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Minimal Projections in Banach Spaces

Problems of Existence and Unic and their Application

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To

Marii Schiffmann (Elenbogen)

Preface

This book results in part from lectures held by the first author at the Pedagogical University of Bydgoszcz in 1984 - 85. A part of the present text (chapters I, II, IV) is a revised version of lecture notes [156], which appeared in 1985 under the same title (in Russian) in 350 copies, in the form of mimeographed notes (a university issue). Chapter III includes new results.

The text is supplemented with an Author Index and Subject Index; each chapter is followed by short comments on the history of results exposed in the chapter.

It is the authors' hope that the book may be of interest to those who are just beginining to deepen their knowledge of Banach space geometry and approximation theory, as well as to specialists. The second chapter can perhaps prove to be interesting also to specialists in mathematical economy.

This book would never have appeared without encouraging help from many persons.

First of all, the authors are grateful to Professor Ward Cheney for his suggestion to submit the manuscript to this series and for his interest and encouragement along the way to its completion.

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Włodzimierz Odyniec and Grzegorz Lewicki

Kraków, April 1990

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Introduction

§ 1. General survey

The first 30 years of our century , the time when the edifice of functional analysis was raised, is also the period during which the foundations of the theory of best approximations in normed space were laid. The name forever connected with the origins of that theory is that of S. Banach's ([7]).Later on the ideas of Banach were developed in the works of S. Mazur, M. G. Krein, S. M. Nikolskii, N. I. Ahiezer, J. Walsh, A. N. Kolmogorov, A. I. Markusevič, R. James, R. Phelps, A. L. Garkavi, I. Singer, E. W. Cheney, S. B. Stečkin, and others (see [72],[124],[183],[192] for detailed references).

The problem of best approximation in a normed linear space X by elements of a subspace D (here and elsewhere, by a subspace we always mean a closed subspace) is formulated as follows. Given $x \in X$, find an element $y_x \in D$ for which the lower bound

$$\rho(x,D) = \inf\{\|x - y\|: y \in D\} \qquad (0.1.1)$$

is attained.

The number $\rho(x,D)$ is called the best approximation, or a distance, to x from D. Element y_x (which need not be unique) is called an element of best approximation from D to x.

The question of vital importance in computing (or estimating) the quantity $\rho(x,D)$, as well as in veryfying is whatever or not the subspace D is complemented in X; that is to say, whether there exists a projection of X onto D. In particular, if P is such a projection, then we have for every $x \in X$

$$\|x - Px\| \le \|I - P\| \cdot \rho(x,D), \qquad (0.1.2)$$

with I denoting the identity operator in X. Since

$$\|I - P\| \le 1 + \|P\|, \qquad (0.1.3)$$

with equality

$$\|I - P\| = 1 + \|P\| \qquad (0.1.4)$$

holding in many important specific cases (see e.g. I. K. Daugavet's paper [54]), the problem which naturally arises is that of estimating from below the norm of P. To serve this purpose, a quantity $\rho(X,D)$ is introduced, called the relative projection constant:

$$\rho(X,D) = \inf\{\|P\|: P \text{ is a projection of } X \text{ onto } D\} \qquad (0.1.5)$$

(see [25],[78],[88],[90]).

Among all projections from X onto D, those are of special interest whose norms coincide with the constant $\rho(X,D)$; they are called **minimal projections** (provided they exist). In other words, a projection P of a Banach space X onto a subspace D is minimal if

$$\|P_1\| \geq \|P\| \qquad (0.1.6)$$

for any other projection $P_1: X \to D$.

The problem of finding a best approximating element, as well as that of finding a minimal projection, is in fact an instance of an optimization problem. The relationship between the two types of problems is not just apparent; it goes quite deep into the inner nature of things, though certain differences occur at some points. Similarly to the general task of finding a best approximation, when dealing with minimal projections we are facing two principal kinds of problems:

(E_m) = problem of existence of minimal projection,

(U_m) = problem of uniqueness of a minimal projection.

In a finite-dimensional space, problem (E_m), just as (E), the problem of existence of a nearest element, always has a solution. This is no more the case for spaces of infinite dimension (see [39],[83]). Both of this existence problems can lack a solution. Let us examine two examples. In each of them X is taken to be c_o ; for D we take two subspaces of codimension 1 defined by two functionals f_1, $f_2 \in (c_o)^* = l_1$.

Example 0.1.1. a) $f_1 = (1/2,1/4,1/8,\ldots,1/2^n,\ldots)$, $x = (2,0,\ldots) \in c_o$, $D = f^{-1}(0)$. It is easy to see that $\inf\{\|x - y\| : y \in D\} = 1$, whereas $\|x - y\| > 1$ for any particular y in D. Strangely enough, there exists a minimal projection onto D, of norm 1, which is unique (moreover it is a SUBA projection) (see [12],[21]).

b) $f_2 = (1/4,1/4,1/4,1/8,1/16,\ldots 1/2^{n+1},\ldots)$, $D = f_2^{-1}(0)$

Then $1 < \rho(X,D) < 2$ and there are no minimal projections onto D (see [21]).

To emphasize the relevance of minimal projections to the problem of best approximation we mention the following result of P. Franck ([68]). Let $\mathcal{L}(X,Y)$ denote the space of all linear operators from X to Y (Banach spaces) and let $\mathcal{L}_V$ be the space of all operators in $\mathcal{L}(X,Y)$ whose kernels contain a given subspace $V \subset X$. Then for any $T \in \mathcal{L}(X,Y)$ we have the inequality

$$\inf\{\|T - T_1\| : T_1 \in \mathcal{L}_V\} \geq \sup\{\|Tx\|_Y / \|x\|_X : x \in V\} \qquad (0.1.7)$$

which turns into an equality whenever there exists a norm one projection of X onto V. (For functionals (0.1.7) holds with an equality sign ([27], [71]).)

Despite the fact that, historically, the term "minimal projection" entered the stage relatively late, the study of minimal projections originated in fact already in the thirties, mainly in the connection with geometry of Banach spaces. Projections with unit norm have been treated most thoroughly; this is chiefly due to the fact that they can be considered as a generalization of orthogonal projections in Hilbert space.

Numerous mathematicians have been dealing with norm one projections; let us mention there A. E. Taylor, H. F. Bohnenblust, L. V. Kantorovich, R. C. James, L. Nachbin, G. P. Akilov, M. Z. Solomjak, J. Lindenstrauss, M. I. Kadeč, Cz. Bessaga, A. Pełczyński, J. Ando, V. I. Gurarii (see[19], [139], [159] for detailed references).

On the other hand, papers dealing with certain minimal projections with nonunit norm appeared already in the forties. We wish to emphasize expecially two of these works, A. Sobczyk [180], 1941, and S. M. Łozinskii [130], 1948.

Using the results of Taylor [186] and Bohnenblust [25], A. Sobczyk [180] has shown that minimal projections of (c) onto (c_o) have norm 2 and there are infinitely many of them. (Recall that (c_o) is of codimension 1 in (c)). To be precise, if $(x) \in (c)$ then, writing $x_o = (1,1,\dots)$, we can find a unique $t \in \mathbb{R}$ such that $x = x^o + t \cdot x_o$.

Now, if we define an operator P by

$$Px = x + t \cdot b, \tag{0.1.8}$$

where

$$b = (b_1, b_2, \dots) \in (c), \quad \lim_{i \to +\infty} b_i = -1 \tag{0.1.9}$$

then clearly P is a projection of (c) onto (c_o); if besides $|b_i| \leq 1$ for all i, then P is a minimal projection.

Łozinskii's paper [130] is connected with the Fourier projection F_n of $C_o(2\pi)$, the space of all continuous, 2π-periodic function, onto π_n, the space of all trigonometric polynomials of order $\leq n$ $(n \geq 1)$. It is defined by the formula

$$F_n(f) = \sum_{k=0}^{2n} \left(\int_0^{2\pi} f(t) \cdot g_n(t)\, dt \right) g_k, \tag{0.1.10}$$

where $(g_k)_{k=0}^{2n}$ is any orthonormal basis in π_n, i.e.

$$\int_0^{2\pi} g_i(t) \cdot g_j(t)\, dt = \delta_{ij}, \tag{0.1.11}$$

(the Kronecker delta). The main result of this paper is that $\rho(C_o(2\pi), \pi_n) = \|F_n\|$; in other words that F_n is a minimal projection (see Theorem 0.1.3

below). Because of the great importance of the Fourier projection F_n we present after [35] some basic properties of this operator. We start with the following

Theorem 0.1.2. (see [35], p.212) The norm of F_n satisfies the following estimation:

$$4/\pi^2 \cdot \ln(n) \le \|F_n\| \le \ln(n) + 3 \qquad (0.1.12)$$

Proof. It is well known (see e.g. [2], p.180) that

$$\|F_n\| = (2/\pi) \cdot \int_0^{\pi} |D_n(t)|\, dt, \qquad (0.1.13)$$

where

$$D_n(t) = \sin((n + 1/2) \cdot t)/\sin((1/2) \cdot t) \qquad (0.1.14)$$

for $t \in [0,2\pi]$. The change of variable $t \to 2x$ and the inequality $\sin(x) \le x$ on $[0,2\pi]$ give us, following (0.1.13)

$$\|F_n\| \ge (2/\pi) \cdot \int_0^{\pi/2} |\sin((2n+1)x)/x|\, dx.$$

The change of variable $x \to \pi \cdot x/(2n+1)$ then yields

$$\|F_n\| \ge (2/\pi) \cdot \int_0^{n+1/2} |\sin(\pi x)|/x\, dx > (2/\pi) \cdot \int_0^{2n} |\sin(\pi x)|/x\, dx.$$

Breaking the interval into n subintervals, we obtain

$$\|F_n\| \ge (2/\pi) \cdot \left(\int_0^1 + \int_1^2 + \ldots + \int_{n-1}^{n} \right) |\sin(\pi x)|/x\, dx =$$

$$= (2/\pi) \cdot \int_0^1 (1/x + 1/(x+1) + \ldots + 1/(x+n-1)) \cdot |\sin(\pi x)|\, dx \ge$$

$$\ge (2/\pi) \cdot \int_0^1 (1 + 1/2 + \ldots 1/n) \cdot \sin(\pi x)\, dx \ge$$

$$\ge (2/\pi) \cdot \ln(n) \cdot \int_0^1 \sin(\pi x)\, dx = (4/\pi^2) \cdot \ln(n).$$

That $1 + 1/2 + \ldots + 1/n > \ln(n+1)$ may be seen from a graph of the function $1/(x+1)$ or by induction.

To prove the second inequality, first observe that

$$D_n(t) = 1/2 + \sum_{i=1}^{n} \cos(it) \qquad (0.1.15)$$

Integrating separately over $[0,1/n]$ and $[1/n,\pi]$ we obtain the following estimations

$$(2/\pi) \cdot \int_0^{1/n} |D_n(t)|\, dt = (2/\pi) \cdot \int_0^{1/n} 1/2 + \sum_{i=1}^{n} |\cos(it)|\, dt \le$$

$$\le (2/\pi \cdot n) \cdot (1/2 + n) < 1$$

and

$$(1/\pi) \cdot \int_{1/n}^{\pi} |D_n(t)|\, dt \le (1/\pi) \cdot \int_{1/n}^{\pi} \pi/t\, dt = \ln(\pi) - \ln(1/n) < 2 + \ln(n)$$

(here we have used the fact that $\sin(t/2) \ge t/\pi$ which is evident from

the graph of $\sin(t/2)$. Combining the above inequalities we derive the second estimation. ***

Theorem 0.1.3 ([35], p. 212) Let P be any projection of the space $C_o(2\pi)$ onto the space π_n. Then $\|P\| \geq \|F_n\|$. In other words F_n is a minimal projection among all projections of $C_o(2\pi)$ onto π_n.

Proof. Define operators T_s and Φ by the equations

$$(T_s f)(x) = f(x+s)$$

$$(\Phi f)(x) = (1/2\pi) \cdot \int_{-\pi}^{\pi} (T_{-s} P T_s f)(x)\, ds.$$

If we can establish that $\Phi = F_n$, then we will be finished because

$$\|F_n f\| = \|\Phi f\| = \max\{|(1/2\pi) \cdot \int_{-\pi}^{\pi} (T_{-s} P T_s f)(x)\, ds : x \in [0, 2\pi]\} \leq$$

$$\leq \|T_{-s} P T_s f\| \leq \|P\| \cdot \|f\|.$$

In order to prove that $F_n = \Phi$ it will be enough to prove that $\Phi f_k = F_n f_k$ where f_k is the function $f_k(x) = e^{ikx}$ $(k = 0, \pm 1, \pm 2, \ldots)$, since this family of functions is fundamental in $C_o(2\pi)$, while the operators Φ and F_n are linear and continuous. If $|k| \leq n$, then $F_n f_k = f_k$. On the other hand, $T_s f_k \in \pi_n$, so that $P T_s f_k = T_s f_k$. Thus $T_{-s} P T_s f_k = f_k$ and $\Phi f_k \quad f_k$. (In the integration, the integrand is independent of s.) Suppose next that $|k| > n$. Then, by (0.1.10) and (0.1.11), $F_n f_k = 0$. Since $T_s f_k = e^{iks} \cdot f_k$, it follows that $(T_{-s} P T_s f_k)(x) = e^{iks} \cdot (P f_k)(x-s)$. But $P f_k \in \pi_n$, and consequently, as a function of s, e^{iks} is orthogonal to $(P f_k)(x-s)$. Hence $\Phi f_k = 0$, which completes the proof. ***

Theorem 0.1.4. ([35], p. 214) Let P be a projection of the even part of $C_o(2\pi$ onto the even part of π_n. Then $\|I - P\| \geq (1/2) \cdot (\|F_n\| + 1)$.

Proof. Define the linear operator

$$(\Phi f)(x) = (1/2\pi) \cdot \int_{-\pi}^{\pi} (T_s (I - P)(T_{-s} + T_s) f)(x)\, ds$$

in which T_s, as in the previous theorem, denotes the translation operator. The crux of the proof is in verifying the equation $\Phi = I - F_n$. After that we write $\|(I - F_n)(f)\| = \|\Phi(f)\| \leq 2 \cdot \|I - P\| \cdot \|f\|$ whence $2 \cdot \|I - P\| \geq \|I - F_n\| = 1 + \|F_n\|$. (The last equality can be obtained using Daugaviet's Teorem from [54].) In order to prove that $\Phi = I - F_n$, it sufficies to prove that $\Phi f_k = (I - F_n) f_k$, where $f_k(x) = \cos(kx)$, since

these functions form a fundamental set and the operators in questions are continuous. Thus we must show that $\Phi f_k = 0$ when $k \le n$ and $\Phi f_k = f_k$ for $k > n$. We have

$$(T_s f_k)(x) = \cos(k(x+s)) = \cos(ks)\cdot\cos(kx) - \sin(ks)\cdot\sin(kx).$$

Hence $(T_{-s} + T_s)f_k = 2\cdot\cos(ks)\cdot f_k$

and $T_s(I - P)(T_{-s} + T_s)f_k = 2\cdot\cos(ks)\cdot T_s(f_k - Pf_k)$.

Now if $k \le n$, $f_k = Pf_k$ so that $\Phi f_k = 0$. If $k > n$, then

$$(\Phi f_k)(x) = (1/\pi)\int_{-\pi}^{\pi} \cos(ks)[\cos(ks)\cos(kx) - \sin(ks)\sin(kx) - (Pf_k)(x+s)]ds$$

Since $(Pf_k)(x-s)$ is a trigonometric polynomial of degree $\le n$ in s, the integral involving it vanishes, by orthogonality. By the orthonormality relations the remaining integration yields $\cos(kx) = f_k(x)$. ***

The Banach-Steinhaus Theorem, Theorems (0.1.2) and (0.1.3) yield the following

Theorem 0.1.5. (Charsziładze-Łozinski Theorem I) For each n let there be given a continuous projection ${}_nP$ of ${}_oC(2\pi)$ onto ${}_n\pi$. Then there exists a function $f \in C_o(2\pi)$ for which $\|P_n f\|$ is unbounded as $n \to \infty$.

Proof. By the Theorems (0.1.3) and (0.1.4) $\|P_n\|$ is unbounded because

$$\|P_n\| \ge \|F_n\| \ge (4/\pi^2)\cdot\ln(n).$$

If $\|P_n f\|$ were bounded for all f, then by the Banach-Steinhaus Theorem, $\|P_n\|$ would be bounded. ***

A simple consequence of Theorem (0.1.4) is

Theorem 0.1.6. (Charsziładze-Łozinski Theorem II) For each n, let P_n denote a projection of $C_{\mathbf{R}}[a,b]$ onto the subspace of algebraic polynomials of degree $\le n$. Then there exists a function $f \in C_{\mathbf{R}}[a,b]$ for which the sequence $\|f - P_n f\|$ is unbounded.

Proof. We define a map M from $C_{\mathbf{R}}[a,b]$ onto the evan part of $C_o(2\pi)$ by

$$(Mf)(\alpha) = f((a+b)/2 + ((b-a)/2)\cdot\cos(\alpha)).$$

The map M is an isometric isomorphism. That is, it is one-to-one, linear, and has the property $\|Mf\| = \|f\|$. Now the operators $P_n^* = MP_nM^{-1}$ are projections of the even part of $C_o(2\pi)$ onto the evan trigonometric polynomials of degree $\le n$. By Theorem (0.1.5), then, $\|I - P\|_n^* \to \infty$. Hence $\|I - P_n\| \to \infty$, and by the Banach-Steinhaus Theorem, $\|f - P_n f\|$ is unbounded for some f. ***

The question of uniqueness of F_n has remained unsolved until 1968 (see [36],[37],and also [89]). The more general situation has been considered in ([172]).

The papers of Sobczyk [180] and Łozinski [130], although bearing very little resemblance to one another, both lie in the two main streams of application of minimal projections: geometry of Banach spaces and theory of best approximations. These two principial topics have become the object of the first general survey by E. W. Cheney and K. H. Price in 1970 (see [39]). One of papers mentioned in that survey deserves a special comment. We mean here the considerable progress towards the solution of problem (E_m) achieved by J. R. Isbell and Z. Semadeni in their joint paper [83]. As it is stated by the authors, inspiration came from B. Grünbaum (see [77]).

The main result of [83] reads as follows

Theorem 0.1.7. (Isbell, Semadeni). Let D be a complemented subspace of a Banach space X and suppose that D is isometrically isomorphic to the dual of a Banach space Y. Then there exists a minimal projection from X onto D. This is the case, in particular, if D is a reflexive subspace of X.

Compare this with the result of R. James' paper [85] (which appeared one year later, in 1964) concerning property (E): it is only in a reflexive space that each hyperplane (codimensiom one sobspace) enjoys property E.

Needless to say, Theorem (0.1.7) does not exhaust all the knowledge on problem (E_m) (see e.g. [22],[43]). Several results have been obtained on the existence of minimal projections onto subspaces which are not the duals of any space. For instance, no infinite dimensional subspace of c_o is a dual space (see [15],Corollary 2). Nevertheless, some of these subspaces admit minimal projection from c_o.

As regards the problem of uniqueness, survey [39] contains just a few theorems, which concern norm 1 projections.

This book is mainly devoted to the solution of uniqueness and strong unicity (see Chapter III) problems for minimal projections (with nonunit norm in general) in Banach spaces; it also exhibits the relevance of this problem to the uniqueness problem (U_{MP}) in mathematical programming and to the problem of characterization of Hilbert spaces.

For quite a long time already spaces with symmetric norm (for instance, l_p^n, l_p, L_p $(1 \le p \le \infty)$, c_o) have been used as a tester for new ideas in functional analysis,although the term "symmetric space" appeared

relatively recently, in Singer's papers [175],[176]. (See e.g. Kadec-Pełczyński [91] or [123], or the papers on the interpolation of linear operators [173],[174], or papers on Banach lattices [1],[87];for detailed bibligraphy see [17],[25],[178]). Therefore, the problem of uniqueness of minimal projections deserves special interest in symmetric spaces. On the other hand, in symmetric space of finite dimension,the uniqueness problem for minimal projections is essentially a problem of mathematical programming.

Now, the question of uniqueness in problems of mathematical programming (we confine attention to linear programming)admits two approaches. The first of them consists in an analysis of the programming algorithm and requires its execution (see e.g. [59]). The other possibility is to inspect carefully the condition of the problem without executing the algorithm. This second way of approach has proved especially fruitful in the solution of several problems in matrix games (see e.g. [15],[23],[26], [69],[70],[129],[158]).

The interrelation between functional analysis and mathematical programming is well known. It suffices to resort for instance to the works of L. V. Kantorovich and his disciples.

In most of these papers, however, methods of functional analysis are applied in solving one or another problem of mathematical programming, whereas in the present book the direction is reversed. Methods of mathematical programming (in particular, the simplex method) are used in Chapter II to the solution of a functional analysis problem, namely, the problem of uniqueness of a minimal projections in a symmetric space of finite dimension.

In view of the natural relation between the two uniqueness problems (U_{MP}) and (U_M), established in this paper, it seems reasonable to hope that it might find an application in mathematical programming.

Let us now turn attention to the question of a characterization of Hilbert spaces within the class of all Banach spaces. This questions belongs among problems of functional analysis which have gained big popularity (see e.g. [58],[100],[161]). The following two questions have been posed in this connection by S. Banach [7] (The second question had been inspired by S. Mazur (see. [8], p.211).

(1.B) Let B be a Banach space (dim B ≥ 3) and k ≥ 3 an integer.Suppose that all k-dimensional subspaces of B are mutually isometric. Is B necessarily isometric to a Hilbert space?

(1.B-M) In a Banach space B, if x,y ∈ B are any elements with unit norm then there exists a linear isometry A carrying B onto itself and such that Ax = y. (A space with this property is sometimes called an isotropic space or a space with a transitive norm (see [50]).) Supposing

B to be separable, must B be isometric to a Hilbert space?

As shown by S. Rolewicz [167], the assumption of separability is necessary for a positive answer to problem (1.B-M).

According to the results of further research [131], if the space is infinite-dimensional, the requirement that the space norm is transitive cannot be replaced by partial transitivity. More precisely, every separable space B is a complemented subspace of a separable space X whose unit sphere contains a dense subset T such that for any $x,y \in T$ there is a linear isometry A of X onto X with $Ax = y$.

As to the first question, it is still lacking a complete solution in the case when $4 \leq \dim B < \infty$. The second question is unanswered for $\dim B = \infty$. For finite dimensional spaces this problem has been positively solved by H. Auerbach (see [6], [7]). For more information the reader is referred to [61],[62],[76],[161].

Along with the questions (1.B) and (1.B-M) we will concern with the following two problems:

(1.B°) Let B be a Banach space and k a positive integer. Suppose that all subspaces of codimension k in B are isometric. Is B necessarily isometric to a Hilbert space?

Evidently, the problem (1.B°) does not differ from the problem (1.B) in the case of $\dim B < \infty$. (If $4 \leq \dim B < \infty$ then $k \leq \dim B - 1$.)

(1.Od) Let B be a reflexive Banach space and k a positive integer. Suppose that each subspace of codimension k in B is the image of minimal projection $P_{(k)}$ with norm $1+\alpha$, where $\alpha \geq 0$ is a constant depending on B alone. (Then B is called (α,k)-space). Is B necessarily isometric to a Hilbert space? (The case $\alpha=0$, $k=1$ is treated in [7], p.254.)

It has to be remarked that problem (1.Od) is closely related to the well known result of Kakutani (see e.g. [58]) which states that if in a Banach space B with $\dim B \geq 3$ every subspace is the image of a projection B with norm $\|P\| = 1$, then B is isometric to a Hilbert space.

Problems (1.B) and (1.B-M) indicate that the task of characterization of Hilbert spaces among all Banach spaces unavoidably leads to investigations of the isometry group of Banach spaces involved. Investigation of that type has been originated by S. Banach with regard to spaces l_p, L_p ($1 < p < \infty$, $p \neq 2$), C(X). Further research has spread to spaces of complex-valued functions and sequences (see [63],[64],[112], [163] to mention just a few) on the one hand and the spaces with a basis and a symmetric norm [167], arbitrary symmetric coordinate spaces [29] and also certain nonsymmetric coordinate spaces defined by W. Orlicz [105],[106],[179], on the other. We have not mentioned here the literature concerning isometries in subspaces of classical spaces (see e.g. [111]), results on the groups of isometries of Banach spaces [5], [167]

and, in particular, groups containing reflections [191].

In this paper we take up the problem of characterization of Hilbert spaces, chiefly, within the class of uniformly smooth strictly normed spaces. We will be concerned with problems (1.B°) and (1.0d), as well as with certain properties, hitherto ignored, of the isometry subgroup connected with a fixed subspace of the space under consideration.

***

We now expose in more detail the results of the book.

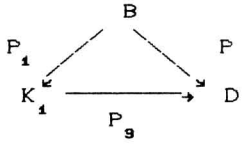
In Chapter I we pursue the uniqueness problem (U_m) in an arbitrary Banach space.

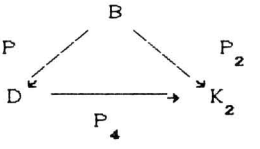
The main result of section 1 states that, in a three-dimensional space, a minimal projection with nonunit norm onto a codimension one subspace always is unique. As regards spaces of higher dimension, an analogous statesment is not true, in general; this is a result of K.Kursten, which we also present (see also section 3 of Chapter II).

In section 2 we examine certain properties of minimal projections. In this context, specific features of projections with nonunit norm are readily seen. The difference between these operators and projections of unit norm is made plain by two circumstances. Firstly, the latter always do attain their norm; and secondly, the question of key significance for the uniqueness of norm one projections onto a subspace D is the possibility of a unique norm-preserving extension of functionals on D to the whole B (see, for instance, Corollary I.2.20; also [13],Theorem 5). In other words, what is of main importance for the uniqueness of a projection with unit norm, is the "smoothness" of points of D in the space B. For minimal projection with nonunit norm, smoothness of that type is is also important, but only in the case when dim B/D ≥ 2. If dim B/D = 1 (dim B ≥ 4), the fundamental part for the problem of uniqueness of minimal projection with nonunit norm is played by the concept of strict convexity of a subspace (see Theorem I.2.3 below).

In the same section we present V. M. Kadec's result on the linearity of a Lipschitz projection projection with unit norm.

Section 3 is mainly devoted to an inspection of the two diagrams





(0.1.16)

with P, P_i (i=1,2,3,4) denoting minimal projections, $\|P\| > 1$, $\|P_2\| > 1$, $\|P_3\| > 1$, $\|P_1\| = \|P_4\| = 1$. It is pointed out, in what way questions concerning the uniqueness of P are influenced by the uniqueness of the P_i's.