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Geometric Topology: Recent Developments

Montecatini Terme, 1990

Editors: P. de Bartolomeis, F. Tricerri

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Geometric Topology: Recent Developments

Lectures given on the 1st Session of the
Centro Internazionale Matematico Estivo (C.I.M.E.)
held at Montecatini Terme, Italy, June 4-12, 1990

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Foreword

The present volume contains the text of three series of lectures given in Montecatini for the period June 4-June 10, 1990, during the C.I.M.E. session "Recent Developments in Geometric Topology and Related Topics".

Geometric Topology can be defined to be the investigation of global properties of a further structure (e. g. differentiable, Riemannian, complex, algebraic etc...) assigned to a topological manifold.
As a result of numerous recent outstanding achievements, which are as complex as they are deep, and always involve a dramatic spectrum of tools and techniques originating from a wide range of domains, Geometric Topology appears nowadays as one of the most fascinating and promising fields of contemporary mathematics.

Our main goal in organizing the session was to gather a distinguished group of mathematicians to update the subject and to give a glimpse on possible future developments.
We can proudly affirm that the lecturers did a superb job.
For an idea of how rich and interesting was the subject-matter that they presented, it is enough to give a brief description of the three main topics.

[1] The geometry and the rigidity of discrete subgroups in Lie groups especially in the case of lattices in semi-simple groups.Two main streams of approaches are considered:
i) the geometry and the dynamics of the action of discrete subgroups on the ideal boundary of the ambient group;
ii) the theory of local and infinitesimal deformations of discrete subgroups via elliptic P.D.E. and Bochner type integro-differential inequalities.
The basics of these two methods are fully described and more advanced materials are covered.
[2] The study of the critical points of the distance function and its application to the understanding of the topology of Riemannian manifolds.
Moving from Toponogov's celebrated theorem and from a complete description of the techniques of critical points of distance function, three basic results in global differential geometry are discussed:
i) the Grove-Petersen theorem of the finiteness of homotopy types of manifolds admitting metrics with bounds on diameter, volume and curvature;
ii) Gromov's bound on the Betti numbers in terms of curvature and diameter;
iii) the Abresch-Gromoll theorem on finiteness of topological type, for manifolds with nonnegative Ricci curvature, curvature bounded below and slow diameter growth.

[3] The theory of moduli space of instantons as a tool for studying the geometry of low-dimensional manifolds.

As main topics, we can quote:

i) the correspondence between instantons over algebraic surfaces and stable algebraic vector bundles, with the investigation of the relations between the geometry of an algebraic surface and the differential topology of its underlying 4-manifold;

ii) the existence of infinitely many exotic C^{∞}-structures on some topological 4-manifolds;

iii) the theory of the decomposition of 4-manifolds along homology 3-spheres.

Finally, it is worthwhile adding that the texts of the present volume capture completely the spirit and the atmosphere of a very successful event.

Paolo de Bartolomeis

Franco Tricerri

TABLE OF CONTENTS

Critical Points of Distance Functions and Applications to Geometry

Jeff Cheeger

0. Introduction

These lecture notes were written for a course given at the C.I.M.E. session "Recent developments in geometric topology and related topics", June 4-12, 1990, at Montecatini Terme. Their aim is to expose three basic results in riemannian geometry, the proofs of which rely on the technique of "critical points of distance functions" used in conjunction with Toponogov's theorem on geodesic triangles. This method was pioneered by Grove and Shiohama, [GrS].

Specifically, we discuss

i) the Grove-Petersen theorem of the finiteness of homotopy types of manifolds admitting metrics with suitable bounds on diameter, volume and curvature; [GrP],

ii) Gromov's bound on the Betti numbers in terms of curvature and diameter; [G],

iii) the Abresch-Gromoll theorem on finiteness of topological type, for manifolds with nonnegative Ricci curvature, curvature bounded below and slow diameter growth; [AGl].

The first two of these theorems are stated in § 3 and proved in § 4 and §§ 5-6, respectively. The third is stated and proved in § 8.

The reader is assumed to have a background in riemannian geometry at least the rough equivalent of the first six chapters of [CE], and to be familiar with basic algebraic topology. For completeness however, the statement of Toponogov's theorem is recalled in § 2. Additional material on finiteness theorems and on Ricci curvature is provied in § 3 and § 7.

1. Critical Points of Distance Functions.

Let M^n be a complete riemannian manifold. We will assume that all geodesics are parametrized by arc length. For $p \in M^n$, we denote the distance from x to p by $\overline{x,p}$ and put

$$\rho_p(x) := \overline{x,p}$$

Note that $\rho_p(x)$ is smooth on $M \setminus \{p \cup C_p\}$, where C_p, the *cut locus* of p, is a closed nowhere dense set of measure zero.

Grove and Shiohama made the fundamental observation that there is a meaningful definition of "critical point" for such distance functions, such that in the absence of critical points, the Isotopy Lemma of Morse Theory holds. They also observed that in the presence of a lower curvature bound, Toponogov's theorem can be used to derive geometric information, from the *existence* of critical points. They used these ideas to give a short proof of a generalized Sphere Theorem, see Theorem 2.5. Other important applications are discussed in subsequent sections.

Remark 1.1. If the *sectional curvature* satisfies $K_M \leq K$ (for $K \geq 0$) and q is a critical point of ρ_p with $\rho_p(q) \leq \frac{\pi}{2\sqrt{K}}$, then there is also a reasonable notion of *index* which predicts the change in the *topology* when crossing a critical level. But so far, this fact has not had strong applications.

Definition 1.2. The point q $(\neq p)$ is a *critical point* of ρ_p if for all v in the tangent space, M_q, there is a minimal geodesic, γ, from q to p, making an angle, $\measuredangle\,(v, \gamma'(0)) \leq \frac{\pi}{2}$, with $\gamma'(0)$. Also, p is a critical point of ρ_p.

From now on we just say that q is a critical point of p.

Remark 1.3. If $q \neq p$ is a critical point of p, then $q \in C_p$. If q is *not* critical, the collection of tangent vectors to all geodesics, γ, as above, lies in some *open* half space in M_q. Thus, there exists $w \in M_q$, such that $\measuredangle\,(w, \gamma'(0)) < \frac{\pi}{2}$, for all minimal γ from p to q.

Put $B_r(p) = \{x \mid \overline{x,p} < r\}$.

Isotopy Lemma 1.4. *If* $r_1 < r_2 \leq \infty$, *and if* $\overline{B_{r_2}(p)} \setminus B_{r_1}(p)$ *is free of critical points of* ρ_p, *then this region is homeomorphic to* $\partial B_{r_1}(p) \times [r_1, r_2]$.*Moreover,* $\partial B_{r_1}(p)$ *is a topological submanifold (with empty boundary).*

Proof: If x is noncritical, then there exists $w \in M_x$ with $\measuredangle\,(\gamma'(0), w) < \frac{\pi}{2}$, for all minimal γ from x to p. By continuity, there exists an extension of w to a vector field, W_x, on a neighborhood, U_x, of x, such that if $y \in U_x$ and σ is minimal from y to p, then $\measuredangle\,(\sigma'(0), W_x(y)) < \frac{\pi}{2}$. Take a finite open cover of $\overline{B}_{r_2}(\rho) \setminus B_{r_1}(\rho)$, by sets, U_{x_i}, locally finite if $r_2 = \infty$, and a smooth partition of unity, $\sum \phi_i \equiv 1$, subordinate to it. Put $W = \sum \phi_i W_{x_i}$. Clearly, W is nonvanishing. For each integral curve ψ of W, the *first variation formula* gives

$$\rho_p(\psi(t_2)) - \rho_p(\psi(t_1)) \leq (t_1 - t_2)\cos(\frac{\pi}{2} - \epsilon) ,$$

for some small ϵ. This holds on compact subsets if $r_2 = \infty$. The first statement easily follows.

To see that $\partial B_{r_1}(p)$ is a submanifold, let $q \in \partial B_{r_1}(p)$, σ a minimal geodesic from q to p, and V a small piece of the totally geodesic hypersurface at q, normal to σ. Then for $z \in V$, sufficiently close to q, each integral curve, ψ, of W through z intersects $\partial B_{r_1}(p)$ in exactly one point, $z' \in \partial B_{r_1}(p)$ (ψ extends on both sides of V). It is easy to check that the map, $z \to z'$, provides a local chart for $\partial B_{r_1}(p)$ at q.

Example 1.5. M compact and q a farthest point from p implies that q is a critical point of ρ_p, obviously, the topology changes when we pass q. This observation was made by Berger, well in advance of the formal definition of "critical point"; [Be].

Example 1.6. If γ is a geodesic loop of length ℓ and if $\gamma \mid [0, \frac{\ell}{2}]$ and $\gamma \mid [\frac{\ell}{2}, \ell]$ are minimal, then $\gamma(\frac{\ell}{2})$ is a critical point of $\gamma(0)$. In particular, if q is a closest point, to p on C_p, and q is not conjugate to p along some minimal geodesic then q is a critical point of p; see Chapter 5 of[CE]. Thus, if p, q realize the shortest distance from a point to its cut locus in M^n, and are not conjugate along any minimal γ, then p and q are *mutually critical.*

Example 1.7. On a flat torus with fundamental domain a rectangle, the barycenters of the sides and the corners project to the three critical points of p, other than p itself.

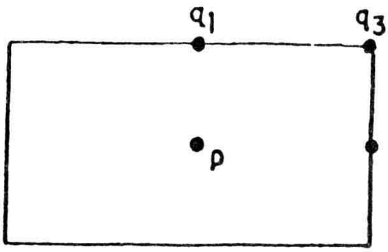

Fig. 1.1

Example 1.8. A conjugate point need not be critical. Here is a concrete example. Write the standard metric on S^2 in the form $g = dr^2 + \sin^2 r\, d\theta^2$, where $0 \leq r \leq \pi$, $0 \leq \theta \leq 2\pi$. Let $f(r,\theta)$ be a smooth function, periodic in θ, such that

i) $f(r,\theta) \equiv 1$, for all (r,θ) satisfying any of the following conditions:

$$0 \leq r \leq \frac{\pi}{4} , \qquad \frac{3}{4}\pi \leq r \leq \pi ,$$
$$\pi - \epsilon \leq \theta \leq \pi + \epsilon .$$

Here we require $\epsilon < \pi/4$.

ii) $f > 1$ elsewhere.

The metric $g' = f\,dr^2 + \sin^2 r\, d\theta^2$ satisfies $g' \geq g$. In fact, if the intersection of a curve, c, with the region, $\pi/4 < r < 3\pi/4$, is not contained in the region $\pi - \epsilon \leq \theta \leq \pi + \epsilon$, then its length with respect to g' is strictly longer than with respect to g. It follows that for the metric g', the only minimal geodesics connecting the "south pole" $(\theta = \theta)$ to the "north pole", $(\theta = \pi)$ are the curves $c(t) = (t, \theta_0)$, $\pi - \epsilon \leq \theta_0 \leq \pi + \epsilon$. Since $2\epsilon < \pi$, it follows that the north and south poles are mutually *conjugate*, but mutually *noncritical.*

We are indebted to D. Gromoll for helpful discussions concerning this example.

Remark 1.9. The *criticality radius*, r_p, is, by definition, the largest r such that $B_r(p)$ is free of critical points. By the Isotopy Lemma 1.4, $B_{r_p}(p)$ is homeomorphic to a standard open ball, since it is homeomorphic to an arbitrarily small open ball with center p.

2. Toponogov's Theorem; first applications.

Denote the length of γ by $L[\gamma]$.

By definition, a *geodesic triangle* consists of three geodesic segments, γ_i, of length $L[\gamma_i] = \ell_i$, which satisfy

$$\gamma_i(\ell_i) = \gamma_{i+1}(0) \bmod 3 \quad (i = 0, 1, 2)\ .$$

The *angle* at a corner, say $\gamma_0(0)$, is by definition, $\measuredangle(-\gamma_2'(\ell_2), \gamma_0'(0))$. The angle opposite γ_i will be denoted by α_i.

A pair of sides e.g. γ_2, γ_0 are said to determine a *hinge*.

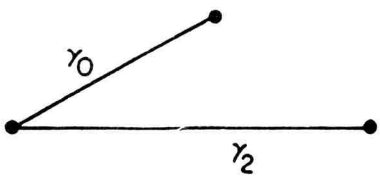

Fig. 2.1

Let M^n_H denote the n-dimensional, simply connected space of curvature $\equiv H$ (i.e. hyperbolic space, Euclidean space, or a sphere).

Toponogov's theorem has two statements. These are equivalent in the sense that either one can easily be obtained from the other.

Theorem 2.1 (Toponogov). *Let M^n be complete with curvature $K_M \geq H$.*

A) *Let $\{\gamma_0, \gamma_1, \gamma_2\}$ determine a triangle in M^n. Assume γ_1, γ_2 are minimal and $\ell_1 + \ell_2 \geq \ell_0$. If $H > 0$, assume $L[\gamma_0] \leq \frac{\pi}{\sqrt{H}}$. Then there is a triangle $\{\underline{\gamma}_0, \underline{\gamma}_1, \underline{\gamma}_2\}$ in M^2_H, with $L[\gamma_i] = L[\underline{\gamma}_i]$ and $\underline{\alpha}_1 \leq \alpha_1$, $\underline{\alpha}_2 \leq \alpha_2$.*

B) *Let* $\{\gamma_2, \gamma_0\}$ *determine a hinge in* M^n *with angle* α. *Assume* γ_2 *is minimal and if* $H > 0$, $L[\gamma_0] \leq \frac{\pi}{\sqrt{H}}$. *Let* $\{\underline{\gamma}_2, \underline{\gamma}_0\}$ *determine a hinge in* M^2_H *with* $L[\gamma_i] = L[\underline{\gamma}_i]$, $i = 0, 2$, *and the same angle* α. *Then*

$$\overline{\gamma_2(0), \gamma_0(\ell_0)} \leq \overline{\underline{\gamma}_2(0), \gamma_0(\ell_0)} .$$

Proof: See [CE], Chapter 2.

Remark 2.2. In the applications of Toponogov which occur in the sequel, the following elementary fact is often used without explicit mention. Consider the collection of hinges, $\{\underline{\gamma}_0, \underline{\gamma}_2\}$ in M^2_H, with fixed side lengths, ℓ_0, ℓ_2 and variable angle α; $0 \leq \alpha \leq \pi$. *Then* $\overline{\underline{\gamma}_0(\ell_0), \underline{\gamma}_2(0)}$ *is a strictly increasing function of* α.

Remark 2.3. If the inequalities in A) or B) are all equalities, more can be said (see [CE]).

By using Toponogov's theorem we can derive geometric information from the existence of critical points.

Let the triangle, $\{\gamma_0, \gamma_1, \gamma_2\}$ satisfy the hypothesis of Toponogov's theorem, and assume $\gamma_0(\ell_0)$ is *critical* with respect to $\gamma_0(0)$. Then (as explained in detail in the applications), we can

i) bound from above the side length ℓ_2 (see Theorems 2.5, 4.2),

ii) bound from below, the excess, $\ell_0 + \ell_1 - \ell_2$ (see Proposition 8.5),

iii) bound from below, the angle α_1 (see Lemma 2.6, Corollaries 2.7, 2.9, 2.10 and 6.3).

Remark 2.4. It is important to realize that in order to obtain the preceding bounds, we do *not* assume $\alpha_2 \leq \pi/2$. The assumption that $\gamma_1(\ell_0)$ is *critical* with respect to $\gamma_0(0)$ implies that $\measuredangle(-\tilde{\gamma}_0'(\ell_0), \gamma_1'(0)) \leq \pi/2$, for *some* minimal $\tilde{\gamma}_0$ from $\gamma_0(0)$ to $\gamma_0(\ell_0)$. This is all that we require.

Theorem 2.5 (Grove-Shiohama). *Let* M^n *be complete*, with $K_M \geq H$, *for some* $H > 0$. *If* M^n *has diameter*, $\text{dia}(M^n) > \frac{\pi}{2\sqrt{H}}$, *then* M^n *is homeomorphic to the sphere*, S^n.

Proof: Let $p, q \in M^n$ be such that $\overline{p,q} = \text{dia}(M^n)$; in particular, p and q are mutually critical (see Example 1.5).

Claim. There exists no $x \neq q, p$ which is critical with respect to p (the same holds for q).

Proof of Claim: Assume x is such a point. Let γ_2 be minimal from q to x. By assumption there exists γ_0, minimal from x to p, with

$$\alpha_1 = \measuredangle(-\gamma_2'(\ell_2), \gamma_0'(0)) \leq \frac{\pi}{2} .$$

Similarly, since p and q are mutually critical, there exist *minimal* γ_1, $\tilde{\gamma}_1$ from p to q such that

$$\measuredangle(-\gamma_0'(\ell_0), \gamma_1'(0)) \leq \frac{\pi}{2}$$

and

$$\measuredangle(-\tilde{\gamma}_1'(\ell_1), \gamma_2'(0)) \leq \frac{\pi}{2} .$$

Note that $L[\gamma_1] = L[\tilde{\gamma}_1] = \overline{p,q} > \frac{\pi}{2\sqrt{H}}$.

Apply A) of Toponogov's theorem to both $\{\gamma_0, \gamma_1, \gamma_2\}$ and $\{\gamma_0, \tilde{\gamma}_1, \gamma_2\}$. Since a triangle in M_H^2 (the sphere) is determined up to congruence by its side lengths, we get a *unique* triangle, $\{\underline{\gamma}_0, \underline{\gamma}_1, \underline{\gamma}_2\}$, in M_H^2, *all* of whose angles are $\leq \pi/2$. By elementary spherical trigonometry, this implies that all sides have length $\leq \frac{\pi}{2\sqrt{H}}$, contradicting $\overline{p,q} > \frac{\pi}{2\sqrt{H}}$.

Given the claim, the proof is easily completed (compare the proof of Reeb's Theorem given in [M]).

The following observation and its corollaries (2.7, 2.10) are of great importance.

Lemma 2.6 (Gromov). *Let q_1 be critical with respect to p and let q_2 satisfy*

$$\overline{p, q_2} \geq \nu \overline{p, q_1} \ ,$$

for some $\nu > 1$. Let γ_1, γ_2 be minimal geodesics from p to q_1, q_2 respectively and put $\theta = \measuredangle(\gamma_1'(0), \gamma_2'(0))$.

i) *If $K_M \geq 0$,*

$$\theta \geq \cos^{-1}(1/\nu) \ .$$

ii) *If $K_M \geq H$, $(H < 0)$* and $\overline{p, q_2} \leq d$, *then*

$$\theta \geq \cos^{-1}\left(\frac{\tanh(\sqrt{-H}d/\nu)}{\tanh(\sqrt{-H}d)}\right) .$$

Proof: Put $\overline{p,q_1} = x$, $\overline{q_1, q_2} = y$, $\overline{p,q_2} = z$. Let σ be minimal from q_1 to q_2. Since q is critical for p, there exists τ, minimal from q to p with

$$\measuredangle(\sigma'(0), \tau'(0)) \leq \frac{\pi}{2} \ .$$

i) Applying Toponogov's Theorem B) to the hinges $\{\sigma, \tau\}$ and $\{\gamma_1, \gamma_2\}$ gives

$$z^2 \leq x^2 + y^2 \ ,$$

$$y^2 \leq x^2 + z^2 - 2xz\cos\theta \quad \text{(law of cosines)}$$

Since $z \geq \nu \cdot x$, the conclusion easily follows.

ii) By scaling, we can assume $H = -1$. Replace the inequalities above by the following ones from hyperbolic trigonometry (see e.g. [Be])

$$\cosh z \leq \cosh x \cosh y \ ,$$

$$\cosh y \leq \cosh x \cosh z - \sinh x \sinh z \cos\theta \ .$$

Substituting the second of these into the first and simplifying gives

$$\theta \geq \cos^{-1}\left(\frac{\tanh x}{\tanh z}\right) ,$$

which suffices to complete the proof.

Corollary 2.7. *Let $q_1, \ldots, q_N$ be a sequence of critical points of p, with*

$$\overline{p, q_{i+1}} \geq \nu \, \overline{p, q_i} \quad (\nu > 1)$$

i) *If $K_{M^n} \geq 0$ then*

$$N \leq \mathcal{N}(n, \nu)$$

ii) *If $K_M \geq H$ $(H < 0)$ and $q_N \leq d$, then*

$$N \leq \mathcal{N}(n, \nu, Hd^2) .$$

Proof: Take minimal geodesics, γ_i from p to q_i. View $\{\gamma_i'(0)\}$ as a subset of $S^{n-1} \subset M_p^n$. Then Lemma 2.6 gives a lower bound on the distance, θ, between any pair $\gamma_i'(0), \gamma_j'(0)$. The balls of radius $\theta/2$ about the $\gamma_i'(0) \in S^{n-1}$ are mutually disjoint. Hence, if we denote by $V_{n-1,1}(r)$, the volume of a ball of radius r on S^{n-1}, we can take

$$\mathcal{N} = \frac{V_{n-1,1}(\pi)}{V_{n-1,1}(\theta/2)} ,$$

where $V_{n-1,1}(\pi) = \mathrm{Vol}(S^{n-1})$ and θ is the minimum value allowed by Lemma 2.6.

Remark 2.8. It turns out that Corollary 2.7 is the only place in which the hypothesis on *sectional* curvature is used in deriving Gromov's bound on Betti numbers in terms of curvature and diameter. For details, see Theorem 3.8 and §§ 5-6.

The following result is a weak version (with a much shorter proof) of the main result of [CG12], compare also § 8.

Corollary 2.9. *Let M^n be complete,* with $K_{M^n} \geq 0$. *Given p, there exists a compact set C, such that p has no critical points lying outside C. In particular M^n is homeomorphic to the interior of a compact manifold with boundary.*

Proof: The first statement, which is obvious from Corollary 2.7, easily implies the second.

Corollary 2.10. *Let $\mathcal{N}(n, \nu, Hd^2)$ be as in* Corollary 2.7, *and let $r_1 \nu^{\mathcal{N}} < r_2$. Then there exists $(s_1, s_2) \subset [r_1, r_2]$ such that $\rho_p^{-1}((s_1, s_2))$ is free of critical points and*

$$s_2 - s_1 \geq (r_2 - r_1 \nu^{\mathcal{N}})(1 + \nu + \cdots \nu^{\mathcal{N}})^{-1}$$

Moreover, the set of critical points has measure at most $(1 - \nu^{-\mathcal{N}}) r_2$.

Proof: Let $r_1 + \ell_1$ denote the first critical value $\geq r_1$; $\ell_2 + \nu(r_1 + \ell_1)$ the first after $\nu(r_1 + \ell_1)$ etc. It is easy to see that in the worst case

$$\ell_1 = \ell_2 = \cdots = \ell ,$$

$$(\cdots(\nu(\nu(r_1 + \ell) + \ell) + \ell \cdots) + \ell = r_2$$

The first assertion follows easily. The proof of the second is similar.

Remark 2.11. The proof of Corollary 2.7 easily yields an explicit estimate for the constant $\mathcal{N}$. For example, in case $K_{M^n} \geq 0$, we get

$$\mathcal{N}(n,\nu) \leq \left(\frac{\pi}{\frac{1}{2}\cos^{-1}(1/\nu)}\right)^{n-1}$$

Thus, for ν close to 1,

$$\mathcal{N}(n,\nu) \leq \left[\frac{2\pi^2}{(\nu-1)}\right]^{(n-1)/2}$$

3. Background on Finiteness Theorems.

The theorems in question bound topology in terms of bounds on geometry. In subsequent lectures we will prove two such results due to Gromov, [G] and Grove-Petersen [GrP]. Before stating these, we establish the context by giving an earlier result of Cheeger [C1], [C3] (see also [GLP], [GreWu], [Pe1], [Pe2], [We] for related developments).

Theorem 3.1. (Cheeger). *Given $n, d, V, K > 0$, the collection of compact n-manifolds which admit metrics whose diameter, volume and curvature satisfy,*

$$\mathrm{dia}(M^n) \leq d\ ,$$
$$\mathrm{Vol}(M^n) \geq V\ ,$$
$$|K_M| \leq K\ ,$$

contains only a finite number, $C(n,, V^{-1}d^n, Kd^2)$, of diffeomorphism types.

Remark 3.2. The basic point in the proof is to establish a lower bound on the length of a smooth closed geodesic (here one need only assume $K_M \geq K$). This, together with the assumption $K_M \leq K$, gives a lower bound on the injectivity radius of the exponential map (see [CE], Chapter 5). Although Theorem 3.1 predated the use of critical points, the crucial ingredient in the Grove-Petersen theorem below is essentially a generalization of the above mentioned lemma on closed geodesics (compare Example 1.6).

Theorem 3.3 (Grove-Petersen). *Given $n, d, V > 0$ and H, the collection of compact n-dimensional manifolds which admit metrics satisfying*

$$\mathrm{dia}(M^n) \leq d\ ,$$
$$\mathrm{Vol}(M^n) \geq V\ ,$$
$$K_M \geq H\ ,$$

contains only a finite number, $C(n, V^{-1}d^n, Hd^2)$, of homotopy types.

Remark 3.4. In [GrPW], the conclusion of Theorem 3.3 is strengthened to finiteness up to homeomorphism ($n \neq 3$) and up to diffeomorphism ($n \neq 3, 4$). The proof employs techniques from "controlled topology". Thus, Theorem 3.3 supersedes Theorem 3.1 (as stated) if $n \neq 3, 4$. However, Theorem 3.1 can actually be strengthened to give a conclusion which does not hold under the hypotheses of Theorem 3.3.

Given $\{M_i^n\}$ as in Theorem 3.1, there is a subsequence $\{M_j^n\}$, a manifold M^n, and diffeomorphisms, $\phi_j : M^n \to M_j^n$, such that the pulled back metrics, $\phi_j^*(g_j)$, converge in the $C^{1,\alpha}$-topology, for all $\alpha > 1$ (see the references given at the beginning of this section for further details).

Example 3.5. By rounding off the tip of a cone, a surface of nonnegative curvature is obtained. From this example, one sees that under the conditions of Theorem 3.3, arbitrarily small metric balls need not be contractible. Thus, the criticality radius can be arbitrarily small (compare Remark 1.9).

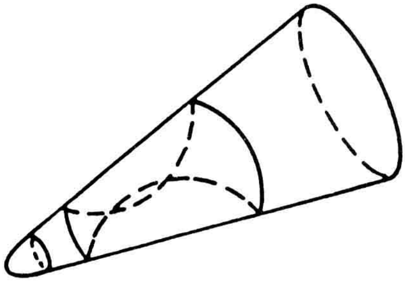

Fig. 3.1

However, it will be shown that the inclusion of a sufficiently small ball into a somewhat larger one is homotopically trivial.

Example 3.6. Consider the *surface* of a solid cylindrical block from which a large number, j, of cylinders (with radii tending to 0) have been removed.

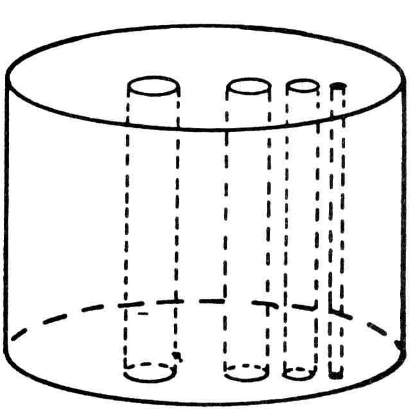

Fig. 3.2

The edges can be rounded so as to obtain a manifold, M_j^2, with $\mathrm{Vol}(M_j^2) \geq V$, $\mathrm{dia}(M_j^2) \leq d$ (but

$\inf K_{M_j^2} \to -\infty$, as $j \to \infty$). For the first Betti number, one has $b^1(M_j^2) = 2j \to \infty$.

Note that the metrics in this sequence can be rescaled so that $K_{M_j^2} \geq -1$, $\mathrm{Vol}(M_j^2) \to \infty$. Then, of course, $\mathrm{dia}(M_j^2) \to \infty$ as well.

Example 3.7. Consider the lens space L_n^3, obtained by dividing

$$S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\} \ ,$$

by the action of $\mathbb{Z}_n = \{1, a, \dots, a^{n-1}\}$, where $a : (z_1, z_2) \to (e^{2\pi i/n} z_1,\, e^{2\pi i/n} z_2)$. Then $\mathrm{dia}(L_n^3) = 1$, $K_{L_n^3} \equiv 1$, but $\mathrm{Vol}(M_n^3) \to 0$, and $H_1(L_n^3, \mathbb{Z}) = \mathbb{Z}_n$. Thus, if the lower bound on volume is relaxed, there are infinitely many possibilities for the first homology group, H_1. Nonetheless, the following theorem of Gromov asserts that for any *fixed* coefficient field, F, the Betti numbers, $b^i(M^n)$ are bounded independent of F.

Theorem 3.8 (Gromov). *Given $n, d > 0$, H, and a field F, if*

$$\mathrm{dia}(M^n) \leq d \ ,$$
$$K_{M^n} \geq H \ ,$$

then

$$\sum_i b^i(M^n) \leq C(n,\, Hd^2) \ .$$

Corollary 3.9. *If M^n has nonnegative sectional curvature, $K_{M^n} \geq 0$, then*

$$\sum_i b^i(M^n) \leq C(n) \ .$$

Remark 3.10. The most optimistic conjecture is that $K_{M^n} \geq 0$ implies $b^i(M^n) \leq \binom{n}{i}$, and hence, $\sum_i b^i(M^n) \leq 2^n$. Note $b^i(T^n) = \binom{n}{i}$ where T^n is a flat n-torus. At present, one knows only that $K_{M^n} \geq 0$ (in fact $\mathrm{Ric}_{M^n} \geq 0$) implies $b^1(M^n) \leq n$. But the method of proof of Theorem 3.8 does not give this sharp estimate; compare also [GLP], p. 72.

In proving Theorems 3.1, 3.3 and 3.8, a crucial point is to bound the number of balls of radius ϵ needed to cover a ball of radius r.

Proposition 3.11 (Gromov). *Let the Ricci curvature of M^n satisfy* $\mathrm{Ric}_{M^n} \geq$ $(n-1)H$. *Then given $r, \epsilon > 0$ and $p \in M^n$, there exists a covering, $B_r(p) \subset \cup_1^N B_\epsilon(p_i)$, $(p_i \in B_r(p))$ with $N \leq N_1(n, Hr^2, r/\epsilon)$. Moreover, the multiplicity of this covering is at most $N_2(n, Hr^2)$.*

Remark 3.12. The condition $\mathrm{Ric}_{M^n} \geq (n-1)H$ is implied by $K_{M^n} \geq H$, in which case, the bound on N_1 could be obtained from Toponogov's theorem. For the proof of Proposition 3.11, see § 7.

Remark 3.13. The conclusion of Theorem 3.8 (and hence of Corollary 2.7) fails if the hypothesis $K_m \geq H$ is weakened to the lower bound on Ricci curvature, $Ric_{M^n} \geq (n-1)H$; see [An], [ShY].

Remark 3.14. S. Zhu has shown that homotopy finiteness continues to hold for $n = 3$, if the lower bound on sectional curvature is replaced by a lower bound on Ricci curvature; [Z]. Whether or not this remains true in higher dimensions is an open problem.

4. Homotopy finiteness.

Pairs of mutually critical points.

The main point in proving the theorem on homotopy finiteness is to establish a lower bound on the distance between a pair of *mutually critical* points (compare Example 1.6). For technical reasons we actually need a quantitative refinement of the notion of criticality.

Definition 4.1. q is *ϵ-almost critical* with respect to p, if for all $v \in M_q$, there exists γ, minimal from q to p, with $\measuredangle\,(v, \gamma'(0)) \leq \frac{\pi}{2} + \epsilon$.

Theorem 4.2. *There exist $\epsilon = \epsilon(n, V^{-1}d^n, Hd^2)$, $\delta = \delta(n, V^{-1}d^n, Hd^2) > 0$, such that if $p, q \in M^n$*

$$\begin{aligned} &\mathrm{dia}(M^n) \leq d\ , \\ &\mathrm{Vol}(M^n) \geq V\ , \\ &K_{M^n} \geq H\ , \\ &\overline{p,q} < \delta d\ , \end{aligned}$$

then at least one of p, q is not ϵ-almost critical with respect to the other.

The proof of Theorem 4.2 uses two results on volume comparison. The first of these, Lemma 4.3, is stated below and proved in the Appendix to this section. The second result, Proposition 4.7 is stated and proved in the Appendix.

For $X \subset Y$ closed, put

$$T_r(X) = \{q \in Y \mid \overline{q, X} < r\}$$

(the case of interest below is $Y = S^{n-1}$, the unit $(n-1)$-sphere).

Recall that the volume of a ball in M^n_H is given as follows. Put

$$\mathcal{A}_{n-1,H}(s) = \begin{cases} (\frac{1}{\sqrt{H}} \sin s\sqrt{H}\, s)^{n-1} & H > 0 \\ s^{n-1} & H = 0 \\ (\frac{1}{\sqrt{-H}} \sinh \sqrt{-H}\, s)^{n-1} & H < 0 \end{cases}$$

$$V_{n,H}(r) = v_{n-1} \int_0^r \mathcal{A}_{n-1,H}(s)\, ds\ ,$$

where $v_{n-1} = V_{n-1,1}(\pi)$ is the volume of the unit $(n-1)$-sphere. Then in M^n_H,

$$\mathrm{Vol}(B_r(\underline{p})) = V_{n,H}(r)\ .$$

Lemma 4.3. *Let $X \subset S^n$ be closed. Then*

a)
$$\frac{\mathrm{Vol}(T_{r_1}(X))}{\mathrm{Vol}(T_{r_2}(X))} \geq \frac{V_{n,1}(r_1)}{V_{n,1}(r_2)}\ .$$