Lecture Notes in Mathematics

Edited by A. Dold, B. Eckmann and F. Takens

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L. Accardi W. von Waldenfels (Eds.)

Quantum Probability and Applications V

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Quantum Probability and Applications V

Proceedings of the Fourth Workshop, held in Heidelberg, FRG, Sept. 26–30, 1988

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INTRODUCTION

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Luigi Accardi
Wilhelm von Waldenfels

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QUANTUM LANGEVIN EQUATION IN THE WEAK COUPLING LIMIT

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0. Introduction

This work is part of a series of papers [2,3,4] where, expanding some heuristic ideas in [11], we develop the theory of the weak coupling limit for open quantum systems. It has been known for fifteen years that the reduced dynamics of an open quantum system converges to a quantum dynamical semigroup in the weak coupling limit (coupling constant $\lambda \to 0$, microscopic time $s \to \infty$, with macroscopic time $t = \lambda^2 s$ held constant) [18,6]. We investigate whether and in which sense the full time evolution of an open quantum system converges, in the weak coupling limit, to an evolution driven by quantum Brownian motion [16,14,15]. In [2] we obtained rigorous results for the time evolution operator $U^{(\lambda)}_{t/\lambda^2}$; in [3] we studied the time evolved observables $j^{(\lambda)}_{t/\lambda^2}(X) = U^{(\lambda)+}_{t/\lambda^2}(X \otimes 1)U^{(\lambda)}_{t/\lambda^2}$, and we proved that $j^{(\lambda)}_{t/\lambda^2}$ converges to a quantum diffusion j_t [10] governed by a quantum Langevin equation.

Here we wish to present the results of [3] in a self-contained way. To this end we adopt a method of proof which is simpler than the one of [3], by reducing the problem to a time-dependent generalization of the derivation of a quantum dynamical semigroup in the weak coupling limit [18,6]. However, the price to be paid for this simplification is that the present method is specific for boson reservoirs with linear coupling to the system of interest, whereas the method of [3] is suitable for generalization to the fermion case [4], to more general interactions, and to the low density limit [9,5].

1. Notations and Preliminaries

We consider a quantum system S, with associated Hilbert space $\mathcal{H}$ and Hamiltonian H, coupled to another quantum system S', with Hilbert space $\mathcal{H}'$ and Hamiltonian H', by an interaction λV, where the coupling constant λ is assumed to be "small". All Hilbert spaces in this paper will be understood to be separable. S is spatially confined, meaning that the Hamiltonian H is self-adjoint in $\mathcal{H}$, bounded from below, and such that $\exp[-\beta H]$ is trace class for all positive β. S' is an infinitely extended quasi-free boson system, meaning that $\mathcal{H}' = \Gamma(\mathcal{H}_1)$ is the symmetric Fock space over the one-particle Hilbert space $\mathcal{H}_1$, and $H' = d\Gamma(H_1)$ is the differential second quantization of the one-particle Hamiltonian H_1, a non-negative self-adjoint operator in $\mathcal{H}_1$ with Lebesgue spectrum. We shall also assume that there exists a nonzero (nonclosed) linear subspace K_1 of $\mathcal{H}_1$ such that, for all $f,g \in K_1$, we have

$$\int_{-\infty}^{+\infty} |\langle f,\exp[iH_1 t]g\rangle| \, dt < +\infty . \qquad (1.1)$$

The annihilation and creation operators in $\mathcal{H}'$ corresponding to the test functions f in $\mathcal{H}_1$ will be denoted by $a(f)$, $a^+(f)$; they satisfy the CCR $[a(f),a^+(g)] = \langle f,g\rangle$ and $a(f)\Phi_\circ = 0$, where $\Phi_\circ$ is the Fock vacuum vector.

By doubling the space $\mathcal{H}_1$ to $\mathcal{H}_1 \oplus \bar{\mathcal{H}}_1$, $\bar{\mathcal{H}}_1$ denoting the conjugate space to $\mathcal{H}_1$, this formalism allows us to consider also a representation of the CCR algebra determined by a gauge-invariant quasi-free state w_Q which is stationary under the time evolution determined by H'. Specifically, let Q be a positive self-adjoint operator in $\mathcal{H}_1$, commuting strongly with H_1 and satisfying $Q \geq 1$, and let w_Q be determined by

$$w_Q(a(f)a^+(g)) = \langle f, \tfrac{1}{2}(Q+1)g\rangle \;, \quad w_Q(a^+(g)a(f)) = \langle f, \tfrac{1}{2}(Q-1)g\rangle \;.$$

Then $a(f)$ is represented by $\pi_Q(a(f)) = a(Q_+ f \oplus 0) + a^+(0 \oplus \overline{Q_- f})$, where

$$Q_+ = [\tfrac{1}{2}(Q+1)]^{\frac{1}{2}} \;, \quad Q_- = [\tfrac{1}{2}(Q-1)]^{\frac{1}{2}} \;; \tag{1.2}$$

we have indeed $\langle \Phi_\circ, \pi_Q(a(f))\pi_Q(a^+(g))\Phi_\circ\rangle = w_Q(a(f)a^+(g)) \; : \; f,g \in H_1$.

We identify $\mathcal{H}_1$ and $\bar{\mathcal{H}}_1$ as sets, assuming that $\mathcal{H}_1$ is mapped onto $\bar{\mathcal{H}}_1$ by a conjugation commuting with H_1 and with Q, so that

$$Q_+ \exp[iH_1 t]f \oplus \overline{Q_- \exp[iH_1 t]f} = \exp[iH_1 t]Q_+ f \oplus \exp[-iH_1 t]\overline{Q_- f} \;.$$

The interaction V between S and S' is assumed to be of the form

$$V = i \sum_{j=1}^{n} [B_j \otimes a^+(g_j) - B_j^+ \otimes a(g_j)] \;, \tag{1.3}$$

where $B_1, \ldots, B_n$ are bounded operators on H, $g_1, \ldots, g_n \in K_1$, and where

$$[H, B_j] = -w_j B_j \;, \quad w_j > 0 \;:\; j = 1, \ldots, n \tag{1.4}$$

$$\langle g_j, \exp[iH_1 t]g_k\rangle = 0 \quad \text{for all } t \text{ if } j \neq k \;. \tag{1.5}$$

Such interactions arise in the so-called rotating wave approximation.

In order to derive a reduced dynamics for the observables of S, we assume that the initial state of S' is a gauge-invariant quasi-free state w_Q. Again, by doubling $\mathcal{H}_1$ to $\mathcal{H}_1 \oplus \bar{\mathcal{H}}_1$ and the set of indices $\{1,\dots,n\}$ to $\{1,\dots,2n\}$, we can reduce this to the case where the initial state of S' is the Fock vacuum: the new expression of the interaction V becomes

$$V = i \sum_{j=1}^{2n} [B_j \otimes a^+(\tilde{g}_j) - B_j^+ \otimes a(\tilde{g}_j)] , \qquad (1.6)$$

where $\tilde{g}_j = Q_+ g_j \oplus 0$, $\tilde{g}_{n+j} = 0 \oplus \overline{Q_- g_j}$, and where $B_{n+j} = - B_j^+$: $j = 1,\dots,n$. In this and in the following two Sections, tildes will be dropped and 2n will be called n again.

Let $H_\lambda = H + H' + \lambda V$ ($= H \otimes 1 + 1 \otimes H' + \lambda V$) be the total Hamiltonian for the composite system S + S', and let $H_\circ = H_{\lambda=0}$. Let also

$$U_t^{(\lambda)} = \exp[iH_\circ t] \exp[-iH_\lambda t] \quad : t \geq 0 \qquad (1.7)$$

be the time evolution operators, in the interaction picture, for state vectors in $\mathcal{H} \otimes \mathcal{H}'$. Then it has been shown [18,6] that, for all u, v in $\mathcal{H}$ and for all X in $B(\mathcal{H})$, one has

$$\lim_{\lambda\to 0} \langle u \otimes \Phi_\circ, U_{t/\lambda^2}^{(\lambda)+} (X \otimes 1) U_{t/\lambda^2}^{(\lambda)} v \otimes \Phi_\circ \rangle = \langle u, T_t(X) v\rangle , \qquad (1.8)$$

where $\{T_t : t \geq 0\}$ is a quantum dynamical semigroup (in the sense of [12,17], or a quantum Markovian semigroup in the sense of [1]) whose infinitesimal generator L is given by

$$L(X) = (\sum_{j=1}^{n} c_j(w_j) B_j^+ X B_j) + K^+ X + X K , \qquad (1.9)$$

where

$$c_j(w_j) = \frac{1}{2} \int_{-\infty}^{+\infty} \langle g_j, \exp[i(H_1 - w_j)t] g_j\rangle \, dt \quad (\geq 0) \qquad (1.10)$$

and where

$$K = -\sum_{j=1}^{n} \int_{-\infty}^{0} \langle g_j , \exp[i(H_1 - w_j)t] g_j \rangle \, dt \; B_j^+ B_j . \tag{1.11}$$

2. Statement of main result

In our previous paper [2] we have shown that there is a precise sense in which $U^{(\lambda)}_{t/\lambda^2}$ (defined by (1.7)) converges as $\lambda \to 0$ to the solution $U(t)$ of a quantum stochastic differential equation (QSDE; in the sense of [16]) of the form

$$dU(t) = \{ \sum_{j=1}^{n} [B_j \, dA_j^+(t) - B_j^+ \, dA_j(t)] + K \, dt \} \, U(t) \tag{2.1}$$

with $U(0) = 1$, where $\{A_j(t), A_j^+(t) : t \geq 0 , j = 1,\dots,n\}$ are mutually independent Fock quantum Brownian motions satisfying

$$dA_j(t) \, dA_k^+(t) = \delta_{jk} \, c_j(w_j) \, dt \tag{2.2}$$

and where K is given by (1.11) (actually, the proof in [2] is just for the case $n = 1$, but an extension to arbitrary n can be easily given). In the present paper, like in [3], we shall prove that there is a precise sense in which

$$j^{(\lambda)}_{t/\lambda^2}(X) := U^{(\lambda)+}_{t/\lambda^2} (X \otimes 1) \, U^{(\lambda)}_{t/\lambda^2} \quad : \quad X \in B(\mathcal{H}) \tag{2.3}$$

converges as $\lambda \to 0$ to

$$j_t(X) := U^+(t) \, (X \otimes 1) \, U(t) \quad : \quad X \in B(\mathcal{H}) , \tag{2.4}$$

where $U(t)$ is the solution of the QSDE (2.1): j_t is a quantum diffusion (in the sense of [10]) satisfying the following quantum Langevin equation:

$$dj_t(X) = \sum_{j=1}^{n} \left(j_t([B_j^+,X])\, dA_j(t) - j_t([B_j,X])\, dA_j^+(t) \right)$$

$$+ j_t(L(X))\, dt \quad , \qquad (2.5)$$

where $dA_j(t)$, $dA_j^+(t)$ are as in (2.1) and where L is given by (1.9).

We introduce the same definitions as in [2,3]. Because of (1.5), we can define a strongly continuous group $(S_t : t \in R)$ of unitary operators on H_1 such that

$$S_t g_j = \exp[i(H_1 - w_j)t]\, g_j \quad : \quad t \in R \; , \; j = 1,\ldots,n \; . \qquad (2.6)$$

We introduce a scalar product $(.|.)$ on K_1 by

$$(f|g) = \int_{-\infty}^{+\infty} \langle f, S_t g\rangle\, dt \quad : \quad f \; , \; g \in K_1 \; , \qquad (2.7)$$

and we denote by K the completion of $K_1/\ker(.|.)$ with respect to the norm got from $(.|.)$. We also introduce collective Weyl operators on H' as $W(h *_\lambda f)$: $h \in L^2(R)$, $f \in K_1$, where, as usual, $W(g) = \exp[a^+(g) - a(g)]$, and where

$$h *_\lambda f = \lambda \int_{-\infty}^{+\infty} h(\lambda^2 t)\, S_t f\, dt \quad : \quad h \in L^2(R) \; , \; f \in K_1 \qquad (2.8)$$

(in [2,3], h has been always taken to be the indicator function of an interval).

In addition to the Fock space $\mathcal{H}' = \Gamma(\mathcal{H}_1)$ we shall also need the Fock space $\mathcal{H}'' = \Gamma(L^2(R) \otimes K)$ over $L^2(R) \otimes K$. Weyl operators on either Fock space will be always denoted by $W(.)$, and the Fock vacuum vector of either space will be always denoted by Φ_o .

Let also $(A_j(t), A_j^+(t) : t \geq 0 \; , \; j = 1,\ldots,n)$ be the annihilation and creation operators in $\mathcal{H}''$ corresponding to the test functions $\chi_{[0,t]} \otimes g_j$: $t \geq 0$, $j = 1,\ldots,n$, g_j being regarded as elements of K ; they are mutually independent Fock quantum Brownian motions satisfying the quantum Ito table (2.2), as shown in [2].

With the use of all the definitions and notations given above, the main result of this paper can be stated as follows:

Theorem 2.1. For all u , v in $\mathcal{H}$, h in $L^2(R)$, f in K_1 , and for all X in $B(\mathcal{H})$ we have

$$\lim_{\lambda \to 0} \langle u \otimes W(h *_\lambda f)\Phi_\circ , j^{(\lambda)}_{t/\lambda^2}(X)\, v \otimes W(h *_\lambda f)\Phi_\circ \rangle_{\mathcal{H}\otimes\mathcal{H}'}$$

$$= \langle u \otimes W(h \otimes f)\Phi_\circ , j_t(X)\, v \otimes W(h \otimes f)\Phi_\circ \rangle_{\mathcal{H}\otimes\mathcal{H}''} , \qquad (2.9)$$

where, in the r.h.s. of (2.9), f is regarded as an element of K .

The proof will be given in the following Section, through a number of Lemmas. It should be noted that the interest of a result like Theorem 2.1 lies in the fact that, in general, linear combinations of coherent states $\langle W(f)\Phi_\circ , \,.\, W(f)\Phi_\circ \rangle$ are dense in the set of all normal states. So the result here is only slightly less general than the one of [3], where it has been shown that, for all u , v in $\mathcal{H}$, h_1 , h_2 in $L^2(R)$, f_1 , f_2 in K_1 , X in $B(\mathcal{H})$, one has

$$\lim_{\lambda \to 0} \langle u \otimes W(h_1 *_\lambda f_1)\Phi_\circ , j^{(\lambda)}_{t/\lambda^2}(X)\, v \otimes W(h_2 *_\lambda f_2)\Phi_\circ \rangle_{\mathcal{H}\otimes\mathcal{H}'}$$

$$= \langle u \otimes W(h_1 \otimes f_1)\Phi_\circ , j_t(X)\, v \otimes W(h_2 \otimes f_2)\Phi_\circ \rangle_{\mathcal{H}\otimes\mathcal{H}''} .$$

3. Proofs

Lemma 3.1. The left-hand side of (2.9) can be rewritten as follows:

$$\langle u \otimes W(h * f_\lambda)\Phi_\circ\, ,\, j^{(\lambda)}_{t/\lambda^2}(X)\, v \otimes W(h * f_\lambda)\Phi_\circ \rangle_{\mathcal{H}\otimes\mathcal{H}'}$$

$$= \langle u \otimes \Phi_\circ\, ,\, j^{(\lambda,h,f)}_{t/\lambda^2}(X)\, v \otimes \Phi_\circ \rangle_{\mathcal{H}\otimes\mathcal{H}'}\ , \qquad (3.1)$$

where

$$j^{(\lambda,h,f)}_{t/\lambda^2}(X) = U^{(\lambda,h,f)+}_{t/\lambda^2}(X \otimes 1)\, U^{(\lambda,h,f)}_{t/\lambda^2} \qquad (3.2)$$

and where

$$U^{(\lambda,h,f)}_{t/\lambda^2} = [1 \otimes W(h * f_\lambda)]^{-1}\, U^{(\lambda)}_{t/\lambda^2}\, [1 \otimes W(h * f_\lambda)]\ . \qquad (3.3)$$

<u>Proof</u>. A straightforward manipulation. ∎

From (1.3) -- (1.5) we have

$$\frac{d}{dt} U^{(\lambda)}_{t/\lambda^2} = -\frac{i}{\lambda} V(t/\lambda^2)\, U^{(\lambda)}_{t/\lambda^2}\ , \qquad (3.4)$$

where

$$V(t/\lambda^2) = i \sum_{j=1}^{n} [B_j \otimes a^+(S_{t/\lambda^2}\, g_j) - B^+_j \otimes a(S_{t/\lambda^2}\, g_j)]\ . \qquad (3.5)$$

Now we derive a differential equation for $U^{(\lambda,h,f)}_{t/\lambda^2}$.

<u>Lemma 3.2.</u> We have

$$\frac{d}{dt} U^{(\lambda,h,f)}_{t/\lambda^2} = -\frac{i}{\lambda} V^{(\lambda,h,f)}(t/\lambda^2)\, U^{(\lambda)}_{t/\lambda^2}\ , \qquad (3.6)$$

where

$$V^{(\lambda,h,f)}(t/\lambda^2) = [1 \otimes W(h * f_\lambda)]^{-1}\, V(t/\lambda^2)\, [1 \otimes W(h * f_\lambda)]$$

$$= V(t/\lambda^2) + \Delta H(\lambda,h,f,t) \qquad (3.7)$$

and where

$$\Delta H(\lambda,h,f,t) = i \sum_{j=1}^{n} [\, \langle h * f_\lambda, S_{t/\lambda^2}\, g\rangle B_j - \langle S_{t/\lambda^2}\, g_j, h * f_\lambda\rangle B^+_j\,]\ . \qquad (3.8)$$

Proof. Note that the map $X \longmapsto [1 \otimes W(h *_\lambda f)]^{-1} X [1 \otimes W(h *_\lambda f)]$ $(X \in B(\mathcal{H} \otimes \mathcal{H}'))$ is an automorphism; then use the commutation relations $[a(f), W(g)] = \langle f,g\rangle W(g)$. ▪

Lemma 3.3. The right-hand side of (2.9) can be rewritten as follows:

$$\langle u \otimes W(h \otimes f)\Phi_\circ , j_t(X)\, v \otimes W(h \otimes f)\Phi_\circ \rangle_{\mathcal{H} \otimes \mathcal{H}''}$$

$$= \langle u \otimes \Phi_\circ , j_t^{(h,f)}(X)\, v \otimes \Phi_\circ \rangle_{\mathcal{H} \otimes \mathcal{H}''} , \qquad (3.9)$$

where

$$j_t^{(h,f)}(X) = U^{(h,f)+}(t)\,(X \otimes 1)\, U^{(h,f)}(t) \qquad (3.10)$$

and where

$$U^{(h,f)}(t) = [1 \otimes W(h \otimes f)]^{-1}\, U(t)\, [1 \otimes W(h \otimes f)] . \qquad (3.11)$$

Proof. A straightforward manipulation. ▪

Lemma 3.4. $U^{(h,f)}(t)$ satisfies the following QSDE (with $U^{(h,f)}(0) = 1$):

$$dU^{(h,f)}(t) = \Big(\sum_{j=1}^{n} [B_j\, dA_j^+(t) - B_j^+\, dA_j(t)] + [K + \Delta K(h,f,t)]dt \Big) U^{(h,f)}(t) \qquad (3.12)$$

where, for each t such that $h(.)$ is continuous at t,

$$\Delta K(h,f,t)dt = [1 \otimes W(h \otimes f)]^{-1} \left[\sum_{j=1}^{n} [B_j\, dA_j^+(t) - B_j^+ dA_j(t)], [1 \otimes W(h \otimes f)] \right]$$

$$= \sum_{j=1}^{n} [\, \overline{h}(t)\, (f|g_j)\, B_j - h(t)\, (g_j|f)\, B_j^+ \,]\, dt \quad . \qquad (3.13)$$

Proof. Note that the map $X \longmapsto [1 \otimes W(h \otimes f)]^{-1} X [1 \otimes W(h \otimes f)]$

$(X \in B(\mathcal{H} \otimes \mathcal{H}''))$ is an automorphism; then use the commutation relations $[a(f), W(g)] = \langle f,g\rangle W(g)$, recalling that $dA_j(t) = a(\chi_{[t,t+dt]} \otimes g_j)$. ▪

Note that $\Delta K(h,f,t)$ is a bounded, skew-adjoint operator.

Lemma 3.5. Let h be a continuous function of compact support. Then

$$\lim_{\lambda\to 0} \left\| -\frac{i}{\lambda} \Delta H(\lambda,h,f,t) - \Delta K(h,f,t) \right\| = 0 \quad , \tag{3.14}$$

uniformly on compact intervals in t .

Proof. It suffices to prove that, for all f , $g \in K_1$ and for all continuous h of compact support, one has

$$\lim_{\lambda\to 0} \frac{1}{\lambda} \langle h_\lambda * f, S_{t/\lambda^2} g\rangle = \bar{h}(t)\,(f|g) \tag{3.15}$$

uniformly on compact intervals in t . Indeed, we have

$$\frac{1}{\lambda} \langle h_\lambda * f, S_{t/\lambda^2} g\rangle = \int_{-\infty}^{+\infty} \bar{h}(\lambda^2 s)\, \langle S_{t/\lambda^2} f, S_s g\rangle\, ds$$

$$= \int_{-\infty}^{+\infty} \bar{h}(\lambda^2 s)\, \langle f, S_{s-t/\lambda^2} g\rangle\, ds$$

(with the change of variable $s-t/\lambda^2 = u$)

$$= \int_{-\infty}^{+\infty} \bar{h}(t+\lambda^2 u)\, \langle f, S_u g\rangle\, du \xrightarrow[\lambda\to 0]{} \bar{h}(t) \int_{-\infty}^{+\infty} \langle f, S_u g\rangle\, du$$

by the dominated convergence theorem. Recalling the definition (2.7) of $(f|g)$ and taking into account that a continuous function of compact support is uniformly continuous, the claim follows. ∎

Lemma 3.6. Let h be continuous of compact support, and let

$$\tilde{j}^{(\lambda,h,f)}_{t/\lambda^2}(X) = \tilde{U}^{(\lambda,h,f)+}_{t/\lambda^2} (X \otimes 1)\, \tilde{U}^{(\lambda,h,f)}_{t/\lambda^2} \quad : X \in B(\mathcal{H}) \ , \tag{3.16}$$

where $\tilde{U}^{(\lambda,h,f)}_{t/\lambda^2}$ is the (unitary) solution of

$$\frac{d}{dt} \tilde{U}^{(\lambda,h,f)}_{t/\lambda^2} = \left[-\frac{i}{\lambda} V(t/\lambda^2) + \Delta K(h,f,t) \right] \tilde{U}^{(\lambda,h,f)}_{t/\lambda^2} \quad , \tag{3.17}$$

with $\tilde{U}^{(\lambda,h,f)}_{0} = 1$. Then, for all $X \in B(\mathcal{H})$, we have

$$\lim_{\lambda\to 0} \| j^{(\lambda,h,f)}_{t/\lambda^2}(X) - \tilde{j}^{(\lambda,h,f)}_{t/\lambda^2}(X)\| = 0 \,, \qquad (3.18)$$

uniformly on compact intervals in t .

Proof. It is a consequence of Lemma 3.5. Indeed, we can write

$$U^{(\lambda,h,f)}_{t/\lambda^2} - \tilde{U}^{(\lambda,h,f)}_{t/\lambda^2} = U^{(\lambda,h,f)}_{t/\lambda^2} Z^{(\lambda,h,f)}_{t/\lambda^2} \,,$$

where $Z^{(\lambda,h,f)}_{t/\lambda^2} = 1 - U^{(\lambda,h,f)+}_{t/\lambda^2} \tilde{U}^{(\lambda,h,f)}_{t/\lambda^2}$ satisfies $Z^{(\lambda,h,f)}_{0} = 0$ and

$$\frac{d}{dt} Z^{(\lambda,h,f)}_{t/\lambda^2} = U^{(\lambda,h,f)+}_{t/\lambda^2}\left(-\frac{i}{\lambda}\Delta H(\lambda,h,f,t) - \Delta K(\lambda,h,f,t)\right)\tilde{U}^{(\lambda,h,f)}_{t/\lambda^2} \,.$$

Since $U^{(\lambda,h,f)}_{t/\lambda^2}$ and $\tilde{U}^{(\lambda,h,f)}_{t/\lambda^2}$ are unitary, we have

$$\| Z^{(\lambda,h,f)}_{t/\lambda^2} \| \le t \left\| -\frac{i}{\lambda}\Delta H(\lambda,h,f,t) - \Delta K(\lambda,h,f,t)\right\| \,,$$

which tends to 0 as $\lambda \to 0$, uniformly on compact intervals in t , by Lemma 3.5. The claim follows. ▪

Lemma 3.7. Let h be continuous of compact support, and let $(T^{(h,f)}_t : t \ge 0)$ be the family of completely positive maps of $B(\mathcal{H})$ such that, for all $X \in B(\mathcal{H})$,

$$\frac{d}{dt} T^{(h,f)}_t(X) = L(X) + \Delta K(h,f,t)^+ X + X\,\Delta K(h,f,t), \qquad (3.19)$$

with $T^{(h,f)}_0(X) = X$. Then, for all $u, v \in \mathcal{H}$, $X \in B(\mathcal{H})$, we have

$$\lim_{\lambda\to 0} \left| \langle u\otimes\Phi_\circ , \tilde{j}^{(\lambda,h,f)}_{t/\lambda^2}(X)\, v\otimes\Phi_\circ\rangle_{\mathcal{H}\otimes\mathcal{H}'} - \langle u, T^{(h,f)}_t(X)v\rangle_{\mathcal{H}}\right| = 0, \qquad (3.20)$$

uniformly on compact intervals in t .

Proof (Sketch). The same arguments as in [18,6] can be used, since the time dependence of $\Delta K(h,f,t)$, being on the "macroscopic time scale", does not give particular problems (see instead [7] for the