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SINGULARITIES IN LINEAR WAVE PROPAGATION

Lars Gårding

Historical introduction The theory of wave propagation started with Huyghens's theory of wave front sets as envelopes of elementary waves. Its first success was the proper explanation of the propagation of light in refracting media. Its modern successor is the theory of boundary problems for hyperbolic systems of partial differential equations. The development which lead to this theory is a story of a search for proper mathematical tools.

The first chapter is the discovery in the eighteenth century of a paradox. The wave equation $u_{tt}-u_{xx}=0$ in one time and one space dimension expresses the movement of the deviation u from rest position for an idealized string. Its general solution $f(x-t)+g(x+t)$ with f and g arbitrary is the sum of two travelling waves with opposite directions. But it was also possible to express the movements of a string fixed at its end points as an infinite sum of sine functions. This raised the question about the nature of functions and how a series with smooth terms could express arbitrary functions. The first efficient solution of this problem came two hundred years later with the theory of distributions.

The nineteenth century made important discoveries about wave propagation. Gemetrical optics was developed to great perfection by Hamilton. It is a theory of normals of wave fronts, in other words of rays rather than waves of light. It gave a very good idea of wave fronts or caustics but not a very clear idea about their intensity or the intensity of light outside the fronts. Other efforts centered around the wave equation, in our notation and with the propagation velocity normalized to 1,

$$u_{tt}-\Delta u = 0,$$

where Δ is Laplace's operator in n space variables $x=(x_1,...,x_n)$. The physically interesting case is of course n=3. In the beginning of the nineteenth century it was observed that the spherical waves $u=f(t-|x|)/|x|$ are solutions for arbitrary functions f and Poisson discovered the remarkable formula, in modern notation,

$$u(t,x) = (4\pi)^{-1} \int f(y)\delta(t-|x-y|)dy/|x-y|,$$

which solves Cauchy's problem $u_{tt}-\Delta u=0$, $u=0$, $u_t=f(x)$ for $t=0$. Its fit with geometrical optics was perfect, the support of the solution at time t is precisely the envelope of spheres with radius t and centers in the support of f. For almost a century, this cemented the idea that geometrical optics contains almost the whole story of wave propagation.

In modern language we can interpret Poisson's formula by saying that the distribution

(1) $E(t,x) = H(t)\ \delta(t-|x|)/4\pi|x|$, $H(t)=1$ when $t>0$ and 0 otherwise,

which solves the wave equation $E_{tt}-\Delta E=\delta(t)\delta(x)$ and vanishes when $t<0$ describes the forward emission of light from a point source. It can also be described as the forward fundamental solution of the wave equation in three space variables.

The most interesting problem of geometrical optics was the problem of double refraction observed and analyzed already by Huygens. A ray of light entering certain kinds of crystals is refracted into two rays whose directions vary with the direction of the incident ray relative to the crystal. According to Huygens's theory of refraction, this means that light in the crystal has two velocities, both direction dependent, which can be measured by the strength of the refraction.

Directions where the two velocities are the same are called optical axes. They appear as double points of the velocity surface which is obtained by taking velocity as distance to an origin along a variable ray. For crystals with one optical axis, Huygens found the velocity surface to be a sphere and an ellipsoid tangent to it.

Crystals with two optical axes remained a mystery until the the French physicist Fresnel found explicit velocity surfaces for them. They turned out to be algebraic of degree 4 depending on three constants varying with the nature of the crystal. The surfaces are symmetric around the origin with two sheets which come together at four double points on the optical axes.

Associated with the velocity surface there is the wave surface consisting of wave fronts at time t=1 emanating from an instantaneous point source of light in the crystal. According to Huygens, the wave surface is the envelope of the velocity surface. Fresnel guessed its analytical form. By a freak of nature, it is identical to the velocity surface with the three constants inverted and hence it has the same general form as the velocity surface. The computations were carried out by, among others, Hamilton. He added an important complement, observing that the tangent planes to the velocity surface through a double point form a circle on the outer sheet of the wave surface which bounds a circular disc covering the inlet to a double point. He predicted from this that an outside ray of light whose direction coincides with an optical axis ought to be broken into a cone of rays. This phenomenon, the conical refraction, was verified by experiment a short time later.

Hamilton made his discovery in the late 1820's. The following decades saw extensive activity with the aim of understanding the nature of light. The first attempts were based on analogy with elasticity theory and resulted among other things in the equations of Lame, a 3X3 hyperbolic system of second order differential equations in four variables, time and three space variables. These equations are identical with what one gets from Maxwell's equations for the electric field in a dielectricum when the magnetic field is elimininated.

To solve the initial value problem for Lame's system was a great challenge taken up by Sonya Kovalevskaya. She had a model to go by, Weierstrass's solution of the Cauchy problem for an analogous system associated with the product of two wave operators with different speeds of light. Led by geometrical optics, she assumed that light from a point source ought to propagate between the two sheets of the wave surface leaving no trace behind. The latter assumption is correct but she did not realize that there is light also between the outer sheet and its convex hull. Her formal calculations where she used the fact that the wave front surface can be parametrized by elliptic functions led her astray. The solution that she deduced is identically zero, a clear contradiction with the Cauchy-Kovalevskya theorem. Her mistake was pointed out a few years later by Volterra. He corrected the formulas but did not arrive at a solution of Cauchy's problem. Earlier, his faith in geometrical optics as a complete clue to wave propagation was shaken when he found that the analogue of the distribution (1) for two space variables is

$$H(t)\ H(t^2 - |x|^2)^{-1/2}/2\pi,$$

which describes propagation of light from an instantaneous point source in a medium with two space variables. Since this distribution does not vanish when $|x|<t$, there is an afterglow behind the wave front on the circle $|x|=t$. In lectures that he gave in Stockholm in 1906, Volterra pointed out that the analytical tools tried so far were not sufficient to treat Lame's equations for the double refraction. One of his listeners, a young mathematics student Nils Zeilon, took notice. His admired teacher Ivar Fredholm had constructed fundamental solutions of elliptic differential operators in three variables using abelian integrals. Zeilon continued his work for other types of equations but using another point of departure, namely the remark that if $P(\xi)$ is a polynomial in n variables, the integral

(2) $$E(x)=(2\pi)^{-n} \int \exp ix.\xi\ d\xi/P(\xi)$$

is, at least formally, a fundamental solution for the operator $P(D)$

where $D=\partial/i\partial x$. In fact,

$$P(D)E(x)=(2\pi)^{-n}\int \exp ix.\xi\ d\xi = \delta(x).$$

The problem is to make sense of the integral (2) which may diverge at 0, at infinity and at the zeros of P. Apart from this, the formal machinery works also when $P(\xi)$ is a square mateix whose elements are polynomials, in particular for the Lame system. Zeilon's mathod of avoiding singularities was to move the chain R^n of integration into C^n. This can be done in various ways. Zeilon's intuition led him right, but his arguments are shady, read by a critical eye. This also applies to his magnum opus, two long articles around 1920 on the problem of conical refraction. But his results are right. The support of the fundamental solution includes the space the outer sheet of the wave surface and its convex hull. This fact has to do with conical refraction, but the precise connection was not clarified until 1961 with a paper by Ludwig.

Zeilon's work did not get much attention. Some years later Herglotz constructed forward fundamental solutions of hyperbolic differential operators wih constant coefficients in any number of variables. For them, the velocity surface has m sheets corresponding to m different propagation velocities. He applied the Fourier transform to the space variables and arrived at very simple formulas covering also the Lame system. He showed that the wave surface in the general case is a system of criss-crossing surfaces of varying dimensions near which the fundamental solution may have a very complicated behavior. Outside the wave surface, i.e. outside its fastest front, the fundamental solution is zero, but it may also vanish in regions inside the fastest front as is does for propagation of light in free space. These regions, the lacunas, attracted the the interest of Petrovsky who published a fundamental paper about them in the forties where he tied the existence if lacunas to topological properties of plane sections of the complex velocity surface. His work was extended to the general case of degenerate velocity surfaces by Atiyah, Bott and Gårding in the early seventies.

Fundamental solutions E_n of the wave equation with an arbitrary number of space variables were constructed by Tedone already in 1889 in the form of solutions of the corresponding Cauchy problems. In terms of the function

$$d(t,x)=t^2-|x|^2,$$

their main properties can be described as follows, They vanish outside the forward light cone where $t\geq 0$, $d(t,x)\geq 0$. Inside the light cone E_n behaves like

$$\text{const } d(t,x)^{(n-3)/2}$$

when n is even. It vanishes there when n is odd >1 and behaves like

$$\text{const } \delta^{((n-3)/2)}(d(t,x))$$

on the light cone outside the origin. Tedone's results were extended to variable coefficients by Hadamard in a famous book, The Cauchy Problem and Hyperbolic Linear Partial Differential Equations, published in 1923 with a French edition in 1932. He replaced the wave operator by an operator

$$P=\Sigma\, g_{ik}(x)u_{ik} + \text{lower terms}$$

where $g=(g_{ik})$ is a symmetric $n\times n$ matrix with Lorentz signature, one plus and the rest minus, and the indices of u indicate second order derivatives with respect to the variables $(x_1,\ldots,x_n)$. The light rays of the wave equation are replaced by the extremals of the indefinite metric corresponding to g. If $d(x,y)$ denotes the square of the corresponding distance from x to a given point y, $d(x,y)=0$ is the equation of a two-sheeted conoid with its vertex at y. For a given half H of the corresponding cone, Hadamard constructed a fundamental solution $E(x,y)$ with pole at y, $PE(x,y)=\delta(x-y)$, with the following properties. It vanishes outside H and behaves inside H as in the constant coefficient case except that the vanishing inside the conoid when n is even (i.e. odd in the previous notation) is replaced by a smooth-behavior up to the boundary. Hadamard guessed the shape of the fundamental solution in the form of an asymptotic series. To verify that the construction yielded a fundamental solution, Green's formula had to be used. This led to difficulties with the singularities on the cone were avoided by a limiting procedure called the method of the finite part. A few years later, Marcel Riesz (1937 and 1949) managed to replace it by a more palatable analytic continuation with respect to a parameter.

It seemed hopeless to extend Hadamard's method to higher order hyperbolic equations. Even an existence proof for Cauchy's problem and with it the existence of fundamental solutions presented problems. Petrovsky managed a very complicated existence proof in the thirties, but an easier one using functional analysis was found in the fifties by Gårding. Still, an analysis of the singularities of the fundamental solutions remained.

The break-through came in 1957 with a paper by Lax. He found out how to make the Fourier method work for variable coefficients by using general oscillatory integrals, ignoring low frequencies and keeping the high ones which are responsible for the singularities. The method was not new in itself. It had been used by physicists under the name of the geometrical optics approximation and, in quantum physics, the semi-classical approximation. But its use for hyperbolic systems and operators of high order was a novelty. The constructions of Hadamard and Lax shared one defect not present in the existence proofs: both were restricted to a neighborhood of the pole of the fundamental solution.

Lax's paper was one of the first that aroused the interest of mathematicians in the analysis of singularities of oscillatory integrals. The outcome has been a vast theory of prime importance whose ingredients are pseudodifferential operators, the notion of wave front set or singularity spectrum for an arbitrary distribution or hyperfunction and a theory of propagation of singularities. Its name is microlocal analysis and it has a host of applications. In one of them, Hörmander and Duistermaat succeded in making the construction of Hadamard and Lax global. The theory of microlocal analysis including many recent results are to be found in Hörmander's books The Analyis of Linear Partial Differential Operators I-IV (Springer 1983-85).

The aim of this series of lectures is to present the use of microlocal theory in the analysis if singularities in linear wave propagation, in the majority of cases represented by the fundamental solutions of linear hyperbolic partial differential and pseudodifferential operators. Chapter 1 deals with forward fundamental solutions of hyperbolic differential operators wit constant coefficients. It presents the theory of lacunas in a general form and has one application to a general form of conical refraction. Chapter 2 about oscillating integrals and wave front sets, Chapter 2 about pseudodifferential operators and Chapter 4 about symplectic geometry present known material necessary for the sequel dealing with the singularities of fundamental solutions of strongly hyperbolic operators and oscillating integrals in general. In Chapter 5 there is a new simple construction of a global parametrix of the fundamental solution of a first order pseudodifferential operator. This construction is basic since parametrices of strongly hyperbolic differential operators are sums of such parametrices paired in a certain way. The final chapters 6 and 7 give a detailed analysis of the singularities of such paired oscillatory integrals.

These lectures were delivered in April and May of 1986 at the Mahematics Institute of Nankai University, Tianjin. The author wants to take this opportunity to thank the Institute for its hospitality and his audience for its patience.

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CHAPTER 1

HYPERBOLIC OPERATORS WITH CONSTANT COEFFICIENTS

Introduction The main object of this chapter is to express the fundamental solutions of homogeneous hyperbolic differential operators as integrals of rational forms over certain cycles. This yields the Petrovsky condition for lacunas. The first step is a section on algebraic hyperbolicity. In a second section inverses of hyperbolic polynomials are studied. The third section deals with intrinsic hyperbolicity, the fourth with fundamental solutions and in the fifth a formula by Gelfand is used to derive the desired results.

1.1 Algebraic hyperbolicity

Let $f(x)=f(x_1,...,x_n)$ be analytic for small x and let $a \neq 0$ be fixed in R^n.

Definition The function f is said to be microhyperbolic with respect to a if

$$\text{Im } t > 0 \Rightarrow f(x+ta) \neq 0$$

for all sufficiently small t and all sufficiently small real x.

Let us develop f at x=0 in series of terms of increasing homogeneity,

$$f(x) = f_0(x)+f_1(x)+...+f_m(x)+... .$$

The first non-vanishing term, say f_m, is called the principal part of f and will be denoted by Pr f.

Examples. When m=0, $f(0) \neq 0$ and f is trivially microhyperbolic with respect to any a. When m=1 and f is real, f is locally hyperbolic with respect to any a with Pr $f(a)\neq 0$.

Lemma Put $h(t,s)= f(ta+sx)$ with small complex t and s. Then, if f is microhyperbolic with respect to a,

(1.1.1) $$h(t,s) =H(t,s) \prod (t+ d_k(sx,a))$$

where $H(t,s)$ is analytic at the origin, $H(0,0)\neq 0$ and the d_k are analytic for small s and vanish when s=0. If

(1.1.2) $$d_k(sx,a) = c_k(s) + \text{higher terms},$$

the numbers c_k are real and the principal part of $h(t,s)$ is

(1.1.3) $$H(0,0) \prod (t+c_k s) = \Pr f(ta+sx).$$

Note. When f is microhyperbolic with respect both a and -a, it is said to be locally hyperbolic with respect to a. It follows from (3) that Pr f has this property when f is microhyperbolic with respect to a. When f is locally hyperbolic with respect to a, the numbers d_k are real for real x and s and hence $f(x)/\Pr f(a)$ is real.

Proof. Choose an x such that Pr $f(x)\neq 0$. Then the principal parts of f and $h(t,s)$ are the same. Disregarding the definition of m, let m be the least k for which $g_k(0)\neq 0$ in the expansion

(1.1.4) $$h(t,s) = g_0(s) + tg_1(s) +\dots t^k g_k(s)+\dots$$

for small s and t. Without loss of generality, we may also assume that m>0. Then, by the properties of power series, (1) holds, the d_k being Puiseux series in s. But since $h(t,s)\neq 0$ when s is real and t is small with Im t >0, these series are actually power series. In fact, the existence of a first term of the Puiseux series with a fractional exponent is easily seen to contradict this assumption. Hence the degree of Pr $h(s,t)$ is m and equal to that of Pr f. In particular, $H(0,0)$ = Pr $h(1,0)$ = Pr $f(a)$ does not vanish, a statement independent of the assumption that Pr $f(x)$ does not vanish. We can then go to (4) again without requiring that Pr $f(x)\neq 0$ and are then sure that $g_m(0)$ does not vanish so that (1) and (2) follow, the formula (3) being a consequence of these two. This finishes the proof.

Let us note that

(1.1.5) $$\Pr f(x) = H(0,0) \prod c_k.$$

for all x. In the sequel we shall assume that $\Pr f(a)=1$.

Definition Let $C(f,a)$, called the hyperbolicity cone of f, be the component of the complement of the real hypersurface $\Pr f(x)=0$ which contains a.

According to (5) this means that x is in $C(f,a)$ precisely when all $c_k = c_k(a,x) > 0$ on $C(a,f)$. In fact, when x=a, all the numbers c_k are 1.

Theorem $C(f,a)$ is an open convex cone. If K is a compact part of it, f is uniformly locally hyperbolic with respect any b in K. More precisely, there is a positive number A such that

(1.1.6) b in K, $|s|, |x| < A$, $\operatorname{Im} s > 0 \Rightarrow f(x+sb) \neq 0$.

Proof. Let us write the formula (2) with x replaced by x+sb and with ta and sb interchanged,

$$f(ta+sb+x) = H(t,s,x) \prod (s + d_k(b,x+ta)).$$

Here $H(t,s,x)$ does not vanish for sufficiently small arguments. Further, since the left side does not vanish when s is real and Im t>0, none of the numbers d_k crosses the real axis. Hence, since

$$d_k(a,sb) = c_k s + \text{smaller}$$

where the c_k are positive, we have

$$\operatorname{Im} t > 0 \Rightarrow \operatorname{Im} d_k(b,x+ta) > 0.$$

when x and s are small enough. Hence the second part of the theorem follows. To prove the first part, note that

$$\Pr f(ta+sb) = \Pr f(a) \prod (t + sc_k(a,b)).$$

It follows that $C=C(a,f)$ contains ta+sb when b is in C and $t,s > 0$. This completes the proof.

Translates

For small real y, let

$$f_y(x) = f(x+y)$$

be the translate of f by y. Our last theorem has the following corollary

Theorem . If f is microhyperbolic with respect to a, so is f_y. The function

$$y \to C(f_y,a)$$

is inner continuous in the sense that if y tends to z, then the right side above contains any compact subset of C(f,a) when z is sufficiently close to y.

Proof. It suffices to prove the theorem when z=0 in which case it follows from the previous one.

Homogeneous hyperbolic polynomials.

When f(x) has a principal part Pr f of order m, then

$$r \to 0 \Rightarrow r^{-m}f(rx) \to \Pr f(x).$$

It follows from this that, if f is microhyperbolic with respect to a, then P(x) = Pr f(x) is a polynomial, homogeneous of order m with the property that

(1.1.7) $\quad \operatorname{Im} s>0,\ x \text{ real} \Rightarrow P(sa+x) \neq 0.$

Such polynomials are said to be hyperbolic with respect to a. The set of those will be denoted by Hyp(a,m). Note that since P is homogeneous, (7) holds with Im s>0 replaced by Im s≠0 so that P is also hyperbolic with respect to -a. It is obvious that if two homogeneous polynomials P,Q are hyperbolic with respect to a, then PQ has the same property and $C(PQ,a)=C(P,a) \cap C(Q,a)$.