π_0 is one of its fiber and $D_{|\Pi_t} = O(-1)$ for every fibers Π_t.

Therefore X is obtained by blowing up a smooth codimension r-1 subvariety of a smooth complex analytic space Y, $\pi : X \longrightarrow Y$.

(1.4.1) **Remark**. It would be interesting to know, if the complex analytic manifold Y we found in the theorem is actually projective: this is not always the case if $r < n/2$ as many examples of Moishezon manifolds can show. The following is an example of a Moishezon manifold obtained by blowing down smoothly a projective manifold with the general fibre of dimension $\geq n/2$; but we do not know if the fibers are embedded as linear $\mathbf{P}^r$'s.

(1.4.2) **Example**. (see also [Ka]) Let r = n-3 in the theorem (1.3); i.e. we have a smooth n-fold X and a small elementary contraction $\varphi: X \longrightarrow Y$ such that the exceptional locus E is the disjoint union of $E_i = \mathbf{P}^{n-2}$, i = 1,...,s, and $N_{E_i/X} = O(-1)^{\oplus 2}$. Suppose moreover that s > 1. For the existence of such a case see the example in [Ka]; this was constructed for n = 4 but it can be generalized to higher dimension in the same way by taking a curve meeting a codimension 2 subvariety (both smooth) in a n-fold V such that K_V is ample.

Blow-up now X along one of E_i, say E_1; the exceptional locus of this blown-up is $\mathbf{P}^{n-2} \times \mathbf{P}^1$ with normal bundle $O_{\mathbf{P}^{n-2}}(-1) \otimes O_{\mathbf{P}^1}(-1)$. Therefore we can blow down this divisor to a smooth $\mathbf{P}_0 = \mathbf{P}^1$ in a complex manifold X'.

The manifold X' is not projective: to see this we will prove that for every Cartier divisor D on X' we have that if $D_{|\mathbf{P}_0} = O(k)$ then $D_{|E_i} = O(-k)$ for every i >1. First notice that by the construction, since $\rho(X/Y) = 1$, we have that $\rho(X'/Y) = 1$. Therefore we need to prove our claim just for D = K_X. By the adjunction formula we see in fact that $K_{X|\mathbf{P}^1} = O(n-3)$ and $K_{X|E_i} = O(-n+3)$ for every i > 1.

(1.4.3) **Remark**. In the situation of (1.4), for r = n-1 or n-2 the manifold Y is projective.

Proof. For n-1 the result is trivial since in this case the map π is just a blow-down of D to a smooth point (see [Io] or [Fu]).

In the other case we consider the line bundle $M := (K_X+rL) + (L+D)$. We prove the remark if we show that M is a good supporting divisor for the map π.

By adjunction $M_{|D} = K_E+(r+1)L_{|D}$; by the theorem (2.7) in [B-S-W] we have that $M_{|D}$ is a good supporting divisor for $\pi_{|D}$ in our hypothesis.

On the other side, for $r = n-2$, we can suppose that $(K_X+(n-2)L)$ is nef: if this is not the case, by the result of [Io] and [Fu], we have that it is not nef on divisor $E_i = \mathbf{P}^{n-1}$ such that $L_{|E_i} = O(1)$ and which are disjoint; by proposition (1.5) they are disjoint from D also. We can therefore contract these E_i's to a smooth n-fold X' and consider (X',L') (the first reduction) instead of (X,L) where L' is the ample line bundle which is the push-forward of L.

Let C be a curve not contained in D: then $(L+D)\cdot C > 0$ since D is effective and L is ample, therefore, since $(K_X+(n-2)L)$ is nef, for every such a curve $M\cdot C > 0$.

This, together with the fact that $M_{|D}$ is a good supporting divisor for $\pi_{|D}$, implies that M is a good supporting divisor for the morphism π in (1.4).

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