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CERTAIN QUESTIONS ON THE GENERAL THEORY OF THE STABILITY OF MOTION IN THE SENSE OF LIAPOUNOFF

Introduction

We will study here the problem of the stability of motion in the sense of Liapounoff[3] from the mathematical point of view. This consists of the following:

Given a system of differential equations

$$\frac{dx_i}{dt} = X_i(t, x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n), \tag{1}$$

where the X_i in the region

$$t \geq t_0, |x_i| \leq H, \quad t_0 \geq 0, \quad H > 0 \quad (i = 1, 2, \ldots, n), \tag{2}$$

are certain continuous functions of $t, x_1, \ldots, x_n$, reducing to zero for $x_1 = \ldots = x_n = 0$.

If we consider t as the time and $x_1, \ldots, x_n$ as certain functions of the coordinates and velocities of a dynamic system, a definite motion of the system will correspond to every solution of equations (1). Since $X_i(t, 0, \ldots, 0) = 0$, equations (1) admit of the obvious solution $x_1 = \ldots = x_n = 0$. We will call the motion corresponding to this solution "undisturbed." Motions corresponding to all remaining solutions of equations (1) will be called "disturbed" and these equations themselves will be called the "equations of disturbed motion."

It is required to find whether it is possible to find for every positive number ϵ, no matter how small, another positive number η such that, if the initial values x_s^0 of the quantities x_s corresponding to $t = t_1 \geqslant t_0$ are chosen in accordance with the conditions

$$|x_s^0| \leq \eta \qquad (s = 1,2,\ldots,n), \tag{3}$$

then for all $t > t_1$ we shall have

$$|x_s| \leq \epsilon \qquad (s = 1,2,\ldots,n). \tag{4}$$

If such a number η exists, the undisturbed motion is called "stable" with respect to the quantities $x_1,\ldots,x_n$. In the contrary case, it is called "unstable."*

Inequalities (3) and (4) define two regions near the origin of coordinates. Later on we shall call the second of these (defined by inequalities (4)), region K.

It may happen that in fulfilling condition (3) we may also satisfy the condition that

$$\lim_{t \to \infty} x_s = 0 \qquad (s = 1,2,\ldots,n),$$

that is, that every disturbed motion sufficiently near the undisturbed motion approaches it asymptotically. In this case we will say that the undisturbed motion is "asymptotically" stable.

*It is possible to investigate a more general problem: that of the stability of the same motion but with respect not to all, but to only certain of the quantities $x_1,\ldots,x_n$, for instance with respect to the quantities $x_1,\ldots,x_m$ $(m < n)$. The treatment of this problem is derived from the preceding by the substitution for inequalities (4) of the following

$$|x_i| \leq \epsilon \qquad (i = 1,2,\ldots,m).$$

Liapounoff divides into two categories all methods which it is possible to indicate for the solution of the problem of stability. He includes in the first category those methods which reduce to the direct consideration of the disturbed motion, that is, to the determination of the general or a particular solution of equations (1). It is usually necessary to search for these solutions in a variety of forms, of which the simplest are those which reduce to the usual method of successive approximations.

Liapounoff calls the totality of all methods of this first category the "first method."

It is possible, however, to indicate other methods of solution of the problem of stability which do not necessitate the calculation of a particular or the general solution of the equations of disturbed motion, but which reduce to the search for certain functions of $t, x_1, \ldots, x_n$, possessing special properties. Liapounoff calls the totality of all methods of this second category the "second method."

Liapounoff states several general propositions as a basis for his second method. But before going into the formulation of these propositions, we will give here certain definitions which we will use in the future.

Let $V(t, x_1, \ldots, x_n)$ represent a certain function of $t, x_1, \ldots, x_n$ which is continuous for all values of these variables lying in the region

$$t \geqslant T, |x_s| \leq H \qquad (s = 1, 2, \ldots, n), \tag{5}$$

where $T \geq t_0$ is an arbitrarily large positive number, and which reduces to zero for $x_1 = \ldots = x_n = 0$.

Definition 1. The function V is called semi-definite if in region (5) it may assume values of only one fixed sign (or also zero values).

Definition 2. The function V, if it is independent of t, is called definite if it is semi-definite in region (5) and reduces to zero only for $x_1 = \ldots = x_n = 0$.

The function V, if it is dependent on t, is called definite if it satisfies one of the following conditions in region (5):

$$V \geq W(x_1,\ldots,x_n), \qquad V \leq -W(x_1,\ldots,x_n),$$

where W is a certain positive definite function, independent of t.

Definition 3. We will say that the function $V(t,x_1,\ldots,x_n)$ admits of an infinitely small upper bound if for every positive number ϵ, no matter how small, a positive number h (different from zero) may be found such that for all values of the variables $t,x_1,\ldots,x_n$ satisfying the conditions

$$t \geq T, \quad |x_s| \leq h \qquad (s = 1,2,\ldots,n)$$

we have

$$|V| \leq \epsilon.$$

Definition 4. A definite function V, whose total derivative with respect to time in virtue of the equations of disturbed motion, that is, the expression

$$V' = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} X_1 + \dots + \frac{\partial V}{\partial x_n} X_n ,$$

either is a semi-definite function opposite in sign to V or is identically equal to zero, is called a "Liapounoff function."

Using these definitions, we are able to express Liapounoff's basic theorems in the following way:

Theorem A. If a Liapounoff function exists for the equations of disturbed motion, the undisturbed motion is stable.

Note. Let us suppose that for the equations of disturbed motion it is possible to find a function $V(t, x_1, \dots, x_n)$, possessing the following characteristics:

1) V reduces to zero when $x_1 = x_2 = \dots = x_k = 0$

2) for all values of the variables lying in the region

$$t \geq T, \quad |x_i| \leq H, \qquad x_s \text{ arbitrary}$$
$$(i = 1,2,\dots,k; \quad s = k+1,\dots,n), \tag{6}$$

the function V satisfies the condition

$$V \geq W(x_1, \dots, x_k)$$

where W is a function of $x_1, \dots, x_k$ independent of t and is positive definite

3) for all values of the variables lying in region (6), $V' \leq 0$.