# Electromagnetic Fields and Waves

## Including Electric Circuits

Third Edition

Paul Lorrain

Dale R. Corson

François Lorrain

# Electromagnetic Fields and Waves

Including Electric Circuits

Third Edition

Paul Lorrain
*Université de Montréal and McGill University*

Dale R. Corson
*Cornell University*

François Lorrain

W. H. Freeman and Company
New York

**Library of Congress Cataloging-in-Publication Data**

Lorrain, Paul.
Electromagnetic fields and waves.

Includes index.
1. Electromagnetic fields. 2. Electromagnetic waves.
I. Corson, Dale R. II. Lorrain, François, 1945–
III. Title

QC665.E4L67 1987 530 1'41 86-31803
ISBN 0-716-71823-5
ISBN 0-716-71869-3 (pbk )



Printed in the United States of America

1 2 3 4 5 6 7 8 9 0 VB 6 5 4 3 2 1 0 8 9 8

# Vector definitions, identities, and theorems

## Definitions

Rectangular coordinates

1. $\nabla f = \dfrac{\partial f}{\partial x}\hat{x} + \dfrac{\partial f}{\partial y}\hat{y} + \dfrac{\partial f}{\partial z}\hat{z}$

2. $\nabla \cdot A = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$

3. $\nabla \times A = \left(\dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}\right)\hat{x} + \left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right)\hat{y} + \left(\dfrac{dA_y}{\partial x} - \dfrac{\partial A_x}{\partial y}\right)\hat{z}$

4. $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

5. $\nabla^2 A = \nabla^2 A_x\hat{x} + \nabla^2 A_y\hat{y} + \nabla^2 A_z\hat{z} = \nabla(\nabla \cdot A) - \nabla \times (\nabla \times A)$

Cylindrical coordinates

6. $\nabla f = \dfrac{\partial f}{\partial \rho}\hat{\rho} + \dfrac{1}{\rho}\dfrac{\partial f}{\partial \phi}\hat{\phi} + \dfrac{\partial f}{\partial z}\hat{z}$

7. $\nabla \cdot A = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho A_\rho) + \dfrac{1}{\rho}\dfrac{\partial A_\phi}{\partial \phi} + \dfrac{\partial A_z}{\partial z}$

8. $\nabla \times A = \left(\dfrac{1}{\rho}\dfrac{\partial A_z}{\partial \phi} - \dfrac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\dfrac{\partial A_\rho}{\partial z} - \dfrac{\partial A_z}{\partial \rho}\right)\hat{\phi} + \dfrac{1}{\rho}\left[\dfrac{\partial}{\partial \rho}(\rho A_\phi) - \dfrac{\partial A_\rho}{\partial \phi}\right]\hat{z}$

9. $\nabla^2 f = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho\dfrac{\partial f}{\partial \rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial \phi^2} + \dfrac{\partial^2 f}{\partial z^2}$

10. $\nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times (\nabla \times A)$ (Sec. 1.11.6).

Spherical coordinates

11. $\nabla f = \dfrac{\partial f}{\partial r}\hat{r} + \dfrac{\partial f}{r\,\partial \theta}\hat{\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \phi}\hat{\phi}$

12. $\nabla \cdot A = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 A_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(A_\theta \sin\theta) + \dfrac{1}{r\sin\theta} - \dfrac{\partial A_\varphi}{\partial \phi}$

13. $\nabla \times A = \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial \theta}(A_\phi \sin\theta) - \dfrac{\partial A_\theta}{\partial \phi}\right]\hat{r} + \dfrac{1}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial A_r}{\partial \phi} - \dfrac{\partial (rA_\varphi)}{\partial r}\right]\hat{\theta}$
$$+ \frac{1}{r}\left[\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi}$$

14. $\nabla^2 f = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial f}{\partial \theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 f}{\partial \phi^2}$

15. $\nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times \nabla \times A$ (Sec. 1.11.6).

**Identities**

1. $(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}=\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$
2. $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=\boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C})-\boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$
3. $\boldsymbol{\nabla}(fg)=f\boldsymbol{\nabla}g+g\boldsymbol{\nabla}f$
4. $\boldsymbol{\nabla}(a/b)=(1/b)\boldsymbol{\nabla}a-(a/b^2)\boldsymbol{\nabla}b$
5. $\boldsymbol{\nabla}(\boldsymbol{A}\cdot\boldsymbol{B})=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A}+(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}+\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A})+\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B})$
6. $\boldsymbol{\nabla}\cdot(f\boldsymbol{A})=(\boldsymbol{\nabla}f)\cdot\boldsymbol{A}+f(\boldsymbol{\nabla}\cdot\boldsymbol{A})$
7. $\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})-\boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})$
8. $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\boldsymbol{A}=\boldsymbol{\nabla}^2\boldsymbol{A}$
9. $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f)=0$
10. $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})=0$
11. $\boldsymbol{\nabla}\times(f\boldsymbol{A})=(\boldsymbol{\nabla}f)\times\boldsymbol{A}+f(\boldsymbol{\nabla}\times\boldsymbol{A})$
12. $\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B})=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A}-(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}+(\boldsymbol{\nabla}\cdot\boldsymbol{B})\boldsymbol{A}-(\boldsymbol{\nabla}\cdot\boldsymbol{A})\boldsymbol{B}.$
13. $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})-\boldsymbol{\nabla}^2\boldsymbol{A}$ (Sec. 1.11.6)
14. $$(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}=\left[A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}\right]\hat{x}$$
$$+\left[A_x\frac{\partial B_y}{\partial x}+A_y\frac{\partial B_y}{\partial y}+A_z\frac{\partial B_y}{\partial z}\right]\hat{y}$$
$$+\left[A_x\frac{\partial B_z}{\partial x}+A_y\frac{\partial B_z}{\partial y}+A_z\frac{\partial B_z}{\partial z}\right]\hat{z}$$

15. $\boldsymbol{\nabla}'(1/r)=\hat{r}/r^2$. This is the gradient calculated at $(x',y',z')$, and $\boldsymbol{r}$ is the vector $\boldsymbol{r}$ pointing from $(x',y',z')$ to $(x,y,z)$.

16. $\boldsymbol{\nabla}(1/r)=-\hat{r}/r^2$. This is the gradient calculated at $(x,y,z)$ with the same vector $\boldsymbol{r}$.

17. $\mathscr{A}=\frac{1}{2}\oint_C \boldsymbol{r}\times d\boldsymbol{l}$, where the surface of area $\mathscr{A}$ is plane. The vector $\boldsymbol{r}$ extends from an arbitrary origin to a point on the curve $C$ that bounds $\mathscr{A}$.

18. $\int_v \boldsymbol{\nabla}f\,dv=\int_{\mathscr{A}} f\,d\boldsymbol{\mathscr{A}}$

19. $\int_v(\boldsymbol{\nabla}\times\boldsymbol{A})\,dv=-\int_{\mathscr{A}}\boldsymbol{A}\times d\boldsymbol{\mathscr{A}}$, where $\mathscr{A}$ is the area of the closed surface that bounds the volume $v$.

20. $\oint_C f\,d\boldsymbol{l}=-\int_{\mathscr{A}}\boldsymbol{\nabla}\times d\boldsymbol{\mathscr{A}}$ where $C$ is the closed curve that bounds the open surface of area $\mathscr{A}$.

**Theorems**

1. The divergence theorem. $\int_{\mathscr{A}}\boldsymbol{A}\cdot d\boldsymbol{\mathscr{A}}=\int_v\boldsymbol{\nabla}\cdot\boldsymbol{A}\,dv$ where $\mathscr{A}$ is the area of the closed surface that bounds the volume $v$.
2. Stokes's theorem: $\oint_C\boldsymbol{A}\cdot d\boldsymbol{l}=\int_{\mathscr{A}}(\boldsymbol{\nabla}\times\boldsymbol{A})\cdot d\boldsymbol{\mathscr{A}}$.

# PREFACE

Like the previous editions, this book is intended primarily for students of physics or electrical engineering at the junior and senior levels. The previous editions have also proved useful for practicing scientists and engineers.

Our aim is to impart to the reader a working knowledge of the basic concepts of electromagnetism. That is why it contains 135 examples and 423 problems. As Alfred North Whitehead stated, over half a century ago, "Education is the acquisition of the art of the *utilization* of knowledge."

This third edition is basically similar to the second, despite many changes. First, we have included four chapters on *electric circuits*: Chapter 7 on *RC* circuits, Chapter 8 on circuit theorems, Chapter 24 on inductance, and Chapter 25 on alternating-current circuits. We have included two chapters on *optical waveguides,* Chapters 35 and 36. This subject ties in well with Chapter 31 on total reflection and with Chapter 34 on hollow rectangular metallic waveguides. Wherever possible, we have simplified the notation and provided simpler proofs. Finally, we have subdivided the material into shorter chapters, 39 in all, versus 14 in the previous editions. This will make the book more palatable for readers, more flexible for teachers, and more convenient as a reference.

Not all readers or teachers will wish to go through this book from cover to cover. *Asterisks* indicate those chapters or sections that can be omitted without losing continuity. They bear no relation to the relative importance of the topic.

The first two chapters on vectors and phasors offer a concise mathematical introduction. There follows a series of 10 short chapters on electric fields, including two on electric circuits Chapters 7 and 8.

*The next five chapters on relativity can be omitted if necessary.* They cover the essentials of special relativity as applied to electromagnetic fields. They are somewhat more thorough than the corresponding chapters of the second edition.

There follow 10 chapters on magnetic fields, including another two on electric circuits, Chapters 24 and 25. By the end of Chapter 26, we have deduced, discussed, and applied Maxwell's equations extensively. Chapter 27 groups these equations and provides a general discussion. This is followed by five chapters on the propagation of plane electromagnetic waves in various media and across interfaces. Then there are four chapters on guided electromagnetic waves, two of which concern planar optical waveguides. The final three chapters discuss the radiation of electromagnetic waves.

As previously, the *problems* form an essential part of the book. Many are new. Their function is not only to illustrate the basic principles but also to show a variety of applications. For convenience, the problems are now classified by section, approximately in order of increasing difficulty. They proceed in short steps. This makes them more instructive and permits the reader to accomplish more.

Teachers will find further, and easier, problems in the companion book *Electromagnetism: Principles and Applications* by the first two authors and by the same publisher.

I am particularly grateful to François Lorrain, who revised most of the text and who rewrote portions of it during the early stages. Joseph Miskin ably revised the final text and most of the problems.

Over the years I have worked on this book, not only at the University of Montréal, but also in several other universities in various countries. I am deeply indebted to the following persons for their hospitality: Prof. Louis Néel of the Université de Grenoble, France; Prof. Maximino Rodriguez-Vidal, of the Universidad de Madrid, Spain; Prof. E. W. J. Mitchell and Dr. F. N. H. Robinson of the Clarendon Laboratory of Oxford University, Great Britain; Prof. Gaston Pouliot of the École Polytechnique, Montréal, Canada; Prof. John Gruzleski of McGill University in Montréal; Prof. Liu Qi Yi of Nankai University in Tianjin and Prof. Zhang of Qing Hua University in Beijing, People's Republic of China; and finally Profs. Robert Martin, Oliver Jensen, and David Crossley for their hospitality at this time in the Geophysics Laboratory of McGill University.

I also owe special thanks to Allen D. Christensen for a Visiting Fellowship at Saint Catherine's College during my sabbatical leave at Oxford in 1981.

I owe thanks to the many people who wrote to me, offering comments and suggestions, particularly to S. Baldursson and S. S. Kristjansdottir, University of Iceland; N. Gauthier, Royal Military College at Kingston, Canada; R. H. Good, California State University at Hayward; R. D. Meyers, University of Maryland; F. Murray, University of Scranton; H. A. Pohl, Oklahama State University; E. H. Rhoderick, University of Manchester; J. R. Ridge, Moravian College; W. M. Saslow, University of Pittsburgh; R. Sevenich, Wisconsin State University; M. S. Tiersen, The City University of New York; and Harold A. Wheeler, Hazeltine Corporation.

Dr. Barry Taylor, of the National Bureau of Standards, Gaithersburg, kindly supplied a list of fundamental constants.

The Computing Science Department of McGill University made one of its computers available for processing the entire text. Charles Snow, of that department, was most helpful throughout this long operation.

Finally, I wish to thank Claire Samson, Ronald Hall, Marjorie Raynor, and Michèle Maschtall for keyboarding the text, for preparing sketches for the figures, and for plotting curves.

I shall be most grateful to those readers kind enough to bring to my attention any misprint or error that may remain, so that further printings can be corrected.

Paul Lorrain

# LIST OF SYMBOLS

## Space, Time, Mechanics

| | |
|---|---|
| Length | $l, L, s, r$ |
| Area | $\mathscr{A}$ |
| Volume | $\boldsymbol{v}$ |
| Solid angle | $\Omega$ |
| Unit vector | $\hat{\boldsymbol{x}}$ |
| Unit vector normal to a surface | $\hat{\boldsymbol{n}}$ |
| Wavelength | $\lambda$ |
| Wavelength in free space | $\lambda_0$ |
| Wavelength of a guided wave | $\lambda_z$ |
| Radian length | $\lambda\!\!\!^{-} = \lambda/2\pi = 1/\beta$ |
| Wave number | $k = \beta - j\alpha$ |
| Attenuation constant | $\alpha$ |
| Attenuation distance, skin depth | $\delta = 1/\alpha$ |
| Time | $t$ |
| Period | $T = 1/f$ |
| Frequency | $f = 1/T$ |
| Angular frequency | $\omega = 2\pi f$ |
| Velocity, speed | $v, \mathscr{V}$ |
| Gamma | $\gamma = \dfrac{1}{[1 - (v/c)^2]^{1/2}}$ |
| Acceleration | $a = dv/dt$ |
| Mass | $m$ |
| Mass density | $\rho$ |
| Curvilinear coordinate | $q$ |
| Four-vector | $\mathbf{r}$ |
| Momentum | $\boldsymbol{p}$ |
| Four-momentum | $\mathbf{p}$ |
| Moment of inertia | $I$ |
| Force | $\boldsymbol{F}$ |
| Torque or moment | $\boldsymbol{T}$ |

| | |
|---|---|
| Pressure | $p$ |
| Energy | $\mathscr{E}$ |
| Power | $P$ |

## Electricity and Magnetism

| | |
|---|---|
| Quantity of electricity | $Q$ |
| Speed of light | $c$ |
| Linear charge density | $\lambda$ |
| Surface charge density | $\sigma$ |
| Volume charge density | $\rho$ |
| Electric potential, scalar potential | $V$ |
| Induced electromotance, voltage | $\mathcal{V}, V$ |
| Electric field strength | $\boldsymbol{E}$ |
| Electric flux density | $\boldsymbol{D}$ |
| Permittivity of vacuum | $\epsilon_0$ |
| Relative permittivity | $\epsilon_r$ |
| Permittivity of a medium | $\epsilon = \epsilon_r \epsilon_0 = D/E$ |
| Electric dipole moment | $\boldsymbol{p}$ |
| Electric quadruple moment | $\mathcal{Q}$ |
| Electric polarization | $\boldsymbol{P}$ |
| Electric susceptibility | $\chi_e$ |
| Electric current | $I$ |
| Mobility | $\mathscr{M}$ |
| Volume current density | $\boldsymbol{J}$ |
| Four-current density | $\mathbf{J}$ |
| Surface current density | $\alpha$ |
| Avogadro's constant | $N_A$ |
| Boltzmann's constant | $k$ |
| Electronic charge | $e$ |
| Planck's constant | $h$ |
| Planck's constant divided by $2\pi$ | $\hbar$ |
| Vector potential | $\boldsymbol{A}$ |
| Four-potential | $\mathbf{A}$ |
| Magnetic flux density | $\boldsymbol{B}$ |
| Magnetic field strength | $\boldsymbol{H}$ |
| Magnetic flux | $\Phi$ |
| Permeability of vacuum | $\mu_0$ |
| Relative permeability | $\mu_r$ |
| Permeability | $\mu = \mu_r \mu_0 = B/H$ |
| Magnetic dipole moment per unit volume | $\boldsymbol{M}$ |
| Magnetic susceptibility | $\chi_m$ |
| Magnetic dipole moment | $\boldsymbol{m}$ |
| Resistance | $R$ |

| | |
|---|---|
| Reactance | $X$ |
| Capacitance | $C$ |
| Self-inductance | $L$ |
| Mutual inductance | $M$ |
| Impedance | $Z = R + jX$ |
| Resistivity | $\rho$ |
| Conductivity | $\sigma$ |
| Poynting vector | $\mathcal{S}$ |

## Mathematical Symbols

| | |
|---|---|
| Approximately equal to | $\approx$ |
| Proportional to | $\propto$ |
| Factorial $n$ | $n!$ |
| Exponential of $x$ | $\exp x$ |
| $(-1)^{1/2}$ | $j$ |
| Arctangent $x$ | $\arctan x$ |
| Complex conjugate of $z$ | $z^*$ |
| Peak value of $E$ | $E_m$ |
| Average value of $x$ | $\langle x \rangle$ |
| Complex number | $z = x + jy$ |
| Real part of $z$ | $\mathrm{Re}\, z$ |
| Imaginary part of $z$ | $\mathrm{Im}\, z$ |
| Modulus of $z$ | $\|z\| = (x^2 + y^2)^{1/2}$ |
| Absolute value of $A$ | $\|A\|$ |
| Decadic log of $x$ | $\log x$ |
| Natural log of $x$ | $\ln x$ |
| Magnitude of a vector $\mathbf{F}$ | $F$ |
| Vector | $\mathbf{E}$ |
| Four-vector | $\mathsf{E}$ |
| Gradient | $\nabla$ |
| Divergence | $\nabla \cdot$ |
| Curl | $\nabla \times$ |
| Quad | $\Box$ |
| Laplacian | $\nabla^2$ |
| Unit vectors in Cartesian coordinates | $\hat{x}, \hat{y}, \hat{z}$ |
| Unit vectors in cylindrical coordinates | $\hat{\rho}, \hat{\phi}, \hat{z}$ |
| Unit vector along $\mathbf{r}$ | $\hat{r}$ |
| Unit vectors in spherical coordinates | $\hat{r}, \hat{\theta}, \hat{\phi}$ |
| Field point | $(x, y, z)$ |
| Source point | $(x', y', z')$ |

# CONTENTS



* Asterisks mean that the chapters can be skipped without losing continuity.

CHAPTER 1

# VECTOR OPERATORS



This introductory chapter is meant to help those readers who are not yet proficient in the use of vector operators.

We shall frequently refer to the fields of electric charges and currents. For example, we shall consider the force between two electric charges to arise from an interaction between either one of the charges and the field of the other.

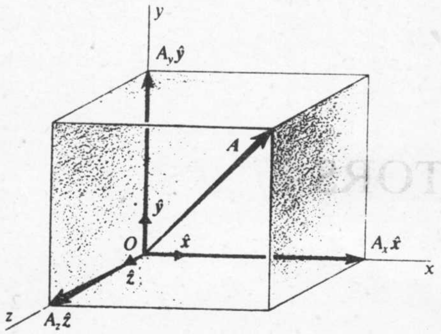


**Fig. 1-1.** A vector $\boldsymbol{A}$ and its three component vectors $A_x\hat{\boldsymbol{x}}$, $A_y\hat{\boldsymbol{y}}$, $A_z\hat{\boldsymbol{z}}$ which, when they are placed end to end, are equivalent to $\boldsymbol{A}$. The unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ point in the positive directions of the coordinate axes and are of unit magnitude.

Mathematically, a *field* is a function that describes a physical quantity at all points in space. In *scalar fields* this quantity is specified by a single number for each point. Temperature, density, and electric potential are examples of scalar quantities that can vary from one point to another in space. In *vector fields* the physical quantity is a vector, specified by both a number and a direction. Wind velocity and gravitational force are examples of such vector fields.

Vector quantities will be designated by ***boldface italic type***, and *unit vectors* will carry a circumflex: $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$.

Scalar quantities will be designated by *lightface italic type.*

We shall follow the usual custom of using right-hand Cartesian coordinate systems as in Fig. 1-1: the positive $z$-direction is the direction of advance of a right-hand screw rotated in the sense that turns the positive $x$-axis into the positive $y$-axis through the 90° angle.

## 1.1 VECTOR ALGEBRA

Figure 1-1 shows a vector $\boldsymbol{A}$ and its three *components* $A_x$, $A_y$, $A_z$. If we define two vectors

$$\boldsymbol{A} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}}, \qquad \boldsymbol{B} = B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\boldsymbol{z}}, \tag{1-1}$$

where $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ are the *unit vectors* along the $x$-, $y$-, and $z$-axes, respectively, then

$$\boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\hat{\boldsymbol{x}} + (A_y + B_y)\hat{\boldsymbol{y}} + (A_z + B_z)\hat{\boldsymbol{z}}, \tag{1-2}$$

$$\boldsymbol{A} - \boldsymbol{B} = (A_x - B_x)\hat{\boldsymbol{x}} + (A_y - B_y)\hat{\boldsymbol{y}} + (A_z - B_z)\hat{\boldsymbol{z}}, \tag{1-3}$$

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z = AB\cos\phi, \tag{1-4}$$

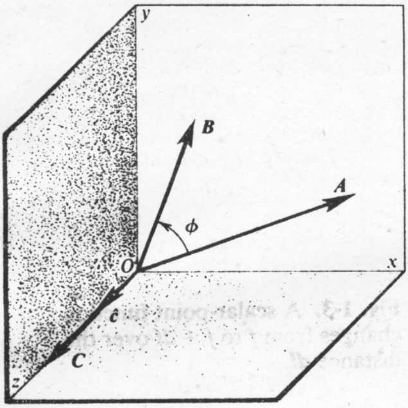


**Fig. 1-2.** Two vectors ***A*** and ***B*** and the unit vector $\hat{c}$, normal to the plane containing ***A*** and ***B***. The positive directions for $\phi$ and $\hat{c}$ follow the right-hand screw rule. The vector product $\boldsymbol{A} \times \boldsymbol{B}$ is equal to $AB \sin \phi\, \hat{c}$, and $\boldsymbol{B} \times \boldsymbol{A} = -\boldsymbol{A} \times \boldsymbol{B}$.

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = AB \sin \phi \boldsymbol{c} = \boldsymbol{C}, \qquad (1\text{-}5)$$

as in Fig. 1-2, where

$$A = (A_x^2 + A_y^2 + A_z^2)^{1/2} \qquad (1\text{-}6)$$

is the *magnitude* of ***A***, and similarly for ***B***.

The quantity $\boldsymbol{A} \cdot \boldsymbol{B}$, which is read "A dot B," is the *scalar,* or *dot product* of ***A*** and ***B***, while $\boldsymbol{A} \times \boldsymbol{B}$, read "A cross B," is their *vector,* or *cross product.*

### 1.1.1 Invariance

The quantities ***A***, ***B***, and $\phi$ are independent of the choice of coordinate system. Such quantities are said to be *invariant.* A vector, say the gravitational force on a brick, is invariant, but its components are not; they depend on the coordinate sytem.

Both the dot and cross products are functions of only ***A***, ***B***, and $\phi$ and are thus also invariant.

The sum and the difference, $\boldsymbol{A} + \boldsymbol{B}$ and $\boldsymbol{A} - \boldsymbol{B}$, are themselves vectors and invariant.

## 1.2 THE GRADIENT $\nabla f$

A *scalar point-function* is a scalar quantity, say temperature, that is a function of the coordinates. Consider a scalar point-function $f$ that is