PART B

Studies in Applied Mechanics

37

# Constitutive Equations for Engineering Materials

## Volume 2: Plasticity and Modeling

Wai-Fah Chen

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STUDIES IN APPLIED MECHANICS 37B

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## Volume 2: Plasticity and Modeling

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# PREFACE

This is the second volume of the two-volume book on constitutive equations for engineering materials. Volume 1 deals with the development of stress-strain models for metals, concrete, and soils based on the principles of elasticity and shows how these models can be applied to engineering practice. This volume extends the elasticity-based stress-strain models to the plastic range and develops plasticity-based models for engineering applications. Here, as in Volume 1, it provides the necessary foundations of the theory of plasticity for civil engineers, develops the constitutive models for metals, concrete, and soils; shows the necessary numerical procedures for computer solutions, and presents extensive finite element results for typical problems in structural and geotechnical engineering applications.

The book is intended as a text as well as a reference book for self-study. It is aimed squarely for civil engineers who do not specialize in this field; yet there is a great demand on them to apply these mathematical models to obtain computer-based solutions of their fast-changing engineering tasks. The two-volume book has been planned to serve such a need. It has been planned as a textbook for the student, as a tool for the practitioner, and as a reference book for the research worker.

Here, as in writing Volume 1, we have endeavored to present the available information on the plasticity and modeling of engineering materials in as elementary a form as possible. For this reason, the book is divided into four parts: Part I on *basic concepts in plasticity* and Part II on *metal plasticity and implementation* contain a reasonably comprehensive treatment of the classical theory of plasticity and its application to metal structures. These two parts can be reasonably covered in a three-hour one-semester course, because it is assumed that the reader has had some contact with the basic stress analysis described in Part 1 on *basic concepts in elasticity* in Volume 1. The first five chapters in Parts I and II serve as a transition to the more complicated problems of Parts III and IV involving concrete and soil materials.

Part III is primarily concerned with *concrete plasticity*, while Part IV deals with *soil plasticity*. Enough is now known about these topics so that a reasonably complete state-of-the-art coverage can be given, such that the reader can follow without difficulty through most of the derivations and implementation. Only in a few cases are some final results given without complete derivations. In these cases, the necessary references to the papers and theses in which the derivations can be found are always given. Thus, the amount of time devoted to Part III or IV can be very flexible, depending upon the background and interest of the reader in which various topics are considered. In the last four chapters, we have also included additional references to relevant topics which were either omitted or only covered very briefly.

As mentioned previously in Volume 1, much of the materials presented in the book is of fairly recent origin and therefore not found in the standard reference works of the field. In fact, much of the research on the constitutive modeling of engineering materials conducted at Purdue University in the last 12 years provided a background for the book and has been drawn on extensively from those Technical Reports and Ph.D. theses, prepared under various phases of research projects on this subject. Those sponsoring this work includes the National Science Foundation, the Exxon Production Research Company, the Purdue Research Foundation, the Federal Highway Administration, and the Defense Nuclear Agency.

In the preparation of this book the contents of many of my former Ph.D. students' theses were used to a large extent. These include: S.S. Hsieh (81), D.J. Han (84), Y. Ohtani (87), E. Yamaguchi (87) and M. Aboussalah (89) in the area of concrete plasticity, and A.F. Saleeb (81), E. Mizuno (81), C.J. Chang (81), M.F. Chang (81), W.O. McCarron (85), and T.K. Huang (90) in the area of soil plasticity. Their contributions to this work is gratefully acknowledged.

W.F. CHEN

*July, 1993*
*West Lafayette, IN*

# NOTATION

Given below is a list of the principal symbols and notations used in the book. All notations and symbols are defined in the text when they first appear. Symbols which have more than one meaning are defined clearly when used to avoid confusion, and usually the correct meaning will be obvious from the context.

**Stresses and Strains**

| | |
|---|---|
| $\sigma_1, \sigma_2, \sigma_3$ | Principal stresses |
| $\sigma_{ij}$ | Stress tensor |
| $s_{ij}$ | Stress deviator tensor |
| $\sigma$ | Normal stress |
| $\tau$ | Shear stress |
| $\sigma_{oct} = \frac{1}{3} I_1$ | Octahedral normal stress |
| $\tau_{oct} = \sqrt{\frac{2}{3} J_2}$ | Octahedral shear stress |
| $\sigma_m = \sigma_{oct}$ | Mean normal (hydrostatic) stress |
| $\tau_m = \sqrt{\frac{2}{5} J_2}$ | Mean shear stress |
| $s_1, s_2, s_3$ | Principal stress deviators |
| $\varepsilon_1, \varepsilon_2, \varepsilon_3$ | Principal strains |
| $\varepsilon_{ij}$ | Strain tensor |
| $e_{ij}$ | Strain deviator tensor |
| $\varepsilon$ | Normal strain |
| $\gamma$ | Engineering shear strain |
| $\varepsilon_v = I_1'$ | Volumetric strain |
| $\varepsilon_{oct} = \frac{1}{3} I'_1$ | Octahedral normal strain |
| $\gamma_{oct} = 2\sqrt{\frac{2}{3} J_2'}$ | Octahedral engineering shear strain |
| $e_1, e_2, e_3$ | Principal strain deviators |

**Invariants**

$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_{ii}$ = first invariant of stress tensor

$J_2 = \frac{1}{2} s_{ij} s_{ij}$

$= \frac{1}{6}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2$

= second invariant of stress deviator tensor

$J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki}$ = third invariant of stress deviator tensor

$\cos 3\theta = \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}$ where $\theta$ is the angle of similarity defined in Figure 5.13

$I_1' = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_v$ = first invariant of strain tensor

$\rho = \sqrt{2J_2}$ = deviatoric length defined in Figure 5.12

$\xi = \frac{1}{\sqrt{3}} I_1$ = hydrostatic length defined in Figure 5.12

$J_2' = \frac{1}{2} e_{ij} e_{ij}$

$= \frac{1}{6}[(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2] + \varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2$

= second invariant of strain deviator tensor

**Material Parameters**

| | |
|---|---|
| $f_c'$ | Uniaxial compressive cylinder strength ($f_c' > 0$) |
| $f_t'$ | Uniaxial tensile strength |
| $f_{bc}'$ | Equal biaxial compressive strength ($f_{bc}' > 0$) |
| $E$ | Young's modulus |
| $\nu$ | Poisson's ratio |
| $K =$ | $\frac{E}{3(1-2\nu)}$ = Bulk modulus |
| $G =$ | $\frac{E}{2(1+\nu)}$ = Shear modulus |
| $c, \phi$ | Cohesion and friction angle in Mohr–Coulomb criterion |
| $\alpha, k$ | Constants in Drucker–Prager criterion |
| $k$ | Yield (failure) stress in pure shear |

**Miscellaneous**

| | |
|---|---|
| { } | Vector |
| [ ] | Matrix |
| $C_{ijkl}$ | Material stiffness tensor |
| $D_{ijkl}$ | Material compliance tensor |
| $f(\ )$ | Failure criterion or yield function |
| $x, y, z$ or $x_1, x_2, x_3$ | Cartesian coordinates |
| $\delta_{ij}$ | Kronecker delta |
| $W(\varepsilon_{ij})$ | Strain energy density |
| $\Omega(\sigma_{ij})$ | Complementary energy density |
| $l_{ij} =$ | $\cos(x_i', x_j)$ = The cosines of the angles between $x_i'$ and $x_j$ axes (see Section 1.11) |
| $\varepsilon_{ijk}$ | Alternating tensor defined in Section 1.10 |

PART ONE

# BASIC CONCEPTS IN PLASTICITY

# CONTENTS

## VOLUME ONE

# VOLUME TWO

# CHAPTER ONE

# Characteristics and Modeling of Uniaxial Behavior

## 1.1 INTRODUCTION

### 1.1.1 Plasticity and Modeling

The failure process or failure mechanism of metal plasticity has been well *identified as the slip or dislocation of crystals.* As a result, the plastic deformation is closely associated with shear deformation, no volume change occurs due to plastic deformation, and plastic behaviors in tension and compression are almost identical. All these characteristic behaviors are common to various metals. Historically, the plasticity theory has been developed in conjunction with these metal behaviors. Thus, the stress–strain relationships of an elastic-plastic material to be described in this chapter represent an idealization of the behavior of these metals.

The internal events taking place in other engineering materials such as concrete, rock and soil are quite different from the microscopic event in metals. For example, the nonlinear behavior of concrete materials is attributed to the development of microcracks. The difference is manifest in the experimental facts that their plastic behaviors involve volume change and that their tensile and compressive behaviors are much different from each other. However, the typical stress–strain curves of these materials under compressive loading exhibit similar characteristics to those of a typical elastic-plastic material. With some modifications, therefore, the concept of the metal plasticity theory is still applicable to this class of materials, and many plasticity-based constitutive models have been proposed for these materials (Chen, 1982; Chen and Mizuno, 1990). A significant advantage of applying the plasticity theory to these materials is that modeling would be logical and concise without loss of mathematical rigorousness.

Microscopically, engineering materials are inhomogeneous, and not all elements yield at the same time. The transition from elasticity to plasticity thus takes place in an homogeneous fashion, and this is why we observe a smooth transition in an overall stress–strain curve obtained in the experiments. However, macroscopically, we may consider these materials homogeneous, whose element yields at the elastic limit and deforms in the way an overall stress–strain response indicates. Most of the plasticity-based constitutive models including those presented in this book have been constructed based on this concept of homogeneous response.

### 1.1.2 Scope

Most of the characteristic behaviors of elastic-plastic materials can be seen in uniaxial material behavior. Therefore, we begin with a discussion of uniaxial behavior; we first describe the essential characteristics of uniaxial elastic-plastic material behavior and against this background, we proceed to describe some material models for elastic-plastic behavior. To this end, this chapter provides the fundamentals of the theory of plasticity, which could directly lead to general formulations of the plasticity theory.

## 1.2 UNIAXIAL STRESS–STRAIN BEHAVIOR

### 1.2.1 Monotonic Loading

Figure 1.1(a) shows a typical stress–strain relationship of an elastic-plastic material under uniaxial loading. Initially, up to a certain stress level $\sigma_0$ at Point P, the strain $\varepsilon$ is proportional to the stress $\sigma$, and the deformation is fully reversible. On further straining beyond Point P, the relationship between stress and strain becomes nonlinear. Point P is therefore called the *proportional limit.*

At Point Q, the material begins to accumulate permanent strain that will not vanish even upon the complete removal of load. This permanent strain is called *plastic strain* in contrast to elastic strain which is recoverable and completely vanishes with the removal of load. Beyond Point Q, the deformation involves both elastic and plastic strains; this process is called either *elastic-plastic deformation, plastic deformation* or *plastic flow*. Point Q is therefore termed the *elastic limit* or the *yield point*. The discrepancy between Points P and Q is usually small and the precise determination of the elastic limit is difficult. Therefore, although various ways of defining the elastic limit have been proposed, the difference is often neglected in particular applications and the proportional limit is generally regarded also as the elastic limit in the construction of constitutive models. In metals, the strain at the elastic limit usually lies between 0.1 and 0.2%.

Beyond the yield point, the slope of the stress–strain curve decreases steadily and monotonically with the load and eventually becomes negative. The nonlinear material behavior in the range with the positive slope ($d\sigma/d\varepsilon > 0$), i.e. before peak

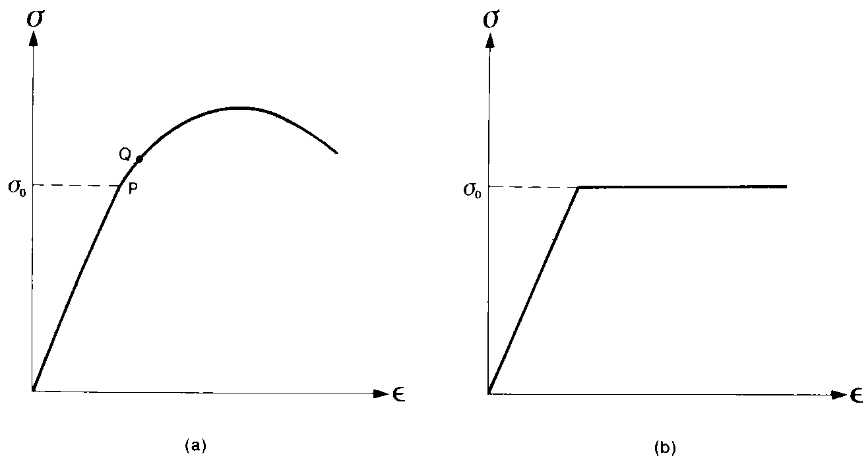


**FIGURE 1.1** Stress–strain behavior in monotonic loading. (*a*) General material. (*b*) Elastic-perfectly plastic material.

load, is called *hardening*, whereas the behavior is called *softening* when the further deformation requires a decrease in load. However, it is often observed in the experiments that the softening behavior is associated with localized and non-homogeneous deformation such as necking in metals. Thus, the softening branch of the stress–strain curve does not always represent a true material response; since it also includes the effect of structural geometrical changes. No considerations will be given to this branch of the stress–strain curve in Part One of this book.

Some class of material such as structural steel possesses an important and unique property called *ductility*. Its stress–strain curve may be represented in an idealized form by two straight lines, as shown in Fig. 1.1(b): up to the yield point the material is elastic; after the yield point has been reached, plastic flow occurs and the strain can increase greatly without any further increase of the stress. This type of behavior is called *elastic-perfectly plastic* and is important in engineering practice. For example, plastic design in steel has been well developed based on this material behavior and there has been a significant application of the method to building frame design (ASCE-WRC, 1971).

In Part One of this book, we shall deal with both hardening and perfectly plastic behaviors. Since perfectly plastic behavior may be considered as the limiting case of hardening behavior, we shall focus our discussion mainly on the hardening response and mention the perfectly plastic response only briefly.

### 1.2.2 Unloading and Reloading

The idealized unloading and reloading behavior of an elastic-plastic material is shown in Fig. 1.2. If the load is reduced during an elastic-plastic process, the strain decreases only elastically with the slope equal to that of the initial elastic response. Even when the load is removed completely, there remains *plastic strain* $\varepsilon^p$, while *elastic strain* $\varepsilon^e$ vanishes. Subsequent reloading proceeds along the same linear elastic relation as that in the unloading process up to Point R where unloading started. Further deformation after Point R produces both elastic and plastic strains,

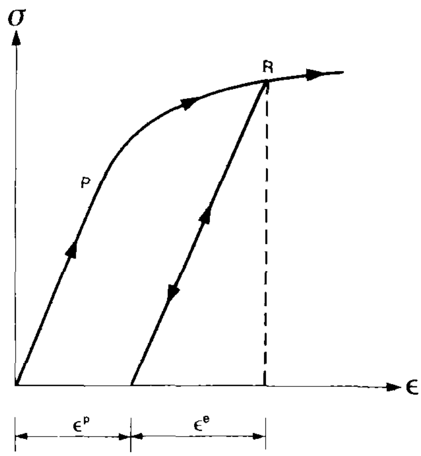


**FIGURE 1.2** Stress–strain behavior in unloading and reloading processes.

and the stress–strain relationship traces the monotonic-loading path as if unloading and reloading had never taken place. Point R thus serves as another yield point. For clarity, the stress at Point R is called the *subsequent yield stress* while the stress at Point P, $\sigma_0$, is called the *initial yield stress*; Point P is the *initial yield point* whereas Point R is the *subsequent yield point*. Description here and Fig. 1.2 clearly indicate that there is no one-to-one correspondence between stress and strain in a plastically deformed solid. The same observations are made in elastic-perfectly plastic materials. However, obviously for this class of materials, the initial yield stress and the subsequent yield stress are both equal to $\sigma_0$.

### 1.2.3 Reversed Loading

As far as monotonic loading is concerned, the initial loading direction makes little difference in the behavior of metal; the behavior of metal in compression is almost the same as that in tension. However, when a hardening type of metal is subjected to loading in tension beyond the initial yield point, it behaves differently in the subsequent reversed loading of compression.

Let the initial yield stresses in tension and compression be $\sigma_{0T}$ and $\sigma_{0C}$, respectively. Thus, in its initial state, the material behaves elastically provided that the stress $\sigma$ lies in the range between $\sigma_{0C}$ and $\sigma_{0T}$. The absolute values of the two yield stresses are the same for metals, although they are not necessarily so in other engineering materials.

Consider now a loading program in which the stress is increased monotonically from zero to the stress in the tensile plastic region, $\sigma_T(\geq \sigma_{0T})$, and then decreased into compression, as shown in Fig. 1.3. During unloading from Point T, linear elastic behavior will persist until some stress $\sigma_C$ is reached, where plastic strain in the opposite direction begins to occur. Point C is thus defined as the subsequent yield point in compression. Subsequent tensile and compressive yield stresses, $\sigma_T$ and $\sigma_C$, are the upper and lower boundaries or limits of the subsequent elastic range, respectively.

The subsequent yield stress in compression, $\sigma_C$, will generally be different from the initial value $\sigma_{0C}$. In particular, $\sigma_C$ is numerically smaller than the initial compressive yield stress $\sigma_{0C}$; i.e. $|\sigma_{0C}| > |\sigma_C|$. This lowering of the compressive yield stress following a plastic preloading in tension is called the *Bauschinger effect*.

By its very nature, plastic deformation is an anisotropic process. The Bauschinger effect is one particular type of directional anisotropy induced by plastic straining, since an initial plastic deformation in one direction reduces the yield stress in the opposite direction during a subsequent reversed loading. This phenomenon has its counterpart in interaction and cross effects between yield stresses in different directions in the case of multiaxial stress; prestraining into the plastic region in any one direction will alter the yield stress values in all directions under multiaxial stress. The Bauschinger effect is thus even more crucial in multidimensional problems involving complex stress histories with significant changes in loading directions, such as stress reversals and cyclic loading conditions.