EQUALITIES AND APPROXIMATIONS

WITH FORTRAN PROGRAMMING

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with fortran programming

dedicated to Winslow S. Parkhurst

PREFACE

This textbook is designed for an enrichment course in mathematics and contains material not regularly given in a secondary-school course or in any traditional first-year course in college.

It is hoped that a course based on this textbook would give a student a broader foundation in mathematics and better prepare him for the calculus; a course in trigonometry would be a necessary prerequisite.

A unique feature is the option of integrating Fortran programming for an electronic computer into the complete course for all the types of problems considered. The book is self-contained and requires no prior knowledge of programming or computers.

However, all of the work on Fortran may be omitted without affecting the continuity of the material in any way.

The material in this textbook has been given by the author in an enrichment program for high-school students since 1961. It also was the basis of a course given in National Science Foundation Summer Institutes for High Ability Secondary-School Students in 1962 and 1963. It may be taught in one or two semesters, depending upon the depth of coverage undertaken, especially in the use of the computer.

I would like to acknowledge the support given me by the National Science Foundation under whose grant NSF-G21180, in the Cooperative College-School Science Program, the original lecture notes were prepared which formed the basis of my book.

I express my deep appreciation to Professor John M. Perry, Clarkson College of Technology, Professor Seymour Schuster, Carleton College, and Professor Roy Dubisch, University of Washington, for their many helpful comments and suggestions.

This book was written while I was professor of mathematics at Clarkson College of Technology.

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ROBERT D. LARSSON

CONTENTS

1 GROUPS

1. Sets and Equations

The concept of a set is of great usefulness in mathematics.

Definition. A set is a collection of numbers, symbols, points, objects, etc., called elements, which have some distinguishing characteristics in common.

We speak of the set of even integers, the set of positive integers, the set of books on a shelf, the set of rational numbers, the set of people in a room, just to cite a few examples.

Now given two elements b and c belonging to a set S, one can ask whether there is an element x, belonging to the set S, such that

$$bx = c.$$

But immediately we have introduced another concept. What do we mean by the expression bx? Is it multiplication in the ordinary sense of the word? If so, then the elements in the set S under consideration must be such that they can be combined by ordinary multiplication.

For example, if our set S was composed of the men in a room, we could not combine one man with another man and obtain a third man. But if our set S was the set of committees made up from men in a room, and b and c each represented a committee, then it might be possible to combine committee b with some other committee x and obtain a committee identical to committee c. However, it also might be impossible.

Thus we see that in any equation one must define not only the set S whose elements appear in the various expressions in the equation, but the operation or operations by which the elements may be combined as well.

We can illustrate this in the following examples:

Given the set S of odd integers and the operation of multiplication, does the equation

$$3x = 15$$

have a solution? In other words, is there an element called x belonging to the set of odd integers, such that the left-hand side of the equation could replace the right-hand side? The answer is obviously yes, since the element x can be selected as the element 5, and 5 is an odd integer.

But suppose that we are given the set of odd integers and the operation of addition, does the equation

$$3 + x = 15$$

have a solution? The answer is no. There is no odd integer, x, which can be added to 3 to give 15. We all know that x would have to be 12, which is not an odd integer.

The linear equation in two variables is written as

$$ax + by = c.$$

If our set S is the set of integers and our operations are multiplication and addition, the linear equation

$$2x + 4y = 3$$

has no solution—where by a solution we mean values of x and y which belong to the set S. While, for example, it is true that $x = 1$ and $y = \frac{1}{4}$ satisfies the equation, $\frac{1}{4}$ is not a member of the set S. Can you see why no solution is possible in S? (Hint: $2x + 4y = 2(x + 2y)$.)

On the other hand, the equation

$$3x + 5y = 1$$

has an infinite number of solutions. It can be verified by direct substitution that

$$x = 2 + 5t$$

and

$$y = -1 - 3t$$

are solutions for all integral values of t.

There are many practical problems of this nature, since in production of various articles there can be no consideration given to part of an article, such as half of a book or a third of a car.

Consider the general quadratic equation

$$ax^2 + bx + c = 0,$$

where a, b, c are defined as coefficients and must all be members of the same set S. Our question is whether or not there is an element x belonging to S which satisfies the equation under the operations as defined.

If we select S as the set of integers and the operations as ordinary addition and multiplication, we can consider the following examples.

$$x^2 + 2x + (-3) = 0.$$

This has two solutions, $x = -3$ and $x = 1$, which are both members of the set S.

But the equation

$$2x^2 + 5x + 2 = 0$$

has only one solution, $x = -2$, belonging to the set of integers. The other, $x = -\frac{1}{2}$, does not. However, if we had taken S as the set of rational numbers, namely all those which can be expressed as the quotient of two integers, our quadratic would have two solutions belonging to the set S.

If we take S as the set of integers, the quadratic

$$x^2 + 2x + 5 = 0$$

has no solutions. If we take S as the set of all real numbers, it still has no solutions belonging to the set S. But if we extend S to include all complex numbers, those of the form $a + bi$, where a and b are real numbers, then the quadratic has two solutions. They are $-1 - 2i$ and $-1 + 2i$, where $i^2 = -1$.

It becomes evident that just because the coefficients in a given equation belong to some set S, the solutions need not belong to the same set S. The operations involved are part of the key as well.

Exercises

1. (*a*) Given the equation

$$2 + x = 12,$$

find at least three sets of numbers to which the elements 2 and 12 belong and to which x belongs also.

(*b*) Change the operation to multiplication and do the problem.

2. (*a*) Given the equation

$$3x = 5,$$

find at least two sets of numbers to which the elements 3 and 5 belong and to which x does not belong.

(*b*) Change the operation to addition and do the problem.

3. Give a set of numbers which contains three other sets of numbers. Define each subset carefully.

4. Given a set of ten men, form them into three committees A, B and C such that

$$A + B = C,$$

where + means "combined with".

5. Given the linear equation

$$2x + 3y = 5,$$

find two solutions in the set of integers. Find two solutions in another set, to which 2, 3 and 5 also belong.

6. Why does the linear equation

$$2x + 4y = 3$$

have no solutions in the set of integers?

7. Form a quadratic equation whose coefficients belong to the set of integers, which has two distinct solutions in that set; no solutions in that set; only one solution in that set.

8. In problem 7, to what other sets do your solutions belong in each case? Do the coefficients belong to the same sets?

2. Properties of a Group

In order to develop a better foundation for the study of equations, we shall begin with the simplest example.

$$b + x = c. \tag{1}$$

We shall ask what the sufficient conditions must be on a set S, containing b and c, in order that there shall exist a unique element x belonging to S and satisfying equation (1).

Now consider the set of even integers and some further properties of this set. If we add two even integers, such as 4 and 12, we obtain the even integer 16 as the sum. And the sum of 10 and 14 is 24, still another even integer. Although many examples may appear to verify some mathematical observation, such as the sum of two even integers is an even integer, one has no proof unless he has been able to exhaust all cases.

Since any even integer, by definition, is divisible by 2, we may write it in the form $2n$, where n is some integer. Then we have

$$2n + 2m = 2(n + m)$$

for the sum of two such even integers. But the sum of two integers is an integer, say p, and we obtain

$$2n + 2m = 2p.$$

This proves the assumption that the sum of any two even integers is an even integer. This leads us to an important definition.

Definition. A set of elements, with some prescribed operation by which any two of these elements may be combined, is said to be closed under that operation if the resulting combination is a member of the set.

Note that both the set and the operation must be prescribed. For example, the set of even integers is closed under multiplication also but not under division, since 2 divided by 2 is 1.

Note also that one counter example is enough to prove a statement false, while a thousand examples will not prove it true unless all the possibilities have been exhausted.

Other examples of closure which come to mind are the set of odd integers under multiplication, the set of all integers under addition, the set of positive rational numbers under division.

The even integer 0 has a special property under addition. For example, 0 plus 4 is 4, and 0 plus 30 is 30. In fact we see that

$$0 + 2n = 2n$$

for all n. Because of this property, the integer 0 is called the identity element under addition for the set of even integers.

We define the identity element as follows.

Definition. An element i belonging to the set is called the identity element if, under the operation, the combination of i with any other element b belonging to the set is the element b.

For the set of odd integers under multiplication, the number 1 is the identity element. Furthermore, it is evident that not all sets have an identity element under an operation. For example, consider the set of even integers under multiplication; 1 is not a member of the set.

To continue with the properties of the even integers, we note that with each even integer there is associated another even integer such that the sum under addition is the identity element. For example, 6 and -6 have a sum of 0. In general we show that

$$(-2n) + (2n) = 0$$

for all n. The element $-2n$ is called the inverse of the element $2n$. We make the following definition.

Definition. An element b^{-1} is called the inverse of an element b in the set if, under the operation, the combination of these two elements is the identity element.

Careful attention is directed here to the symbol b^{-1}. One should not think of the $^{-1}$ as being an exponent. The student is already familiar with other examples of this kind. In trigonometry we have

$$y = \sin x$$

and then, solving for x, we write

$$x = \arc \sin y$$

or

$$x = \sin^{-1} y,$$

where the $^{-1}$ is not an exponent, but denotes the inverse trigonometric function.

Unfortunately, we run out of convenient symbols many times and have to use duplicate ones. For example, the symbol (a, b) sometimes represents the coordinates of a point, sometimes the greatest common divisor of a and b and sometimes simply an ordered pair. One needs to be sure of the definition an author has made for such a symbol when he first used it.

Now we recall the associative law. If one is given three even integers, such as 2, 6 and 14 to add, it is evident that one can combine only two at a time. Then the sum of that pair is added to the remaining number. We have two ways of doing this:

$$(2 + 6) + 14 = 8 + 14 = 22$$

or

$$2 + (6 + 14) = 2 + 20 = 22.$$

We know that for all triples of integers this law, called the associative law, holds under both addition and multiplication. It does not hold under division, however, as observed in the following.

$$(2 \div 4) \div 14 = \tfrac{1}{2} \div 14 = \tfrac{1}{28}$$

$$2 \div (4 \div 14) = 2 \div \tfrac{2}{7} = 7$$

Note once again, and this point cannot be overemphasized, that most laws and properties hold under certain operations and not under others for a given set of elements. Not only must the set be defined but also the operation by which the elements are to be combined must be defined.

We define the associative law as follows.

Definition. Under addition, the associative law states that

$$(a + b) + c = a + (b + c)$$

and under multiplication, the associative law states that

$$(ab)c = a(bc).$$

We now define a group.

Definition. A set S of elements $a, b, c, \ldots$ forms a group, under an operation, say $\oplus$, if:

1. The set S is closed under $\oplus$.
2. There is a member of the set S called i such that $i \oplus a = a$ for all a in the set S.
3. The associative law holds.
4. Corresponding to each element b in the set S there is an element b^{-1} in S such that

$$b^{-1} \oplus b = i.$$

It is evident that the set of even integers under addition forms a group, as does the set of all integers under addition. We assume that the associative law holds for all integers under addition and multiplioation and that the set is closed as well.

Exercises

In the following assume that the associative law holds for all real numbers under addition and multiplication.

1. Prove that the set of all integers under addition forms a group. Does it under multiplication?
2. Prove that the set of rational numbers, except 0, forms a group under multiplication.
3. Does the set of numbers of the form $a + b\sqrt{3}$, with a and b integers, form a group under addition?
4. Is the set of numbers of the form $a + b\sqrt{3}$, a and b integers, closed under multiplication?
5. Is the set of numbers of the form $a + b\sqrt[3]{3} + c\sqrt[3]{9}$, a and b integers, closed under multiplication?
6. What is the inverse of the complex number $a + bi$, a and b real numbers, under addition? Under multiplication?
7. Given the three elements a, b and c with the following multiplication table (Table 1), prove that they form a group under multiplication:

Table 1

$\times$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

8. Construct a group of four elements a, b, c, d.

3. Some Basic Theorems

Now we shall prove some fundamental theorems concerning groups. We shall answer such questions as whether $i \oplus a = a \oplus i$. In other words is a left identity also a right identity? Also, is the identity unique? But, and most important of all, we shall determine under what conditions an equation involving only one operation can be assured of having a unique solution.

For convenience we introduce the use of the "member of" sign $\in$ as follows. If a belongs to the set S, we write $a \in S$.

If elements a, b and c belong to a set S, we write $a, b, c \in S$.

We shall use the capital letter G to refer to a group.

Theorem 1. If $a, b, c \in G$ and

$$c \oplus a = c \oplus b$$

then $$a = b.$$

PROOF. By (4) of our definition of a group, there exists an element $c^{-1} \in G$ such that $c^{-1} \oplus c = i$.

	$c^{-1} \oplus (c \oplus a) = c^{-1} \oplus (c \oplus b)$	Equals added to equals and closure.
$\therefore$	$(c^{-1} \oplus c) \oplus a = (c^{-1} \oplus c) \oplus b$	Associative law holds.
$\therefore$	$i \oplus a = i \oplus b$	Property of inverse.
$\therefore$	$a = b$	Property of identity.

Theorem 1 is called the left-hand cancellation law.

Theorem 2. If $b \in G$, then

$$i \oplus b = b \oplus i = b.$$

PROOF. By (4) of our definition of a group, there exists an element $b^{-1} \in G$.

	$b^{-1} \oplus (b \oplus i) = (b^{-1} \oplus b) \oplus i$	Associative law and closure.
	$(b^{-1} \oplus b) \oplus i = i \oplus i$	Property of an inverse.
	$i \oplus i = i = b^{-1} \oplus b$	Property of an identity and inverse.
$\therefore$	$b^{-1} \oplus (b \oplus i) = b^{-1} \oplus b$	Equals substituted for equals.
$\therefore$	$b \oplus i = b$	Theorem 1.

This proves that the left identity is a right identity.

Theorem 3. If $b \in G$, then

$$b^{-1} \oplus b = b \oplus b^{-1} = i.$$

PROOF.

	$b^{-1} \oplus (b \oplus b^{-1}) = (b^{-1} \oplus b) \oplus b^{-1}$	Associative law and closure.
	$= i \oplus b^{-1}$	Property of an inverse.
$\therefore$	$b^{-1} \oplus (b \oplus b^{-1}) = b^{-1} \oplus i$	Theorem 2.
$\therefore$	$b \oplus b^{-1} = i$	Theorem 1.

This proves that the left inverse is a right inverse.

Theorem 4. If $b, c \in G$, then the equations

(2) $b \oplus x = c$

and (3) $y \oplus b = c$

have unique solutions in G.

PROOF. Equation (2) has a solution in G,

$$x = b^{-1} \oplus c,$$

since

$$b \oplus (b^{-1} \oplus c) = c.$$

Suppose equation (2) has two solutions x and x'. Then

$$b \oplus x = b \oplus x' = c$$

$$\therefore \quad x = x'$$

by Theorem 1, and the solution is unique.

In the same way equation (3) may be proved to have a unique solution in G.

From Theorem 4, we may prove that the identity i is unique for a group and that each element has a unique inverse.

We can conclude from Theorem 4 that every equation in which the elements are members of a group G, and in which there is a single operation, namely the operation under which the group is defined, always has a unique solution.

Note that this says nothing about solutions of an equation of the form $ax = b + c$ which involves two operations. We shall investigate this later.

It should be remarked that all of the preceding theorems could have been proven just as easily using any symbol for the operation, rather than $\oplus$. This symbol was selected only as a convenience and was not meant to imply that it even meant addition under the ordinary sense of the word. Many texts use the symbol for multiplication, whatever that may be. We might have called it little o.

Exercises

1. Prove the right-hand cancellation law for a group.
2. Complete the proof of Theorem 4.
3. Prove that the identity element of a group is unique.
4. Prove that each element of a group has a unique inverse.
5. Prove Theorems 1–4 using multiplication as the operation.
6. Why is there no distributive law for a group?

4. Finite Groups

To correct the possible impression that groups always have an infinite number of elements, we offer the following examples.

Suppose that we divide the set of integers into classes, or subsets, such that each class consists of members whose smallest nonnegative remainders, after division by some positive integer, are the same. For example, consider division by the integer 4. The class of integers whose remainder after division by 4 is 0 would be

$$[\ldots, -8, -4, 0, 4, 8, 12, \ldots] = C_0.$$