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DIFFERENTIAL SYSTEMS

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To

D. M.

PREFACE

The primary purpose of this book is to develop the theory of systems of partial differential equations and that of pfaffian systems so as to exhibit clearly the relation between the two theories. The questions treated concern almost exclusively the existence of solutions and methods of approximating them rather than their properties, whose study seems to belong to the theory of functions.

In writing the book the author has been guided by a desire for generality in results and conciseness in subject matter and proofs. As a consequence, the postulational method seemed to force itself upon him. Roughly, the plan has been to take a few existence theorems as postulates and construct the theory upon them. A consistency proof is included by proving the postulates in particular cases. The original plan included extensions of the consistency proofs, but the pressure of other duties prevented carrying this out.

The ideas and nomenclature of modern algebra, as developed, for instance, in van der Waerden's admirable treatise, have been freely used. Some modifications of certain topics, essential for our purposes, have been included, but no systematic development of the theory of commutative polynomial rings has been made. On the other hand, the theory of a certain non-commutative polynomial ring, called here a Grassmann ring, is developed in detail from the postulates in Chapter III, which together with Chapter IV develops ideas introduced by Grassmann and brought to such a high degree of perfection by Cartan. A combination of Cartan's notation, the tensor calculus, and modern algebraic concepts seems very effective. Incidentally, the results about determinants and linear dependence, which are needed, can be proved directly from the postulates as readily as the manner of stating them in the literature can be modified to fit the case in hand.

The treatment of the algebraic case is the author's. Although it has close connection through the highest common factor with Ritt's excellent discussion, which is based on the division algorithm, it differs radically in several respects from that work because of a difference in purpose and viewpoint. In the first place, the basis of our method is algebra, rather than analysis. Secondly, reducibility, which plays such a prominent rôle in Ritt's developments, is of little importance in ours. With existence theorems as our chief objective, the important thing for us is to eliminate multiple roots. A polynomial's having two factors, for example, does not prevent the application of the implicit function theorem, if the factors are distinct, and making that theorem applicable is the chief purpose of the reduction process. Incidentally, it might be well to point out that the term "reducible" has slightly different meanings in the two theories. The system y^2, which Ritt classes as irreducible, is reducible in ours.

Another feature of our treatment, which assumes its most elegant and satisfactory form in the algebraic case, although employed in the whole work, is the admission of the inequation on an equal footing with the equation. This, together with the use of resultants of all orders (subresultants), obviates the necessity of making the preliminary linear transformation of the indeterminates, which is an essential step in Kronecker's method of solution of algebraic systems.

Finally, the algebraic case furnishes the model for treating the elimination problem for systems of functions. This is done in Chapter VIII. The method is subject to certain limitations. First, there is no algorithm for determining the zeros of an analytic function in a given region. The difficulty of removing this restriction can be appreciated if the zeros of the Riemann ζ-function are cited. Second, there may exist zeros which are not the centers of regions where assumption W is true. These zeros may be termed singular. Their determination and study seem destined to remain for some time a highly complex problem, only to be solved in special cases by special methods. In this respect they resemble the solutions of a system of partial differential equations in the neighborhood of a singular point. In spite of these limitations, the general method of elimination given here seems to furnish a definite result, which is perhaps as satisfactory as can be obtained at present.

In addition to bringing Cartan's existence theorem for pfaffian systems into the scheme, Chapter IX shows clearly that it has limitations because it does not give the singular integral varieties unless substantially modified. The same chapter also gives what is believed to be the only method yet developed for finding and making a partial classification of the singular integral varieties. The method ultimately—and it seems essentially—depends on Riquier's fundamental researches.

In order not to interrupt the continuity of the development, the illustrative examples have been segregated in Chapter XI. The reader may find it convenient to study them at the appropriate place in the text.

The author has drawn freely from the work of Cartan, Goursat, and Janet, but he is particularly indebted to Riquier's treatise. The book also incorporates many suggestions made by students in his courses during the past nine years; the present neat statement of the rule of signs in Theorem 9.1, for example, was suggested by Mr. Alexander Makarov. The author is even more indebted to all those who have listened to his lectures for sustaining his interest in the subject by their sympathetic attention.

J. M. Thomas

July, 1936

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TABLE OF CONTENTS

Chapter VI

Algebraic systems

Chapter VII

Algebraic differential systems

Chapter VIII

Function systems and differential systems

Chapter IX

Pfaffian systems

Chapter X

Consistency examples

Chapter XI

Illustrative examples

CHAPTER I

INTRODUCTION

The developments in this book are founded upon two types of algebra which we shall in general regard as having a purely formal nature. Each of them is concerned with a set of given symbols, which it combines by four processes called addition, multiplication, identification, and substitution. The following significance, and nothing further, is to be attached to these names. The addition of two symbols A, B in the order indicated means writing them thus: $A + B$. Their multiplication in the order A, B means writing them thus: AB, or when desired, $A \cdot B$. By these two processes compound symbols, such as the AB, for example, are formed.

At the basis of either type of algebra is a set of symbols denoted by $\mathfrak{R}$. Two symbols of $\mathfrak{R}$, such as A and B, which have different appearance are not necessarily distinct. Every symbol of $\mathfrak{R}$ belongs to one and only one of two important subsets of $\mathfrak{R}$ which will be denoted by $\mathfrak{O}$ and $\mathfrak{N}$. All symbols in $\mathfrak{O}$ are to be regarded as identical with the particular symbol 0. Those in $\mathfrak{N}$ are distinct from 0.

The set $\mathfrak{O}$ in particular acquires some of its members when the operations of addition and multiplication are subjected to assumptions, sometimes called laws, which are essentially conventions to the effect that certain compound symbols will be regarded as identical. Identity is denoted by the sign = which will be read "equals" or "is." The assumptions have as logical consequences other statements of identity which they do not formulate explicitly. It is, moreover, often convenient to introduce a new symbol B for a given (compound) symbol A and augment the set of identities $\mathfrak{O}$ by $A - B$. *Identification* is the process of replacing a symbol (in general, occurring in a compound symbol) by another symbol known to be identical with it; applied to a symbol it gives an equal symbol. *Substitution* is the replacement of a symbol by an arbitrarily chosen symbol; it will be applied in particular to the indices on symbols as well as to the symbols themselves. The study of algebra to be made consists in manufacturing compound symbols by the four processes just described and in proving identities among them.

It will be unnecessary to formulate explicitly the assumptions about addition and multiplication of symbols in $\mathfrak{R}$ because that has been done elsewhere in a form which is both elegant and suited to our purpose. We can specify $\mathfrak{R}$ by saying that it is an *integrity domain* [23, I, 39][1] containing an identity symbol with respect to multiplication. Such an $\mathfrak{R}$ can always be imbedded [23, I, 47]

[1] The first number in square brackets refers to the bibliography at the end of the book; the roman number to the volume; and the second arabic number to the page.

in a *commutative field* $\mathfrak{R}^*$, called its *quotient field*. It may happen that $\mathfrak{R} = \mathfrak{R}^*$. This may also be arranged by choosing for $\mathfrak{R}$ a commutative field at the outset.

At times, we shall also regard $\mathfrak{R}$ (or $\mathfrak{R}^*$) as imbedded in another ring $\mathfrak{R}_C$ or $\mathfrak{R}^N$, which in addition to all the properties possessed by $\mathfrak{R}$ have certain others to be specified at the appropriate place. These larger rings are divided into sets $\mathfrak{O}_C$, $\mathfrak{N}_C$, etc.

A symbol y which does not belong to $\mathfrak{R}$ but which behaves in the formal processes of addition and multiplication as if it did will be called an *indeterminate*.

The adjunction of a finite number of indeterminates y to $\mathfrak{R}$ gives a *polynomial ring* $\mathfrak{R}[y_1, \cdots, y_r]$. The algebra of such a ring is the first type of algebra to be considered. The properties of polynomial rings are discussed at length in treatises on modern algebra. Only those results which need to be presented in a special form for our purposes will be developed here and no systematic treatment of the subject will be made.

The adjunction of a finite number of non-commutative *marks* u, which are to be defined later, gives a *Grassmann ring* $\mathfrak{R}[u_1, \cdots, u_n]$, whose algebra constitutes the second type and will be developed systematically from a set of assumptions.

The sum, difference, and product of any two symbols of a ring belong to the ring, which accordingly is said to be closed under addition, subtraction, and multiplication, called the *ring operations*.

CHAPTER II

GENERALITIES ON SYMBOLS AND SYSTEMS

It seems desirable to give in the present chapter certain definitions and theorems in sufficiently general form to answer all our purposes. The chapter can be omitted on a first reading, and the definitions of the terms can be consulted with the aid of the index as they are encountered in subsequent chapters.

1. **Functions of n variables.** Let $y_1, y_2, \cdots, y_n$ be a finite set of symbols, which will be called *variables.* The *scope* of the variables is a set $\mathfrak{A}$ of symbols each of which has the form $(a_1, \cdots, a_n)$, where each a_i belongs to $\mathfrak{R}_C$.

If with the equations

$$y_i = a_i \tag{1.1}$$

is associated the equation

$$f = \text{any symbol of } \mathfrak{B}, \tag{1.2}$$

where the set $\mathfrak{B}$ is determined when a's belonging to $\mathfrak{A}$ are given, and $\mathfrak{B}$ is a subset of $\mathfrak{R}_C$ for all such a's, the symbol f is called a *function* of the variables y. The set $\mathfrak{B}$ is called the *value* of the function.

If there is exactly one symbol on the right of (1.2) for every member of $\mathfrak{A}$, that is, if $\mathfrak{B}$ reduces to a single symbol, so that (1.2) becomes

$$f = b, \tag{1.3}$$

where b is a unique symbol of $\mathfrak{R}_C$, then f is a *single-valued function* of the y's. The word "function" used alone will usually mean "single-valued function."

More generally, if the set $\mathfrak{B}$ contains only a finite number k of symbols, the function is said to have *type* k. Similarly, a set of functions f_i has *type* k if the symbol $(f_1, f_2, \cdots, f_r)$ has associated with it for every $(a_1, a_2, \cdots, a_n)$ from $\mathfrak{A}$ a symbol $(c_1, c_2, \cdots, c_r)$ from a set $\mathfrak{B}$ of k such symbols.

THEOREM 1.1. *If f_i form a set of r functions of $y_1, \cdots, y_n$ whose type is k and g_i form a set of s functions of $y_1, \cdots, y_n$ and $f_1, \cdots, f_r$ whose type is l, then g_i form a set of s functions of $y_1, \cdots, y_n$ whose type is kl.*

The proof consists simply in the remark that the set $\mathfrak{B}$ for g_i as functions of $y_1, \cdots, y_n$ is obtained by combining with an arbitrary member of the $\mathfrak{B}$ for the f's an arbitrary member of that for the g's as functions of both the y's and f's. There are kl such symbols.

The set of all symbols in $\mathfrak{R}_C$ which are not in $\mathfrak{B}$ is called the *complement* of $\mathfrak{B}$ (in $\mathfrak{R}_C$) and is denoted by $\bar{\mathfrak{B}}$. Likewise, the function whose value is $\bar{\mathfrak{B}}$ is called the complement of f (in $\mathfrak{R}_C$) and is denoted by $\bar{f}$.

Formulas (1.1) define a *substitution*, which replaces each y by the a having the same subscript.

The notation $f(y_1, \cdots, y_n)$ for the function f defined above puts in evidence the variables y. Let $f(a_1, \cdots, a_n)$ be used to denote the right member of (1.2). The latter symbol arises from the former by the substitution (1.1).

2. Systems. A finite set S of functions each of which has attached to it the name *equation*[2] or *inequation* is called a *system*. Two functions which are both equations or both inequations are said to have the *same nature*.

The inequations are designated by placing bars over them. Thus if S comprises two equations f, g and one inequation h, we write

$$S = f + g + \bar{h}. \tag{2.1}$$

Strictly speaking, we should employ a new symbol rather than the $+$ in (2.1), for S may contain a compound symbol in which the $+$ has already been employed in another sense. We shall avoid this difficulty by enclosing any compound symbol in non-removable parentheses. Thus $S = (f + g)$ will denote a system with the single function $f + g$ and $S = f + g$ will consist of the two functions f, g. More generally, if S and T are two systems, $S + T$ is the system which contains all the equations of S and T as equations and all their inequations as inequations. Likewise, if T is a subsystem of S, then $S - T$ denotes the system obtained by omitting from S the functions of T.

Let a substitution replace the variables in a function f by symbols from their scope. The substitution is called a *zero* or *non-zero* of f according as the result belongs to $\mathfrak{O}_C$ or $\mathfrak{N}_C$.

A substitution is a *root* of S if it is a zero of every equation and a non-zero of every inequation in S. The totality of the roots of S is its *content*. A system S_1 *implies* S_2 and we write $S_1 \geqq S_2$, if every root of S_1 is a root of S_2. Two systems S_1 and S_2 are *equivalent* and we write $S_1 = S_2$, if each implies the other, that is, if they have the same content. If S_1 implies S_2 but is not equivalent to it, we write $S_1 > S_2$.

The system S is said to be *factored* into the two systems S_1, S_2 according to the equation

$$S = S_1 S_2, \tag{2.2}$$

if every root of S is a root of at least one of the factors, and every root of S_1 and every root of S_2 are roots of S. If no sum is involved, it is unnecessary to distinguish between (fg) and fg.

If f is a function, it is clear that $f + \bar{f}$ has no zero and $f\bar{f}$ has no non-zero. Hence we write

$$f + \bar{f} = 1, \qquad f\bar{f} = 0 \tag{2.3}$$

[2] This terminology will save the introduction of additional names, and will lead to no confusion, although "equation" in the ordinary sense means the result of equating the function to zero and not the function itself.

for purposes of manipulation. The equivalences expressed by the following identities are also useful:

$$S^n = S \qquad (n > 1), \tag{2.4}$$

$$S + ST = S, \tag{2.5}$$

$$(S + \bar{f})(S + f) = S. \tag{2.6}$$

In these, S, T represent any systems and f any function.

A system is *consistent* or *inconsistent* according as it has a root or not. In harmony with (2.3) we write $S = 1$, if S is inconsistent. This symbol 1 may be suppressed if it occurs in a product with other factors. It has the further property that $S + 1 = 1$.

A system is inconsistent if it contains a symbol from $\mathfrak{N}$ *as equation or a symbol from* $\mathfrak{O}$ *as inequation.*

As will be seen later, a symbol y may have associated with it certain other symbols called its *derivatives.* If each member of a substitution (1.1) is replaced by its derivative of a given type and the result is adjoined to the original substitution, an *extended substitution* results.

If some of the variables are selected and called *unknowns* and the others are interpreted as definite derivatives of those unknowns, a system S becomes a *differential system.* A *solution* of S is a substitution *on the unknowns* which when properly extended becomes a root of S. The definitions of content, equivalence, etc. given above apply to differential systems if the word "root" is replaced by "solution." Thus the *(differential) content* of a differential system is the totality of its solutions, etc.

3. **Ordering symbols.** When clarity will not be impaired, we shall often refer to the symbol $i_1 i_2 \cdots i_n$ as i. The equality $i = j$ will mean

$$i_1 = j_1, \quad i_2 = j_2, \quad \cdots, \quad i_n = j_n. \tag{3.1}$$

Likewise, the inequality $i > j$ (to be read "i is greater than j" or "i follows j") will mean the existence of a positive integer $\lambda \leqq n$ such that

$$i_1 = j_1, \quad \cdots, \quad i_{\lambda-1} = j_{\lambda-1}, \quad i_\lambda > j_\lambda, \tag{3.2}$$

and the inequality $i < j$ (read "i is less than j" or "i precedes j") will mean

$$i_1 = j_1, \quad \cdots, \quad i_{\lambda-1} = j_{\lambda-1}, \quad i_\lambda < j_\lambda. \tag{3.3}$$

If the letters in (3.1), (3.2), and (3.3) represent certain of the rational integers and the signs $=$, $>$, $<$ are given their usual meaning, one and only one of the relations (3.1), (3.2), and (3.3) is verified by any pair i, j. Hence in this case the symbols i, j are said to be *ordered* [23, I, 192].

The above definition can be applied inductively to order complex symbols $\rho = \rho_1 \cdots \rho_m$, where each ρ_α is taken from a previously ordered set $\mathfrak{A}_\alpha$, which may vary with α. In the case that interests us, the symbols $i_1 \cdots i_n$, where each i_λ is a non-negative rational integer, are ordered first; the symbols $\rho_1 \cdots \rho_m$.

where each ρ is a non-negative complex integer $i_1 \cdots i_n$, next; and so on. The ordering is called *lexicographical* because it is used to order the words in dictionaries.

Important properties are given in the following easily proved theorems.

THEOREM 3.1. *Lexicographical ordering is transitive: if $i > j$ and $j > k$, then $i > k$.*

THEOREM 3.2. *Every decreasing sequence of lexicographically ordered symbols is finite.*

THEOREM 3.3. *If $(i_1 \cdots i_m) > (j_1 \cdots j_m)$, then $(i_1 \cdots i_m i_{m+1} \cdots i_n) > (j_1 \cdots j_m j_{m+1} \cdots j_n)$ for arbitrary $i_{m+1}, \cdots, i_n, j_{m+1}, \cdots, j_n$.*

At times, it is better to use the parentheses around the symbol $i_1 \cdots i_n$.

The *sum* of the symbols i, j is defined to be $(i_1 + j_1, \cdots, i_n + j_n)$. Their difference is similarly defined.

4. **Reduction algorithm for systems.** With each symbol of a system S let us associate one of the above symbols $i_r \cdots i_1$, which we shall for convenience temporarily call its *rank* because it becomes the rank in an important special case (§30). Likewise we shall say that a symbol of S has *ordinal*[3] k if its rank is $0 \cdots 0\, i_k \cdots i_1$, with i_k, which will be called the *grade,* not equal to zero. Let the symbols of ordinal k form the subset S_k of S.

An operation P_k is called a *reduction algorithm* for systems if it has the following properties:

(i) It is applicable to S so long as S_k contains at least two symbols.

(ii) It leaves unaltered every S_l for $l > k$.

(iii) Each symbol of S_k is omitted or replaced by a symbol not exceeding it in grade. A symbol of ordinal k may be added to S_k provided such symbols added by successive applications of P_k have decreasing rank.

(iv) S_l for $l < k$ is replaced by a finite set of symbols of ordinal l.

(v) There exists a non-negative rational integer a such that P_k^a (i.e., P_k applied a times) replaces at least one symbol of S_k by one with smaller i_k.

(vi) S_k is made to contain at most one inequation by replacing two or more inequations by their product.

We shall next prove: *If $P_r^{c_r} P_{r-1}^{c_{r-1}} \cdots P_1^{c_1}$, where P_k is a reduction algorithm and the c's are appropriately chosen non-negative rational integers, is applied to S, there results a system for which each S_i contains at most one function.*

As P_r is successively applied, the number of symbols in S_r ultimately ceases to increase because by (iii) the additional symbols introduced have decreasing rank and hence by Theorem 3.2 are finite in number. Suppose that, no matter how often P_r is applied, the S_r contains an equation and m other symbols. The grades of these functions form $m + 1$ non-increasing sequences. If only distinct terms are retained, these sequences become decreasing and by Theorem

[3] We do not follow Ritt in attaching the name "class" to this notion because it is necessary to use "class" in a different sense in Chapter IV.

3.2 are finite. Let their minimum members be $d_0, d_1, \cdots, d_m$. There exists an e such that P_r^e gives an S_r with $m + 1$ members having these grades. On the other hand, there exists by (v) an f such that P_r^f decreases at least one d. This contradiction gives the desired result for S_r. The same argument can be successively applied to each of the other S's, and the statement is proved.

We have also proved the useful result contained in

THEOREM 4.1. *A given reduction algorithm can be applied only a finite number of times to a system.*

5. **Ordering by cotes.** The lexicographical ordering has to be modified in order to meet all our needs. Although this modification can be made from a purely abstract viewpoint, we shall develop it in connection with the derivatives

$$\frac{\partial^{i_1+\cdots+i_n} z_\alpha}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \tag{5.1}$$

because the phraseology will be simpler.

The derivatives (5.1) can be given the same order as the complex integers $\alpha i_n \cdots i_1$. In particular, when derivatives of a *single unknown* z are being considered, the symbol $i_n i_{n-1} \cdots i_1$ will be found very useful and will be called the *rank*. This type of ordering will not serve all purposes, however, because a given derivative may have an infinite number of predecessors: thus

$$\frac{\partial z}{\partial x_2} > \frac{\partial^i z}{\partial x_1^i}$$

for all values of i.

The difficulty may be avoided as follows. Let complex integers,[4] called *cotes,*

$$(\gamma_1^\alpha, \cdots, \gamma_s^\alpha), \qquad (1, c_2^j, \cdots, c_s^j) \tag{5.2}$$

be associated with z_α and x_j, respectively, the γ's and c's being non-negative rational integers. The cote of a derivative is defined as the sum of the cotes of the unknown and the independent variables, each of the latter being added as many times as differentiation occurs with respect to that variable. Accordingly, the cote of (5.1) is

$$\begin{aligned}(\gamma_1^\alpha + i_1 + \cdots + i_n, \gamma_2^\alpha + c_2^1 i_1 + \cdots + c_2^n i_n, \cdots,& \\ \gamma_s^\alpha + c_s^1 i_1 + \cdots + c_s^n i_n).&\end{aligned} \tag{5.3}$$

The derivative with greater cote is defined to be the *follower*. Two different derivatives may, however, have the same cote according to the above

[4] Riquier, who introduced cotes, applied the name to the component rather than to the complex number as we prefer to do.