Electrodynamics

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Lectures on Theoretical Physics, Vol. III

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PREFACE

Heinrich Hertz's great paper on the "Fundamental Equations of Electrodynamics for Bodies at Rest" has served as model for my lectures on electrodynamics ever since my student days (see §1). Following this example I proceed in Part I from Maxwell's equations as an axiomatic basis, expressed not as with Hertz, in the coordinates and in differential form, but in vectorial integral form. In Part II the several classes of phenomena, in static, stationary, quasistationary, and rapidly variable fields, are derived from these equations, as in Hertz's paper. After I had heard Hermann Minkowski's lecture on "Space and Time" in 1909 in Cologne, I carefully developed the four-dimensional form of electrodynamics as an apotheosis of Maxwell's theory and at the same time as the simplest introduction to the theory of relativity; in return, this has always met with an enthusiastic reception on the part of my audience. This four-dimensional electrodynamics is presented in Part III. Its title "Theory of Relativity and Electron Theory" requires the comment that it is limited to the *special* theory of relativity on the one hand and to the theory of the *individual* electron on the other. The statistics of electrons in metals and electrons in insulators belong in Vols. IV* and V* of the Lectures. Schwarzschild's principle of action, which establishes the fundamental relationship between Maxwell's theory and the dynamics of the individual electron (or individual electrons), is presented at the end of Part III with certain modifications which appear necessary from our point of view. In Part IV is developed the electrodynamics of moving media, again following Minkowski rather closely. As the most important application the fields of unipolar induction are discussed and are calculated as an exercise for a particularly simple example.

Part II constitutes the main portion of the Lectures. I distinguish between summation and boundary-value problems in electrostatics and magnetostatics. The computation of the electric potential for given charge distribution and the calculation of the magnetic potential for given magnetization are examples of the first; the theory of the permanent magnet, insofar as it falls within the competence of Maxwell's theory rather than atomic theory, becomes simple and clear from this standpoint. On the other hand, the solution of the electric and magnetic boundary-value problems properly belongs in Vol. VI; only the most important cases are treated in the present volume. The calculation of stationary fields for a given distribution of the current density, either by the method of the vector potential in §15 or that of the magnetic shell in §16, is also a simple summation prob-

* See p. xii for list of *Lectures on Theoretical Physics.*

lem. Among the rapidly variable fields those of the wire-wave type are treated with some completeness. The principal wave on a single wire in §22 (symmetric electrical type) serves as primary example; however, because of their recent practical applications (theory of wave guides in §24) and their utilization in the theory of the Lecher system, the magnetic type and the asymmetric secondary waves as well as wire waves on nonconductors are also dealt with in §23. As conclusion of Part II the Lecher system is treated fully for arbitrary separation and dimensions of the two parallel wires, employing bipolar coordinates for the exterior of the wires and ordinary polar coordinates for their interior. It is only assumed that the two wires are rather good conductors.

The dimensional character of the field entities is taken seriously throughout. We do not accept Planck's position, according to which the question of the real dimension of a physical entity is meaningless; Planck states in §7 of his Lectures on Electrodynamics that this question has no more meaning that that of the "real" name of an object. Instead, we derive from the basic Maxwell equations the fundamental distinction between entities of *intensity* and entities of *quantity*, which has heretofore been applied consistently in the excellent textbooks of G. Mie. The Faraday-Maxwell induction equation shows that the magnetic induction **B** is an entity of intensity along with the electric field strength **E**; **B**, rather than **H**, deserves the name *magnetic field strength*. **H**, like **D**, is best designated as "excitation." div **H** represents the magnetic density, just as div **D** represents the electric charge density. Hertz's distinction between "true" and "free" electricity becomes pointless, since div **E** is, dimensionally, not a charge, but a divergence of lines of force. The same applies to the distinction between "true" and "free" magnetism, particularly since div **B** is everywhere zero. The current density **J**, the electric polarization **P**, and the magnetization **M**, are entities of quantity like **H** and **D**. Energy quantities always take the form of products of an entity of quantity and an entity of intensity, e.g. $\frac{1}{2}\mathbf{D}\cdot\mathbf{E}$, $\frac{1}{2}\mathbf{H}\cdot\mathbf{B}$, $\mathbf{J}\cdot\mathbf{E}$, $\mathbf{E}\times\mathbf{H}$. The fact that **B** and **E**, and **H** and **D**, belong together follows unambiguously from the theory of relativity, in which the quantities $c\mathbf{B}$ and $-i\mathbf{E}$, and **H** and $-ic\mathbf{D}$, respectively, are coupled together in a six-vector (antisymmetric tensor). We call the first the field tensor F, the second, the excitation tensor f.

The introduction of a *fourth electric unit*, independent of the mechanical units, is decisive for the fruitfulness of these dimensional considerations. We choose for this the unit of charge Q, which, as a matter of convenience we may identify with the coulomb if we wish. In this manner we avoid the "bed of Procrustes" of the cgs-units, in which the electromagnetic quantities are forced to take on the well known unnatural dimensions. Since we must definitely give up the hope of a mechanical interpretation of electrical quantities, we must regard the charge as a basic, irreducible entity which can claim a dimension of its own. We shall refer to the "electrostatically" or "electromagnetically" measured charge only in passing, and exclusively for historical reasons. With the particular unit of charge Q = 1 coulomb the electric current has the customary unit amperes = Q/sec.

As mechanical units, following the suggestion of G. Giorgi, we shall employ the units meter M, kilogram (mass) K, and second S. The unit of energy then becomes 1 joule (without a power of ten as factor!) and that of power 1 joule S^{-1} = 1 watt. Furthermore, the powers of ten disappear also for the electric units volt, ohm, farad, and henry; we have 1 volt = 1 joule/Q, 1 ohm = 1 joule S/Q^2, 1 farad = 1 Q^2/joule and 1 henry = 1 joule S^2/Q^2.

On the other hand powers of ten must appear as factors when the units of the magnetic field strength **B** and the magnetic excitation **H** are expressed in terms of the gauss and the oersted, respectively, which as we shall see in §8, have been adapted to the cgs-system. As expected, the unit Q automatically drops out of the energy densities $\frac{1}{2}\mathbf{D}\cdot\mathbf{E}$, $\frac{1}{2}\mathbf{H}\cdot\mathbf{B}$, and $\mathbf{J}\cdot\mathbf{E}$ referred to above; their dimension becomes joule/M^3 directly, whereas that of the energy flux $\mathbf{E}\times\mathbf{H}$ is joule/(M^2S).

With our dimensional differentiation between entities of intensity and entities of quantity the dielectric constant and the permeability evidently become *dimensional* quantities and therefore cannot be set equal to 1 in vacuum. Their choice, in which we accept electrical engineering practice, happily permits us to meet the demand for "rational units" without difficulty. It is only necessary to set,

$$\mu_0 = 4\pi\cdot 10^{-7}\,\frac{\Omega S}{M}$$

in accord with international conventions, and to derive ε_0 from the relation $\varepsilon_0\mu_0 = 1/c^2$, verified by Hertz's experiments. With this choice the 4π's disappear wherever they do not belong, as in Poisson's equation and the energy expressions in Poynting's theorem, and appear where they belong, as in Coulomb's law and for the spherical condenser. We thus avoid the desperate expedient by which Lorentz achieves rationalization in his articles in the Enzyklopaedie, namely the introduction of the factor $\sqrt{4\pi}$ in the definition of the charge and of magnetism.

At the same time, with this choice of ε_0 and μ_0, the square root of the ratio of μ_0 and ε_0 evidently becomes a resistance, namely the so-called "wave resistance of vacuum." This quantity occurs in Part II as a factor wherever the wave fields **E** and **H** enter into formulas of the same dimensions. It occurs again in the theory of relativity in the relation between the excitation tensor f and the field tensor F, which in vacuum assumes the simple form for all six components

$$f = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,F.$$

These questions of units, dimensions, and rationalization, often discussed to excess in recent years, are disposed of as briefly as possible in the lectures; however, the reader is repeatedly urged in them to convince himself of the dimensional logic of formulas. In numerical computations our MKSQ system of units is found convenient throughout since it is adapted to the practical and legal units, volt, ampere, etc. We leave the question open as to whether it is also appropriate for atomic physics. So as to permit an effortless transition to the Gaussian system ($\varepsilon_0 = \mu_0 = 1$), which is customary in this case, we explain Cohn's system in §9, which in our interpretation is based on the five units MKSQP (P = magnetic unit pole).

The wonderful simplicity and beauty of the Maxwell equations, which is most striking in their relativistic formulation for vacuum, lead to the conviction that these equations, along with the equations of gravitation (§38), are the demonstration of an all-inclusive world geometry. Approaches to this by no means resolved problem are summarily discussed in §37. The amazingly simple representation of the general theory of relativity in §38 is based on a derivation of Schwarzschild's line element kindly made available to me by W. Lenz. In this manner the three tests of the theory open to astronomical observation may be treated without tensor calculus.

This volume is based on lecture notes prepared by H. Welker in the winter semester of 1933/34, at which time I first abandoned the cgs system and passed over to the more general system of the four units. In the final formulation of Parts I and II I have had the benefit of the constant advice of Professor J. Jaumann, whose electrotechnical experience and point of view have been of great advantage to this volume. I am grateful to Messrs. P. Mann and E. Gora and to my colleague F. Bopp for critical remarks and suggestions for improvements. Dr. W. Becker has kindly assisted me in reading the proof of this, as of preceding volumes.

Munich, April 1948

Arnold Sommerfeld

Translator's Note

A minimum number of changes has been made in this translation of Sommerfeld's "Elektrodynamik" (the third volume of the Lectures on Theoretical Physics) to adapt it for use in English-speaking countries. As far as possible, the same conventions regarding notation are employed as in G. Kuerti's translation of Volume II, "Mechanics of Deformable Bodies." Thus vectors are represented by bold-face letters, vector components and scalars (as well as tensors and their components) by italics; this in spite of the fact that the Gothic letters employed in the original text for both vectors and vector components were used even in Maxwell's Treatise. To avoid confusion a few additional changes of symbols were required in consequence of this major change.

E. G. R.

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VOLUME: VI: Partial Differential Equations in Physics. Translated by Ernst G. Straus

PART I

FUNDAMENTALS AND BASIC PRINCIPLES OF MAXWELL'S ELECTRODYNAMICS

§1. *Historical Review. Action at a Distance and Action by a Field*

I can best give you an idea of the sweeping changes in viewpoint brought about by the theory of Faraday and Maxwell by telling you of the time I spent as a student, 1887–1891.

My native city, Königsberg, was the earliest fountainhead of mathematical physics in Germany, thanks to the activity of the revered Franz Neumann, 1798–1894. At the University of Königsberg he taught, in addition to crystallography, theoretical physics which was not at the time given elsewhere in Germany. His students, of whom Gustav Kirchhoff of Königsberg was the most prominent, spread the teachings of the master to the other German universities. Through the seminar in mathematical physics, founded by him and C. G. J. Jacobi, he also saw to it that the East Prussian secondary-school teachers received a particularly thorough preparation. This may bear some relation to the fact that the Gymnasium in the Altstadt graduated the mathematician Hermann Minkowski and the physicists Max and Willy Wien shortly before my final examination, while at the same time the only slightly older David Hilbert and Emil Wiechert were attending other Königsberg schools. Neumann's greatest successes in research were achieved in the elastic theory of light and in the physics of crystals; his mathematical formulation of the induction currents discovered by Faraday will be discussed in §15. Simultaneously with Neumann and Jacobi, and almost outshining them, F. W. Bessel taught in Königsberg.

My time of study coincided with the period of Hertz's experiments. At first, however, electrodynamics was still presented to us in the old manner—in addition to Coulomb and Biot-Savart, Ampère's law of the mutual action of two elements of current and its competitors, the laws of Grassmann, Gauss, Riemann, and Clausius, and as a culmination the law of Wilhelm Weber, all of which were based on the Newtonian concept of action at a distance. The total picture of electrodynamics thus presented to us was awkward, incoherent, and by no means self-contained. Teachers and students made a great effort to familiarize themselves with Hertz's experiments step by step as they became known and to explain them with the aid of the difficult original presentation[1] in Maxwell's Treatise.

It was as though scales fell from my eyes when I read Hertz's great paper:[2] "Über die Grundgleichungen der Elektrodynamik für ruhende Körper." Here Maxwell's equations, purified by Heaviside and Hertz, were made the axioms and the starting point of the theory. The totality of electromagnetic phenomena is derived from them systematically by deduction. Coulomb's law, which formerly provided the basis, now appears as a necessary consequence of the all-inclusive theory. Electric currents are always closed. Current elements arise only as mathematical increments of line integrals. All effects are transmitted by the electromagnetic field, which may be represented by force-line models. *Action at a distance* gives way to field action,[3] the "constructable representation" of a space-time propagation postulated already by Gauss.[4]

I have held to the order of Hertz's paper in all my lectures on Maxwell's theory. In this presentation, too, we shall not begin with electrostatics, as is done so commonly and also in Maxwell's Treatise, but treat it merely as an extreme simplification of the general field theory. We shall deviate from Hertz only insofar as we shall start not from Maxwell's equations in differential form, but in integral form. It goes without saying that we shall replace the rather extensive coordinate calculations of Hertz by vector algebra, which is perfectly suited to the electromagnetic field. We shall see that this algebra, extended to four dimensions, leads directly to the special theory of relativity. The latter will provide an approach to the electrodynamics of moving bodies, which Hertz unsuccessfully sought to master in the second paper cited. In agreement with Hertz we see in Maxwell's *equations* the essence of his theory. We need not discuss the *mechanical pictures*, which guided Maxwell in the setting up of his equations. We have discussed one such picture in Vol. II, §15 of these lectures.

[1] The great student of electrolysis, Wilhelm Hittorf, who had heard much of the new theory of electricity, in advanced years attempted to study the Treatise, but was unable to find his way through the unfamiliar mass of equations and concepts. He was thus led into a state of deep depression. His colleagues in Münster persuaded him to take a vacation trip to the Harz Mountains. However when just before his departure they checked his luggage they found in it—the two volumes of the Treatise on Electricity and Magnetism by James Clerk Maxwell. (As told by A. Heidweiller.)

[2] Göttinger Nachr. March 1890 and Ann. Physik, Vol. 40; continued in Ann. Physik, Vol. 41: "Über die Grundgleichungen der Elektrodynamik für bewegte Körper."

[3] We avoid the alternative term "near action" which signifies merely action at a small distance, and by our notation direct attention to the medium transmitting the effect, namely the field.

[4] In a letter to Wilhelm Weber, of 1845. See Collected Works, Vol. V, p. 627.

Biographical Notes

Michael Faraday, 1791–1867

He was born as son of a blacksmith in impecunious circumstances. The family belonged to the pious sect of the Sandemanians, to which Faraday remained faithful to his death. His high ethical concept of life and human kindness derived from the religious spirit of his family. He was first newspaper carrier, then bookbinder. In science and letters he was entirely self-taught. The lectures of Sir Humphry Davy at the Royal Institution were decisive for his career; he wrote them up carefully and found an opportunity to present them to the great chemist. He became his laboratory assistant in the Royal Institution. His first important work was "the rotation of a current about a magnet and the rotation of a magnet about a current," and also the liquefaction of chlorine. This work brought about his election as Fellow of the Royal Society and later the indirect succession to Davy at the Royal Institution. In 1832 he began the publication of the "Experimental Researches." His discoveries recorded in these extend to the most diverse fields of physics, electrochemistry, and the study of materials. We mention as most significant for us: The discoveries of the law of electromagnetic induction in 1831, the dielectric constant, para- and diamagnetic behavior, and the picture of electric and magnetic lines of force. His magneto-optical discoveries are discussed in Vol. IV. The failing of his memory forced many pauses in his work, as well as the repetition of experiments made at an earlier date. It is uncertain whether this is to be attributed to mental overexertion or, as is commonly assumed today, to mercury poisoning in the poorly ventilated basement rooms of the Royal Institution. Certainly his purely intuitive method of working, devoid of any mathematical aid, required tremendous mental concentration. In his last years a restful summer retreat in the royal palace, Hampton Court, was made available to him at the suggestion of the Prince Consort, Albert. At his death there were found ninety-five honorary diplomas of learned societies, bound with his own hand.

James Clerk Maxwell, 1831–1879

He came from a prominent Scottish family (the father's name was Clerk, the added name Maxwell being derived from his mother) and was given the best in education that his time offered, both in the field of letters and that of science and mathematics. Thus, at an early date, he could translate Faraday's pictures of lines of force into a mathematical form which could be generally understood. See his paper of 1855 "On Faraday's Lines of Force" (translated into German by Boltzmann in Ostwald's Klassiker Nr. 69). In the preface to his Treatise he states: "Faraday, in his mind's eye, saw lines of force traversing all space where the mathematicians (from the preceding discussion it is apparent that he refers particularly to Gauss, Wilhelm Weber, Riemann, Franz and Carl Neumann) saw centres of force attracting at a distance: Faraday saw a medium where they saw nothing but distance: Faraday sought the seat of the phenomena in real actions going on in the medium, they were satisfied that they had found it in a power of action at a distance impressed on the electric fluids. When I had translated what I considered to be Faraday's ideas into a mathematical form, I found that in general the results of the two methods coincided, . . . but that . . . several of the most fertile methods of research discovered by the mathematicians could be expressed much better in terms of ideas derived from Faraday than in their original form."

The Treatise appeared in 1873. Its greatest achievement is the unification of optics and electrodynamics. The simplified form of the Maxwell equations, later rediscovered by Heaviside and Hertz, is to be found already in Part III of his paper for the Royal Society of 1864. Almost as important as his electromagnetic papers are those on the kinetic theory of gases (Maxwellian velocity distribution) and on general statistics, to which belongs also his theory of the rings of Saturn. He is also the author of purely mathematical papers (on cycloidal surfaces, the theory of the top, and the determination of magnitudes in Helmholtz's color triangle) and of an important paper on lattice structures (see Vol. II of these Lectures, p. 310).

After a brief teaching engagement in Aberdeen he became the first director of the newly founded Cavendish Laboratory in Cambridge; he died there at an early age.

André Marie Ampère, 1775–1836

We shall add a biographical note on Ampère not on account of the fundamental law already mentioned, nor because of the classical experiments, which enabled him to derive it with the simplest possible means, but for his discovery of the general relationship between the magnetic field and electric currents.

Born in Lyon, as a precocious boy he occupied himself with philological and mathematical studies. His father was a victim of the Revolution. Because of his mathematical papers he was named professor at the École Polytechnique in Paris in 1804. Here he soon directed his attention to chemistry, where he was able to compete with Avogadro in the field of atomism. There follow five years in which he is concerned primarily with psychology and metaphysics, though accepted as a mathematician into the Academy of Sciences of Paris. His interest in physics is not awakened until 1820, when he hears of Oersted's discovery. In a few weeks he verifies his belief that electricity in motion, and not electricity at rest, has a magnetic effect. The years 1820–1826 he spent elaborating his concept of the connection between the magnetic field and the electric current, which is equivalent to half of Maxwell's equations provided that the concept of the electric current is extended by the addition of Maxwell's displacement current. We shall hence denote this portion of the Maxwell equations (in integral form) in §3 directly as Ampère's law. From this point of departure Ampère recognized the equivalence of a solenoid traversed by current to a permanent magnet. The strengthening of the magnetic field by a soft-iron core placed in the solenoid is also to be attributed to him. Ampère may thus be regarded as the father of the "electromagnet." We may mention in addition Ampère's molecular currents and the elegant method of the magnetic sheet.

When, in 1826, however, Ampère obtained a professorship in physics at the Collège de France his interests changed once more: he returned to philosophy and logic and devoted himself finally to biology and comparative anatomy. Altogether a scientific career of extraordinary breadth and depth, of intensity and versatility! (This material has been taken from an essay by Louis de Broglie in his book Continu et Discontinu, Paris, 1941.)

Heinrich Hertz, 1857–1894

He was born in Hamburg the son of a respected merchant family; his father was in later years Senator of the Free City. Initially his great modesty prevented Heinrich Hertz from entering upon the career of a scholar; instead, he turned to engineering at the Technische Hochschule in Munich. Soon, however, he begged his father to permit him to transfer to pure physics. He studied first in Munich, then in Berlin, and became the favorite student and assistant of Helmholtz. The relationship between teacher and student was the closest imaginable and finds touching expression in the memorial addressed to him by Helmholtz (reprinted in Vol. I of Hertz's Collected Works). A prize problem set up by Helmholtz directed him to the testing of Maxwell's theory. After a short term as Privatdozent in Kiel he was called to the Technische Hochschule in Karlsruhe.

Even the earliest papers of Hertz show his mastery in relating theory and experiment. Several of them received the warm recognition of his colleagues, as his quantitative determination of hardness among engineers, and his description of the condensation processes in rising air currents among meteorologists. His years in Karlsruhe, from 1885 to 1889, represent the high point in his creative activity. We mention in particular his paper of 1888: "Forces of electrical oscillations treated by Maxwell's theory." It provides the characteristic solution now generally designated as the Hertzian vector and shows the familiar force-line pictures of the