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Edited by A. Dold and B. Eckmann

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Rational Representations of Algebraic Groups

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Rational Representations of Algebraic Groups:

Tensor Products and Filtrations

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Author

Stephen Donkin
School of Mathematical Sciences, Queen Mary College
Mile End Road
London E1 4NS, England

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To My Mother.

Table of Contents

Introduction

Let G be a connected, affine algebraic group over an algebraically closed field and B a Borel subgroup of G. For each one dimensional rational B-module L we have the induced G-module $Ind_B^G L$. These modules are of fundamental importance in the representation theory of G; in characteristic 0 one obtains every simple rational G-module in this way and in arbitrary characteristic $Ind_B^G L$, when non-zero, has a simple socle and each simple G-module occurs as the socle of some such induced module. The formal character of $Ind_B^G L$ is independent of the characteristic, being given by Weyl's Character Formula, but the submodule structure depends very heavily upon characteristic and very little is known about this structure in characteristic p. For G semisimple, the induced modules also have an interpretation as the global sections of line bundles on the quotient variety G/B and so provide a bridge between the representation theory of G and the geometry of G/B.

We make the following key definition. A good filtration of a rational G-module V is an ascending chain of submodules $0 = V_0, V_1, V_2, \ldots$ of V such that V is the union of the V_i and, for each $i > 0$, V_i/V_{i-1} is either 0 or isomorphic to $Ind_B^G L$ for some rational one dimensional B-module L. It was shown in [25] that for G semisimple and simply connected each rationally injective indecomposable G-module has a good filtration. In this monograph we study, for a connected, affine algebraic group G over an algebraically closed field k, the following hypotheses.

Hypothesis 1 *For all rational* G-*modules* V,V' *which have a good filtration the tensor product* $V \otimes V'$ *has a good filtration.*

Hypothesis 2 *For every rational* G-*module* V *which has a good filtration and every parabolic subgroup* P *of* G *the restriction of* V *to* P *has a good filtration.*

The hypotheses are mainly of interest when G is semisimple (we reduce to the semisimple, simply connected case in Chapter 3) and hypothesis 1 has been considered in this case by Wang Jian-pan, [52]. In that paper hypothesis 1 is shown to be true when G is of type A, for the other classical groups when k has characteristic $p \geq h-1$ (h is the Coxeter number of G), for $p \geq 5$ in type G_2, $p \geq 31$

in type F_4 , $p \geq 29$ in type E_6 , $p \geq 59$ in type E_7 and $p \geq 151$ in type E_8 . We show that both hypotheses are satisfied provided that either the characteristic of k is not 2 or G involves no component of type E_7 or E_8 . There are many technical advantages - which will emerge in due course - in allowing G to be an arbitrary (rather than semisimple) connected algebraic group and also in considering both hypotheses together.

Apart from the intrinsic appeal, there are several good reasons for trying to establish the hypotheses. It is enough to do this with $V = Ind_B^G L$, $V' = Ind_B^G L'$ for L and L' one dimensional B-modules. Thus we are attempting to say something about the tensor product of induced modules and the restriction of an induced module. Viewed from this standpoint what we are trying to do is to find substitutes, in the context of algebraic groups, for the Mackey Tensor Product Theorem and the Mackey Subgroup Theorem for finite groups ((44.2) and (44.3) of [21]). The search for Mackey type theorems is particularly tantalizing since one has, by the Bruhat decomposition (28.3 Theorem of [35]) an especially good indexing set for the double cosets of B in G , namely the elements of the Weyl group. As noted by Wang Jian-pan, [52], it follows from hypothesis 1 that the natural map $\Gamma(G/B,L) \otimes \Gamma(G/B,L') \to \Gamma(G/B,L \otimes L')$, between global sections of dominant induced line bundles on G/B , is surjective. When G is classical, that is G has type A, B, C or D, this is a result of Lakshmibai, Musili and Seshadri, [44] - we also obtain the result for F_4 , G_2 , E_6 in arbitrary characteristic and for E_7 and E_8 if the characteristic is not 2 . It seems likely that hypothesis 2 will be useful in the representation theory of reductive groups. We give a small application, in section 11.3 to homomorphisms between Weyl modules. Hypothesis 2 does seem to provide real insight into these modules and, together with hypothesis 1, makes possible many cohomological calculations. One can see this throughout the text but Chapter 7, treating F_4 , is particularly rich in examples of this kind. We also hope that, when k has prime characteristic, hypothesis 1 will be useful in connexion with the structure of the modules $St_n \otimes Y(\lambda)$ (St_n denotes the n^{th} Steinberg module and $Y(\lambda)$ is the G-module induced from a one dimensional B-module λ). Components of these modules and their duals are frequently used ([10], [19], [24], [26], [36], [37], [41], [42]) to compare the representation theory of G with that of its infinitesimal subgroups.

When G is semisimple, the dual of a module induced from a one dimensional B-module is a Weyl module (see section 1 of [40]). The first time that I became aware of the hypotheses was in conversation with Humphreys and Jantzen at the Durham Symposium on Finite Simple Groups in 1978, where we discussed, for semisimple groups, hypothesis 1 in its dual formulation in terms of Weyl modules.

Before giving a synopsis of each chapter we briefly describe the overall strategy of the proof. We show that the hypotheses hold for a connected group G if and only if they hold for the quotient of G by its soluble radical. Moreover the hypotheses hold for a semisimple group G provided that they hold for the semisimple, simply connected group of the same type. Also, the hypotheses hold for a direct product of groups if and only if they hold for each factor. Thus we may assume G to be semisimple, simply connected with an indecomposable root system and by induction we may assume that the hypotheses hold for every proper parabolic subgroup. It suffices to prove that, for each fundamental dominant weight λ_i ($i = 1,2,\ldots,\ell$ where ℓ is the rank of G) and each parabolic subgroup P, $Y(\lambda_i)|_P$ has a good filtration and, for each rational G-module V with a good filtration, $Y(\lambda_i) \otimes V$ has a good filtration. The first property is proved initially for P of largest possible dimension, so that the number of successive quotients in a good filtration of $Y(\lambda_i)|_P$ will be as small as possible (and assuming the result for $j < i$ in the classical case). Then it is proved for an arbitrary maximal parabolic subgroup Q, such that P and Q contain a common Borel subgroup, by examining the effect of restriction from G to $P \cap Q$ followed by induction from $P \cap Q$ to Q.

In Chapters 1, 2 and 5 we deal with general results on group cohomology and the derived functors of induction which are needed for the specific calculations in Chapters 4, 6, 7, 8, 9 and 10. The main purpose of Chapter 1 is to establish the notation and explain the relationship between various left exact functors. Chapter 2 contains results relating the cohomology of some modules for parabolic subgroups to the cohomology of other modules for other parabolic subgroups. This Chapter also contains a deduction from Kempf's Vanishing Theorem of Weyl's Character Formula for the character of $Ind_B^G L$ for a reduction group G and one dimensional B-module L. The formula is used extensively in the later calculations and we were unable to find a suitable reference (for reductive groups). The proof of Weyl's Character Formula given is quite short, it is not obtained by reduction from characteristic zero (c.f. section 1 of [40]) and, as in the paper [48] treating essentially the semisimple characteristic zero case, deals directly with the group rather than the Lie algebra (not even the Casimir operator is used). In deriving the results in Chapter 2 (and elsewhere) we use three different cases of the Grothendieck spectral sequence (section 2.4 of [30]) relating the derived functors of the composite of two left exact functors to the composites of the derived functors. The first application arises from the expression of Ind_H^G (the induction functor from a closed subgroup H to G) as the composite $Ind_K^G \circ Ind_H^K$ (transitivity of induction) for a closed subgroup K containing H. The second application is the Lyndon-Hochschild-Serre spectral sequence expressing G cohomology in terms of the cohomology of a closed normal subgroup N and the quotient G/N. The final application arises from the expression of the fixed point functor F_H, from rational H-modules to k-spaces, as $F_G \circ Ind_H^G$ (reciprocity of induction).

In Chapter 3 we make various reductions to the hypotheses so that they become susceptible to the case by case analysis which follows.

In Chapter 4 we prove the hypotheses for the classical groups. The argument here is independent of the characteristic and it has been possible to treat the groups of type B, C and D in a unified manner. Fortunately the restriction of $Y(\lambda_i)$ to a proper parabolic subgroup of maximal dimension has only 4 successive quotients in a good filtration (for i in "general position") and the module structure is much the same in all three types B, C and D.

Some additional homological algebra is needed to deal with the exceptional groups and this is given in Chapter 5. The hypotheses are proved for G_2 in Chapter 6 however it would not be difficult to treat this case without the benefit of Chapter 5. In Chapter 7, treating F_4, we found it necessary to consider separately the cases of odd and even characteristic. It takes just 11 pages to deal with F_4 for characteristic $p \neq 2$ but needs another 33 pages to give the additional arguments necessary to cover the case $p = 2$ (this may be omitted at a first reading). Briefly, the reason why $p = 2$ is a special case is that the exterior square of a module is not usually a summand of the tensor square. And the reason why we manage to prove the hypotheses in characteristic 2 is that we are able to use Andersen's strong linkage principle, [4], to analyse components of the tensor product of induced modules, the point being that, by Chapter 3, a summand of a module with a good filtration has a good filtration.

The subject, E_6, of Chapter 8 is altogether easier but again there is a division in the proof into odd and even characteristics.

Chapters 9 and 10 are devoted to the remaining exceptional groups E_7 and E_8. The procedure here is to analyse first the modules $Y(\lambda_i)$, corresponding to the terminal vertices α_i of the Dynkin-Diagram, and then exterior powers of these modules are used to deal with $Y(\lambda_r)$ for an arbitrary fundamental dominant weight λ_r. In fact one only ever needs to go as far as the fourth exterior power and the second stage of the procedure works very smoothly provided that the characteristic p is at least 5 (because the exterior power is a summand of the tensor power in this case). In characteristic 3 a consideration of exterior powers is insufficient and some block theory is needed. We have unfortunately not been able to prove the hypotheses in characteristic 2. In the case of E_7 and $p = 2$ we have at least, in Chapter 9, a satisfactory analysis of $Y(\lambda_i)$ for α_i a terminal vertex but for E_8 (in characteristic 2) we have not given an analysis of $Y(\lambda_8)$ (see section 8.1 for the labelling of the Dynkin-Diagrams of type E). Moreover we have no way of going from the terminal $Y(\lambda_i)$ to an arbitrary $Y(\lambda_r)$. At several points in the text (notably in sections 9.3 and 10.2) it is necessary to know that various dominant weights belong to different linkage classes (so the corresponding modules belong to different blocks) and this involves many routine but lengthy calculations which we have omitted. These calculations were done by hand and later checked, on a Sinclair ZX 81 Microcomputer, by J.F. Blackburn to whom I am extremely grateful.

Chapter 11 opens with an example of a reductive subgroup H of a reductive group G and a G-module V such that V has a good filtration but the restriction of V to H does not. The remainder of the chapter is devoted to applications of the hypotheses to rational cohomology, homomorphisms between Weyl modules, canonical products on induced modules and filtrations over $\mathbb{Z}$ of Weyl modules for Kostant's $\mathbb{Z}$-form $\mathcal{U}_{\mathbb{Z}}$ of the enveloping algebra $U(\underline{g})$ of a complex semisimple Lie algebra $\underline{g}$.

The final chapter is given over to a number of observations on issues not directly concerned with the hypotheses but which nevertheless have mainly arisen in the course of our work on the hypotheses. The issues discussed are the injective indecomposable modules for a parabolic subgroup of a reductive group, Kempf's Vanishing Theorem for rank 1 groups, Kempf's Vanishing Theorem in characteristic zero and the exactness of induction.

These notes are a substantially rewritten version of a winning Adams Prize Essay in Algebra for 1981/2. In content the notes differ from the essay only in that a proof of the hypotheses in the cases E_8 with $p = 3$ and 7 has been added and that the final chapter is new.

I wish to thank Tracy Kelly for the excellent job she has done in typing my manuscript.

1. Homological Algebra

1.1 Induction

We recall the induction functor for algebraic groups discussed more fully in [18].

Let G be an affine algebraic group over an algebraically closed field k. By a rational G-module we mean a left kG-module which is the union of its finite dimensional submodules and such that for each finite dimensional submodule W the induced map $G \to GL(W)$ is a morphism of algebraic groups. A morphism of rational G-modules is simply a kG-module homomorphism. We denote by M_G the category of rational G-modules.

Let H be a closed subgroup of G. For $V \in M_H$ we denote by $Map(G,V)$ the set of maps $f:G \to V$ such that the image $f(G)$ lies in a finite dimensional subspace W of V and the induced map $f:G \to W$ is a morphism of varieties. For $x \in G$, $f \in Map(G,V)$ the map $xf:G \to V$, defined by $(xf)(y) = f(yx)$ for $y \in G$, also belongs to $Map(G,V)$. Thus $Map(G,V)$ is naturally a G module which is, as one may easily verify, rational. We set

$$Map_H(G,V) = \{f \in Map(G,V) \mid f(hx) = hf(x) \text{ for all } h \in H,\ x \in G\}.$$

The space $Map_H(G,V)$ is in fact a G submodule of $Map(G,V)$ called the induced module and more often written as $Ind_H^G V$. If $\phi:V \to V'$ is a morphism of rational H modules then the map $Ind_H^G \phi:Ind_H^G V \to Ind_H^G V'$ defined by $Ind_H^G \phi(f) = \phi \circ f$ is a G homomorphism and $Ind_H^G:M_H \to M_G$ is a left exact functor.

We now note for future reference the main properties of induction and of the derived functors of induction. Proofs of these properties of induction may be found in [18] (see also [23] for a more Hopf theoretic treatment). For a left exact functor $F:A \to B$ between abelian categories with A having enough injectives we denote by R^nF $(n \geq 0)$ the derived functors (defined by means of injective resolutions in A). For $V \in M_H$ let $e_V:Ind_H^G(V) \to V$ be the H-module map defined by $e_V(f) = f(1)$.

(1.1.1) *For each* $n \geq 0$, $R^n Ind_H^G$ *commutes with direct sums and direct limits.*

For $n = 0$ this follows fairly directly from the definition of induction. The argument involved in extending the result to arbitrary $n \geq 0$ is that used in Proposition 2.9, Ch.III of [32] to prove that sheaf cohomology on a Noetherian space commutes with direct limits.

(1.1.2) (Reciprocity of induction) *For any* $V' \in M_G$ *the map* $Hom_G(V', Ind_H^G(V)) \to Hom_H(V'|_H, V)$ *taking* θ *to* $e_V \circ \theta$ *is a k-isomorphism.*

(1.1.3) (Transitivity of induction) *For a closed subgroup* K *containing* H , Ind_H^G *is naturally isomorphic to* $Ind_H^G \circ Ind_H^K$.

(1.1.4) Ind_H^G *takes injective objects to injective objects;* $Ind_H^G(k[H])$ *is isomorphic to* $k[G]$.

The coordinate ring $k[G]$ is equal to $Map(G,k)$ and so has the structure of a rational left G-module as above. Taking $H = 1$, the identity subgroup, we see from (1.1.2) that $Ind_1^G(k) = k[G]$ is an injective object in M_G , moreover a rational G-module is injective if and only if it is a direct summand of a direct sum of copies of $k[G]$ (see section 1.5 of [29]) so that the first part of (1.1.4) is a consequence of the second part. The second part is a consequence of (1.1.3).

(1.1.5) *For a closed subgroup* K *containing* H *there is a Grothendieck spectral sequence* $E_r^{\ell,m}$ *converging to* $(R^n Ind_H^G)V$ *with* $E_2^{\ell,m} = (R^\ell Ind_K^G) \circ (R^m Ind_H^K)V$.

The existence of the Grothendieck spectral sequence (see section 2.4 of [30]) is guaranteed by the left exactness of induction, (1.1.3) and (1.1.4).

In particular we have the following.

(1.1.6) *Let* $V \in M_H$ *and* $n \in \mathbb{N}$ *be such that* $(R^m Ind_H^K)V = 0$ *for* $m \neq n$. *Then* $(R^m Ind_H^G)V$ *is zero for* $m < n$ *and is isomorphic to*

$(R^{m-n}Ind_K^G) \circ (R^n Ind_H^K)V$ *for* $m \geq n$.

(1.1.7) (Tensor identity) *For* $V' \in M_G$, $V \in M_H$ *and* $n \geq 0$, $(R^n Ind_H^G)(V' \otimes V) \cong V' \otimes (R^n Ind_H^G)V$.

For $n = 0$ see [18]. It may be proved for arbitrary $n \geq 0$ by dimension shifting or a Grothendieck spectral sequence argument.

(1.1.8) *There is a Grothendieck spectral sequence* $E_r^{\ell,m}$ *converging to* $H^n(H,V)$ *with* $E_2^{\ell,m} = H^{\ell}(G, R^m Ind_H^G V)$ *, where* H^n *denotes Hochschild cohomology* (see [33]).

Let $\mathcal{D} = Ind_H^G$, let $E : M_G \to \underline{k\text{-sp}}$ (the category of k-vector spaces) be the fixed point functor and let $F : M_H \to \underline{k\text{-sp}}$ be the H fixed point functor. Taking V to be the trivial one dimensional G-module in (1.1.2) gives a natural isomorphism $E \circ \mathcal{D} \to F$. Moreover E , $\mathcal{D}$ are left exact and $\mathcal{D}$ takes injective objects to acyclic objects (by (1.1.4)) so there is a Grothendieck spectral sequence (section 2.4 of [30]) as required.

In particular we have the following.

(1.1.9) *Let* $V \in M_H$ *and* $n \in \mathbb{N}$ *be such that* $(R^m Ind_H^G)V = 0$ *for* $m \neq n$. *Then* $H^m(H,V)$ *is* 0 *if* $m < n$ *and is isomorphic to* $H^{m-n}(G,(R^n Ind_H^G)V)$ *for* $m \geq n$.

Combining (1.1.9) and (1.1.7) one obtains:

(1.1.10) *Suppose that* $V_1 \in M_G$, $V_2 \in M_H$ *and* $n \in \mathbb{N}$ *such that* $(R^m Ind_H^G)V_2 = 0$ *for* $m \neq n$. *Then* $H^m(H, V_1 \otimes V_2)$ *is* 0 *if* $m < n$ *and is isomorphic to* $H^{m-n}(G, V_1 \otimes (R^n Ind_H^G)V_2)$ *if* $m \geq n$.

We call (G,H) a vanishing induction pair (VIP for the sake of brevity) if the trivial one dimensional module k satisfies $(R^n Ind_H^G)k = 0$ for $n > 0$ and is k for $n = 0$. From (1.1.7) we obtain: