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Günther J. Wirsching

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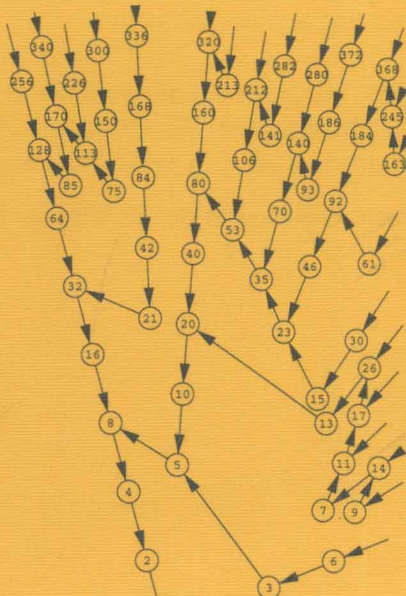

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THE DYNAMICAL SYSTEM ON THE NATURAL NUMBERS GENERATED BY THE $3n + 1$ FUNCTION

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INTRODUCTION

Among the most fascinating mathematical problems are those which are easily formulated, but withstand for a long time sophisticated attacks for solving them. One instance of this kind of problems is the by now famous $3n+1$ problem: Let $f(n) := n/2$, if n is even, and $f(n) := 3n+1$, if n is odd. Choosing a natural number x as starting number and applying f repeatedly produces a sequence of natural numbers, which is called *f-trajectory* of x and denoted by

$$\mathcal{T}_f(x) := (x, f(x), f(f(x)), \ldots, f^k(x), \ldots).$$

For example, taking $x = 13$ gives the f-trajectory

$$\mathcal{T}_f(13) = (13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \ldots)$$

which continues periodically with the cycle $(4, 2, 1)$. All f-trajectories which have been calculated up to now have this limiting behaviour, and there are many starting numbers which have been tested (see section I.3). This leads to the $3n+1$ *conjecture* which asserts that any f-trajectory eventually runs into the limiting cycle $(4, 2, 1)$. Is there a rigorous proof that this is the only possible limiting behaviour of a sequence of natural numbers generated by f?

Many authors consider the $3n+1$ conjecture as intractably hard—and they may well be right, as the problem is still open. On the other hand, the problem is not new in the mathematical literature. Its—somewhat foggy—origin dates back to the 1930's; but since the 1970's we observe a rapidly growing interest in this problem and mathematics which people consider to be connected to it. (*Proof*: see the bibliography at the end of these notes.) If the $3n+1$ conjecture itself appears to be intractable, what is, then, the mathematics people do around it, and is it really justified to claim that there is some progress towards a solution of the original problem? I do not plan to give an answer to this question in a few words, I just describe the facts and leave the final judgement to the reader.

The strategy for finding interesting things about an "intractable" problem is threefold: translate the conjecture into as many different contexts as you can, formulate weaker statements implied by the conjecture in question and try to prove some of them, and wait for flashes of genius giving new and interesting insights. In the case of the $3n+1$ problem, the conjecture has been reformulated, for instance, in terms of formal languages (see section I.12), and even in terms of analytic functions in the complex unit disk (see section I.13), leading to problems which seem as intractable as the original one. Following the second device, an "intermediate" conjecture is:

FINITE CYCLES CONJECTURE. *There are only finitely many cyclic numbers, i.e., the number of integers $y > 0$ such that $f^n(y) = y$ for some $n \in \mathbb{N}$ is finite.*

By now, this is also unproved. But there are some results in this direction: for example, R. P. Steiner proved in 1978, using a deep result of A. Baker on linear forms in logarithms, that there is just one cycle of a special type which had been called *circuit* (see section I.9).

A priori, an f-trajectory can either turn out to be eventually cyclic, or it must grow to infinity (this is due to the fact that f produces a *unique* successor to each number). Even the following consequence of the $3n + 1$ conjecture is unsolved.

(NO) DIVERGENT TRAJECTORY CONJECTURE. *There is no divergent $3n + 1$ trajectory, i.e., there is no $y \in \mathbb{N}$ such that $\lim_{n\to\infty} f^n(y) = \infty$.*

But, also in this case there is a partial result: J. C. Lagarias showed in 1985 that, if a divergent f-trajectory happens to exist, then it cannot grow too slowly (see section I.6).

* * *

The point of view on the $3n+1$ problem adopted here is based on a nice idea due to L. Collatz: he represented an arbitrary integer function $g : \mathbb{N} \to \mathbb{N}$, say, as a directed graph Γ_g with the domain $\mathbb{N}$ of g as infinite set of vertices, and with all pairs $(n, g(n))$ as directed edges. This graph derived from the function g is now called the *Collatz graph* of g. Taking $g := f$, with the integer function f defined above, we clearly have the following equivalence:

the $3n + 1$ conjecture holds $\iff$ the graph Γ_f is (weakly) connected.

For a general integer function $g : \mathbb{N} \to \mathbb{N}$, the *(discrete) dynamical system* on $\mathbb{N}$ *generated* by g consists of all possible g-trajectories. Now it is clear that any g-trajectory must remain in some weak component of Γ_g. Moreover, two g-trajectories $\mathcal{T}_g(x)$ and $\mathcal{T}_g(y)$ *coalesce*, i.e. there are integers $n, m \geqslant 0$ such that $g^n(x) = g^m(y)$, if and only if x and y belong to the same weak component of the graph Γ_g. Taking a fixed g-trajectory $\mathcal{T}_g(x)$, the *domain of attraction* of this trajectory consists of all starting numbers $y \in \mathbb{N}$ whose g-trajectory coalesces with $\mathcal{T}_g(x)$. So we infer that a domain of attraction of the dynamical system on $\mathbb{N}$ generated by g is just a (weak) component of Γ_g. This means: the study of a dynamical system on $\mathbb{N}$ is equivalent to the study of a Collatz graph.

The topic of interest here is the dynamical system on $\mathbb{N}$ which is generated by the $3n + 1$ *function*

$$T : \mathbb{N} \to \mathbb{N}, \qquad T(n) := \begin{cases} T_0(n) := n/2 & \text{if } n \text{ is even}, \\ T_1(n) := (3n+1)/2 & \text{if } n \text{ is odd}. \end{cases}$$

This function T replaces the function f defined above without loss of information: if n is even, then $T(n) = f(n)$, and if n is odd, then $T(n) = f(f(n))$ (as $3n + 1$

is even whenever n is odd). In this sense T "shortens" the f-trajectories; several authors prefer to deal with T instead of f. To study the dynamical system on $\mathbb{N}$ generated by T, we emphasize the *predecessor sets*

$$\mathcal{P}_T(a) := \{b \in \mathbb{N} : a \in \mathcal{T}_T(b)\} = \{b \in \mathbb{N} : \text{ some } T\text{-iterate of } b \text{ hits } a\}.$$

As any domain of attraction may be written as a union of predecessor sets, to study dynamical systems can mean to study predecessor sets. An interesting point about a predecessor set is any information refering to its *size*. In this number-theoretic setting, all information concerning the size of a set of natural numbers is contained in the *counting function* of that set. Here we consider counting functions of predecessor sets,

$$Z_a(x) := Z_{\mathcal{P}_T(a)}(x) := \big|\{n \in \mathcal{P}_T(a) : n \leqslant x\}\big|.$$

For technical reasons, it is easier to deal with predecessor sets of *non-cyclic* numbers, i.e. to natural numbers which do not pertain to a T-cylce. This is not really a restriction, as any domain of attraction can be written as a disjoint union of a trajectory and some predecessor sets of non-cyclic numbers (which can be chosen pairwise disjoint). For example, the domain of attraction of the T-cycle $(1,2)$ is given by $\{1,2\} \cup \mathcal{P}_T(4)$ (observe that $a \in \mathcal{P}_T(a)$ for each $a \in \mathbb{N}$). At this stage, we are still very close to the $3n+1$ conjecture, as there are the equivalences:

$$\begin{aligned} \text{the } 3n+1 \text{ conjecture holds} \quad &\Longleftrightarrow \quad \mathcal{P}_T(4) = \mathbb{N} \setminus \{1,2\} \\ &\Longleftrightarrow \quad Z_4(x) = x - 2 \quad \text{for integers } x \geqslant 2. \end{aligned}$$

But the $3n+1$ conjecture itself may be intractable. So, we have to look for less ambitious assertions for treatment. Let us first state some properties which a general dynamical system on $\mathbb{N}$ given by $g : \mathbb{N} \to \mathbb{N}$ may or may not have.

POSITIVE PREDECESSOR DENSITY PROPERTY FOR FIXED $a \in \mathbb{N}$:

$$\liminf_{x \to \infty} \frac{Z_a(x)}{x} > 0\,.$$

UNIFORM POSITIVE PREDECESSOR DENSITY ON $A \subset \mathbb{N}$:

$$\liminf_{x \to \infty} \left(\inf_{a \in A} \frac{Z_a(ax)}{x} \right) > 0\,.$$

Note that the $3n+1$ conjecture implies that the $3n+1$ function shares the positive predecessor density property for $a = 4$. Uniform positive density is only interesting for infinite sets A; it seems reasonable to take $A := \{a \in \mathbb{N} : a \not\equiv 0 \mod 3\}$, as the predecessor sets of multiples of 3 are easily calculated:

$$\mathcal{P}_T(3k) = \{2^n\, 3k : n \in \mathbb{N}_0\}, \qquad \text{hence} \qquad Z_{3k}(x) = \left\lfloor \log_2 \frac{x}{3k} \right\rfloor.$$

Whether or not the $3n+1$ function has the uniform positive density property seems to be quite independent from the $3n+1$ conjecture.

Neither of these two properties is known for the $3n+1$ function. But there has been some progress in a much weaker formulation of the uniform positive predecessor density property.

FIND GOOD EXPONENTS $c > 0$ SUCH THAT

$$\liminf_{x\to\infty}\left(\inf_{a\not\equiv 0 \mod 3}\frac{Z_a(ax)}{x^c}\right) > 0 .$$

There has been a certain industry in improving the exponent c. The first who established that there is such an exponent $c > 0$ was R. E. Crandall (1978); actually Crandall derived his estimate only for the counting function $Z_1(x)$, but we shall see in section II.6 that his method also proves the relation above. Crandall's method has been pushed further by J. W. Sander (1987) to give $c = \frac{1}{4}$ (who also formulated it only for $Z_1(x)$). Finally, D. Applegate and J. C. Lagarias (1995) called Crandall's approach *tree-search method* and improved it to produce a computer-aided proof for $c = 0.654$. The tree-search method is related to the approach given here. In section II.6, we discuss tree-search in our terminology established in chapter II, showing that, in fact, the above uniform lim inf-relation is what has been proved.

The best result in this direction known up to now is the theorem of Applegate and Lagarias (1995) stating that, for each $a \not\equiv 0 \bmod 3$, there is a constant $c_a > 0$ such that $Z_a(x) \geqslant c_a\, x^{0.81}$ for each $x \geqslant a$. The computer-assisted proof is based on a different idea called *Krasikov inequalities* and initiated by I. Krasikov (1989). Although Krasikov inequalities appear more powerful in improving the constant c, this method is not discussed in these notes because it is not directly related to the approach prosecuted here.

One of the main results of these notes is the reduction theorem linking distribution properties of sums of mixed powers to dynamic properties like those stated above. To be more explicit about this, let j, k denote two non-negative integers. Then we are concerned with sums of mixed powers

$$2^{\alpha_0} + 2^{\alpha_1}3 + 2^{\alpha_2}3^2 + \cdots + 2^{\alpha_j}3^j ,$$

where $j + k \geqslant \alpha_0 > \cdots > \alpha_j \geqslant 0$. The set of all such sums will be denoted by $\mathcal{R}_{j,k}$. Then the cardinality of this set will be proved to be just the number of possible choices of integers $\alpha_0, \ldots, \alpha_j$ satisfying the condition above. This is elementary combinatorics:

$$|\mathcal{R}_{j,k}| = \binom{j+k+1}{j+1} .$$

Now the question is: given an integer $\ell \geqslant 1$, for which indices j, k does the set $\mathcal{R}_{j,k}$ meet all prime residue classes to modulus 3^ℓ (observe that an element of $\mathcal{R}_{j,k}$ cannot be divisible by 3, hence $\mathcal{R}_{j,k}$ is contained in the union of the prime residue classes to modulus 3^ℓ)? Technically, we do not want to deal with two indices j, k, but we want to deal with large sets $\mathcal{R}_{j,k}$. Hence, let us restrict attention to the sets $\mathcal{R}_{j-1,j}$ where the binomial coefficient is $\binom{2j}{j}$. The unsolved problem is the following.

COVERING CONJECTURE FOR MIXED POWER SUMS. *There is a constant $K > 0$ such that, for every $j, \ell \in \mathbb{N}$, the following implication holds:*

$$|\mathcal{R}_{j-1,j}| \geqslant K \cdot 2 \cdot 3^{\ell-1}$$
$$\Longrightarrow \quad \mathcal{R}_{j-1,j} \text{ covers the prime residue classes to modulus } 3^{\ell}.$$

This conjecture seems reasonable: as there are precisely $2 \cdot 3^{\ell-1}$ prime residue classes modulo 3^ℓ, the precondition says that the set $\mathcal{R}_{j-1,j}$ has sufficient elements to put at least K of them into each prime residue class modulo 3^ℓ. If the distribution of $\mathcal{R}_{j-1,j}$ among those prime residue classes is not too unbalanced, one should expect that, for large K, we find at least one of the mixed power sums of $\mathcal{R}_{j-1,j}$ in each prime residue class. Of course, the essential content of the covering conjecture is in the asymptotics $\ell \to \infty$. If the conjecture is true, then it ensures that there is a sequence $(j_\ell)_{\ell \in \mathbb{N}}$ satisfying the two conditions:

(i) each set $\mathcal{R}_{j_\ell-1,j_\ell}$ covers the prime residue classes modulo 3^ℓ, and

(ii) $\displaystyle\lim_{\ell\to\infty} \frac{j_\ell}{\ell} = \log_4 3 .$

These two conditions will be technically essential in the proof of the following reduction theorem: *If* the covering conjecture for mixed power sums is true, *then* the dynamical system on $\mathbb{N}$ generated by the $3n+1$ function has the following

UNIFORM SUB-POSITIVE PREDECESSOR DENSITY PROPERTY:

$$\liminf_{x\to\infty} \left(\inf_{a \not\equiv 0 \mod 3} \frac{Z_a(ax)}{x^\delta} \right) > 0 \qquad \textit{for any } \delta \in \mathbb{R} \textit{ satisfying } \quad 0 < \delta < 1 .$$

In fact, the implication remains valid if the covering conjecture for mixed power sums is slightly weakened. We need not assume that there is a *constant* $K > 0$ with the required property. It suffices to assume that K is a function of ℓ which grows *sufficiently slowly* when ℓ tends to infinity; more explicitly, it suffices to assume that $K(\ell)e^{-\gamma\ell}$ remains bounded for any constant $\gamma > 0$.

The proof given here for the reduction of the uniform sub-positive predecessor density property to a covering conjectures for mixed power sums requires asymptotic analysis of binomial coefficients. In addition, we make use of integration theory on the compact topological group $\mathbb{Z}_3^*$ of invertible 3-adic integers. The proof gives, in addition, some argument why we cannot prove a uniform *positive* predecessor density property on the basis of a covering conjecture for mixed power sums like that given above. This has to do with the fact that the precise asymptotics of the binomial coefficient $\binom{2j}{j}$ is less than a constant times 2^{2j}. So the feeling arises that the $3n+1$ predecessor sets may not have the uniform positive density property. In this context, we also give a technical condition on the distribution of mixed power sums which is sufficient for the positive predecessor density property for an individual $a \in \mathbb{N}$ (not divisible by 3, of course).

The intuitive content of the reduction is the following. The sums of mixed powers described above can be seen as the "accumulated non-linearities" occuring when iterating the $3n+1$ function T. Then the conjecture on the distribution of power-sums states that the accumulated non-linearities of the $3n+1$ function behave chaoticly. On the other hand, the uniform sub-positive predecessor density property would mean some regular behaviour of the dynamical system generated by T. So the reduction theorem means, intuitively, the more chaotic the behaviour of the accumulated non-linearities, the more regular is the behaviour of the dynamical system.

There are also many other results about the mathematical nature of the Collatz graph of T, or, equivalently, about the dynamical system generated by T— which justifies the title. Let us first briefly describe what is basic for collecting mathematical information about this dynamical system. A dynamical system consists of its trajectories. As there are difficulties in describing the limiting behaviour of the trajectories, we are forced to restrict our attention, at least at the beginning, on finite portions of trajectories. A finite portion of a T-trajectory, from b to a, say, has the form

$$b \xrightarrow{T} T(b) \xrightarrow{T} T^2(b) \xrightarrow{T} \dots \xrightarrow{T} a = T^{k+\ell}(b).$$

Let us assume that on the $k+\ell$ steps from b to a, the function T takes precisely k times the branch T_0 and precisely ℓ times the branch T_1. The number of possible such (k,ℓ)-step portions of T-trajectories terminating at $a \in \mathbb{N}$ will be denoted by (observe that, if k and ℓ are fixed, then there is a one-one-correspondence between the (k,ℓ)-step portions and their initial vertices b, even if a is an element of a T-cycle)

$$e_\ell(k,a) := \left|\left\{b \in \mathbb{N} : T^{k+\ell}(b) = a, k \text{ times } T_0, \ \ell \text{ times } T_1.\right\}\right|.$$

These quantities $e_\ell(k,a)$ constitute the basic objects for most of the research presented in this book. They are linked to $3n+1$ predecessor density estimates via the implication, which is valid for non-cyclic numbers $a \in \mathbb{N}$ (theorem III.2.5),

$$\begin{aligned} & \liminf_{n\to\infty} \frac{s_n(a)}{\beta^n} > 0 \\ \Longrightarrow \quad & Z_a(x) \geqslant C\left(\frac{x}{a}\right)^{\log_2\beta} \quad \text{for some constant } C>0 \text{ and large } x, \end{aligned}$$

where $s_n(a)$ is the *n-th estimating series*

$$s_n(a) := \sum_{\ell=1}^{\infty} e_\ell\left(n + \left\lfloor \ell \log_2 \frac{3}{2} \right\rfloor, a\right).$$

An important observation is that $e_\ell(k,a)$ depends on a only through its residue class to modulus 3^ℓ (this is one reason why we use ℓ as an index), which implies that we are concerned with a family of functions

$$e_\ell(k,\cdot) : \mathbb{Z}_3 \to \mathbb{N}_0 \qquad \text{where } k,\ell \text{ run through } \mathbb{N}_0;$$

here $\mathbb{Z}_3$ denotes the group of 3-adic integers. A simple consideration shows that $e_\ell(k,a) = 0$ whenever $\ell \geqslant 1$ and $3 \mid a$. Hence, the set $\{e_\ell(k,\cdot) \mid \ell \geqslant 1, k \geqslant 0\}$ is a family of functions on the compact topological group $\mathbb{Z}_3^*$ of invertible 3-adic integers. The use of 3-adic integers in the context of the $3n+1$ problem first appeared in [**Wir3**] (1994); the group of invertible 3-adic integers has also been connected to $3n+1$ iterations, in a somewhat different setting, by Applegate and Lagarias [**AL3**] (1995).

As the domain of definition $\mathbb{Z}_3^*$ of our basic functions $e_\ell(k,\cdot)$ is a compact topological group, it admits a unique normalized Haar measure. It turns out that the $e_\ell(k,\cdot)$ are integrable w.r.t. this Haar measure, with the *3-adic average*

$$\overline{e}_\ell(k) := \int_{\mathbb{Z}_3^*} e_\ell(k,a)\, da = \frac{1}{2\cdot 3^{\ell-1}}\binom{k+\ell}{\ell}.$$

We obtain, for instance, to the following results:

(1) The estimating series given above give rise to a sequence of functions $s_n : \mathbb{Z}_3^* \to \mathbb{N}_0$ which turns out to be discontinuous (theorem III.2.7) but perfectly Haar integrable (lemma III.3.6).

(2) The following is true (theorem III.5.2):

$$\liminf_{n\to\infty} \frac{1}{2^n}\int_{\mathbb{Z}_3^*} s_n(a)\, da > 0.$$

This means: If a number $a \in \mathbb{N}$ with $a \not\equiv 0 \bmod 3$ happens to have the property $e_\ell(k,a) = \overline{e}_\ell(k)$ for an appropriate portion of the pairs (k,ℓ), then the predecessor set $\mathcal{P}_T(a)$ has positive asymptotic density. Of course, the conclusion remains valid if, for an appropriate portion of the pairs (k,ℓ), $e_\ell(k,a)$ is sufficiently close to the 3-adic average.

(3) The numbers $e_\ell(k,a)$ can be constructed inductively without reference to the Collatz graph (corollary II.4.4). After an—admittedly complicated—normalization procedure (section IV.2), it turns out that the essential ingredient is an *asymptotically homogeneous* Markov chain in the sense that sequences of transition measures converge vaguely (theorem IV.4.1). Moreover, the limiting transition probability is averaging over $a \in \mathbb{Z}_3^*$.

(4) The limiting transition probability is shown to admit exactly one invariant density (theorem IV.5.1), which comes from a $\mathcal{C}^\infty$ function on $\mathbb{R}$ which is a polynomial on each interval outside the classical Cantor set (lemma IV.5.3).

The techniques to prove result (2) are essential for the proof of the reduction theorem discussed above. Result (3) embodies a first vague idea of "asymptotic self-similarity" of the Collatz graph. It would be nice to know more about this asymptotic self-similarity, and to compare it to the phenomena occurring in the context of discrete-time dynamical systems in the complex plane.

In these notes, I almost everywhere resisted the temptation to generalize a result to other functions than the $3n+1$ function. A natural candidate for such generalizations would be a function T_q defined like the $3n+1$ function T, but with $T_q(n) := (qn+1)/2$ for odd n, where q is a previously fixed odd natural number. If q has the property that 2 generates the multiplicative group of prime residue classes to modulus q^ℓ for each $\ell \in \mathbb{N}$, then a good part of the results presented here admit a straightforward generalization to apply to iterations of T_q.

* * *

The plan of the book is as follows. Chapter I gives a brief survey of some strains of research on the problem. We already find a broad variety of different mathematical methods which have been used to attack the $3n+1$ problem, including probability analysis, continued fractions, formal languages, and holomorphic functions in the complex unit disc.

In Chapter II, the essential notions for discrete dynamical systems on $\mathbb{N}$ and for the Collatz graph are given. Then the counting functions $e_\ell(k, a)$ and some variations thereof are introduced and discussed. These counting functions are linked to lower estimates for the predecessor counting functions $Z_a(x)$ via a series similar to, and, in fact, the model for the estimating series described above. It is shown that known density estimates (e.g., by Crandall [**Cra**] (1978) or Applegate and Lagarias [**AL1**] (1995)) for $3n+1$ predecessor sets perfectly fit into our framework, which leads to slightly stronger formulation of those estimates.

Chapter III mainly deals with the use of 3-adic numbers. Motivated by the estimate for $Z_a(x)$ given in chapter I, the estimating series are introduced and discussed. The remaining part of chapter III studies their 3-adic averages, coming across a rigorous counterpart of the usual $3n+1$ heuristics saying that in an "average" finite portion of a $3n+1$ trajectory, the steps arising from T_0 and those arising from T_1 are, more or less, balanced. The chapter concludes with a proof of result (3) mentioned above, and with a short discussion of possible generalizations to $pn+1$ functions for so-called Wieferich primes p.

Chapter IV contains all the stuff pertaining to the asymptotically homogeneous Markov chain mentioned above. The construction of this Markov chain is given explicitly, including descriptions of the transition probabilities in terms of combinatorial number theory. These explicit descriptions are used to formulate and prove the result indicated under (2) above.

Finally, chapter V takes up essential ideas from the previous chapters to prove the reduction theorem.

It is a great pleasure for me to include a long list of people who stimulated and supported my research on the $3n+1$ problem in one way or another. First of all, let me express special gratitude to K. P. Hadeler, J. C. Lagarias, K. R. Matthews, G. Meinardus, D. Merlini, and H. A. Müller, for giving me access to important information around the $3n+1$ problem, which I hardly could have received otherwise. I am indebted to L. Berg and J. C. Lagarias for valuable comments on a preliminary version of this. For further stimulating information, discussions on the topic, or encouragement to proceed, I also thank M. Kudlek, J. W. Sander, P. Schorn, and the following (in part former) mathematicians at the university of Eichstätt: A. Cornea, S. Deschauer, V. Krafft, J. Rohlfs, H.-G. Weigand, R. Wittmann, and especially W. Rump who originally drew my attention to the problem. Finally, I am grateful to those people who answered my request for sending reprints or preprints of their papers on the problem, and who are not yet mentioned: J. Błazewicz, S. Eliahou, G.-G. Gao, I. Korec, G. T. Leavens, and B. Schuppar.

Eichstätt, october 1997

G. J. Wirsching

CHAPTER I

SOME IDEAS AROUND $3n+1$ ITERATIONS

The $3n+1$ problem can be found in many places. It is presented in D. R. Hofstadter's well-known book *Gödel, Escher, Bach* **[Hof]** (1980), pp. 400–402, where a natural number satisfying the $3n+1$ conjecture (see section 1 for a precise statement) is called a *wondrous* number. The problem has been described in M. Gardner's article **[Grd]** (1972) in *Scientific American* and in C. S. Ogilvy's book *Tomorrow's math* **[Ogi]** (1972), p. 103f. It found entrance in R. K. Guy's problem book **[Guy1]** (1981); Guy also wrote some further introductory articles about $3n+1$ iterations **[Guy2]** (1983), **[Guy3]** (1986).

In addition, there are more than fifty research articles containing substantial results around the $3n+1$ problem. This chapter includes hints to some of the most important strains of research about this topic; thereby we come across a wealth of different mathematical ideas. The material is organized roughly according to themes, and inside a special topic according to date of publication.

1. The problem

The set of natural numbers (starting with 1) is denoted by $\mathbb{N} = \{1, 2, 3, \ldots\}$. If we want to include 0, we write $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. The set of integers is denoted by $\mathbb{Z} = -\mathbb{N} \cup \mathbb{N}_0$. For an arbitrary integer function $f : \mathbb{Z} \to \mathbb{Z}$, we denote by $f^k = f \circ f^{k-1}$ the k-fold iterate of f, for each $k \in \mathbb{N}$, with the (natural) convention $f^0 = \text{id}$. If $n \in \mathbb{Z}$, the *f-trajectory* of n (or with *starting number* n) is the sequence

$$\mathcal{T}_f(n) := \left(f^k(n)\right)_{k \geqslant 0} = \left(n, f(n), f \circ f(n), f \circ f \circ f(n), \ldots\right).$$

An *f-cycle* is generated by an integer a with the property $f^k(a) = a$ for some $k \in \mathbb{N}$. For notational definiteness, we choose the minimal period k and write a cycle as a k-vector,

$$\Omega_f(a) := \left(a, f(a), \ldots, f^{k-1}(a)\right).$$

In the german retroversion of **[Col2]** (1986),* L. Collatz calls the function

$$f(n) = \begin{cases} 3n+1 & \text{für ungerade } n \\ \frac{n}{2} & \text{für gerade } n, \end{cases} \tag{1.1}$$

*This paper originally was written in german, and then translated into chinese by Ren Zhiping; only the chinese translation is published. I am grateful to G. Meinardus, who sent me both the chinese version, and a retranslation into german by Zhangzheng Yu (1991).

the "$3n+1$"-*Funktion*. Meanwhile it turned out to be more convenient to use instead the function

$$(1.2) \qquad T : \mathbb{N} \to \mathbb{N}, \qquad T(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+1)/2 & \text{if } n \text{ is odd,} \end{cases}$$

(cf., for instance, [**Ter1**], [**Lag1**], [**BeM**], and most of the articles cited in our bibliography. Henceforth in these notes, this function T will be called the $3n+1$ *function*, and we shall refer to that function f as the *Collatz function*. The letter T will be reserved for the $3n+1$ function, but we do not reserve the letter f for the Collatz function (even in [**Col2**], f is also used to denote other functions). In some papers, T is called $3x+1$ function, but I prefer the name $3n+1$ function to emphasize that the problem is to deal with *natural* numbers.

The famous problem about the $3n+1$ function is the following

$3n+1$ CONJECTURE. *For any starting number in* $\mathbb{N}$, *the* T*-trajectory eventually ends in the cycle* $(1,2)$.

A fallacious "proof" of this conjecture has been published by M. Yamada [**Yam**] (1980). The error has been described by J. C. Lagarias in his review (see also [**Yam**] for a citation). Prizes have been offered for its solution: \$ 50 by H. S. M. Coxeter in 1970, then \$ 500 by Paul Erdős , and £ 1000 by B. Thwaites [**Lag1**].

2. About the origin of the problem

The exact date of the first occurence of the $3n+1$ conjecture is unclear. L. Collatz reports in [**Col2**] (1986) that he represented integer functions by graphs (for the precise definition, see chapter II) already in his student days from 1928 to 1933. He considered a certain classification of the possible graphs and tried to find simple examples for each type. Looking for a graph containing a "Kreis" (which is a cycle in our terminology) and representing a function f which should be as simple as possible, he was led to the observation that necessarily $f(n) < n$ for certain numbers n, and $f(n) > n$ for others. The first attempt was

$$(2.1) \qquad \widehat{f}(n) := \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ n+1 & \text{if } n \text{ is odd,} \end{cases}$$

which gives only the cycle $(1,2)$, as is easily shown. The second attempt $\widehat{f}(n) := 2n+1$ for odd n does not give any cycle at all, as odd numbers are mapped to larger odd numbers. And the next attempt is the Collatz function (1.1), of which Collatz reports that the only cycle he found was "der triviale Kreis" $(4,2,1)$. He writes that he did not publish the problem because he was unable to solve it.

Collatz also reports that he told the problem to his colleague H. Hasse in 1952. Hasse apparently circulated it by mouth during a visit to Syracuse university in the 1950's, where he proposed the name *Syracuse problem*. Later on, the problem also received the names *Kakutani's problem* and *Ulam's problem*, see [**Lag1**].