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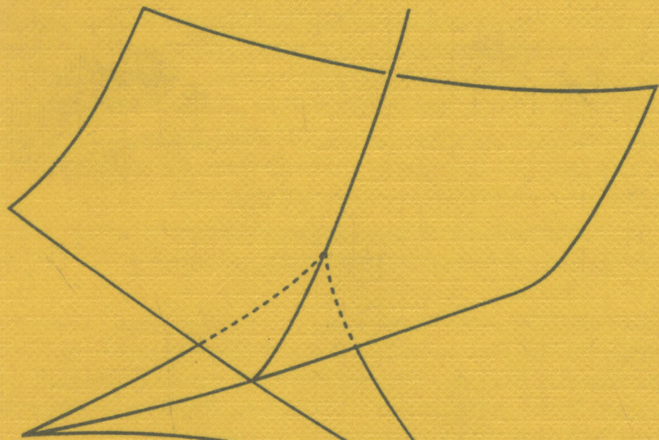

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PREFACE

The meeting on "Real Algebraic Geometry" was held in La Turballe, on the seashore not far from Rennes, from June 24 to 28, 1991. It took place ten years after the first meeting on "Géométrie Algébrique Réelle et Formes Quadratiques" (*). These Proceedings contain survey papers on some of the developements of real algebraic geometry in the last ten years, and also contributions by the participants. Every paper has been submitted to a referee, and we want to thank all of them for their collaboration.

The meeting, and the collaboration between the european teams which made it possible, received support from the Université de Rennes 1, the GDR Mathématiques-Informatique (CNRS), and the programs Réseau Européen de Laboratoires, Acces, Alliance, Actions Intégrées Franco-Espagnoles. We would like also to thank Springer-Verlag for publishing this volume, and to express our gratitude to Ms. Yvette Brunel, for her precious help for the secretary of the meeting.

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(*) Lecture Notes in Mathematics 959, Springer (1982)

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Manfred Knebusch

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§1 Brumfiel's program

Before discussing the subject named in the title it seems appropriate to outline the situation in semialgebraic topology in 1981, at the time of the first Rennes conference on real algebraic geometry.

Already in the seventies, in the long introduction to his book "Partially ordered rings and semialgebraic geometry" [B], G.W. Brumfiel had laid down a program for what we now call "semialgebraic topology". Here Brumfiel advocated a new way of handling topological problems which is closer to the spirit of algebraic geometry than traditional topology. Let me just quote the following passage:

"It thus seems to me that a true understanding of the relations between algebraic geometry and topology must stem from a deeper understanding of real algebraic geometry, or, actually, semi-algebraic geometry. Moreover, real algebraic geometry should not be studied by attempting to extend classical algebraic geometry to non-algebraically closed ground fields, nor by regarding the real field as a field with an added structure of a topology. Instead, the abstract algebraic treatment of inequalities originated by Artin and Schreier should be extended from fields to (partially ordered)

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algebras, with real closed fields replacing the algebraically closed fields as ground fields" [B, p.2].

In the main body of the book [B] Brumfiel develops a "real algebra" by studying partially ordered commutative rings and various sorts of convex ideals, with the perspective that this real algebra should perform a similar role in semialgebraic geometry as commutative algebra does in present day algebraic geometry. But the book does not go very far into semialgebraic topology.

§2 The two approaches

Even today not much semialgebraic topology has been done using Brumfiel's rather intricate real algebra from the seventies. Around 1979 two other approaches to semialgebraic topology emerged independently which turned out to be successful. These are the "abstract" approach by M. Coste and M.F. Roy, and the "geometric" approach by H. Delfs and M. Knebusch.

Before we get into this let me remind you of what are perhaps the two most serious difficulties which one encounters if one works over a real closed base field R different from $\mathbb{R}$.

a) R^n is totally disconnected in the strong topology (i.e. the topology coming from the ordering of R).

b) R^n has very few reasonable (i.e. geometrically relevant) compact subsets. In particular, the closed unit ball in R^n is not compact.

Let M be a semialgebraic subset of some R^n. In the abstract approach one adds to M "ideal points" which turn M into an honest (albeit not Hausdorff) topological space. More precisely, one passes from M to the corresponding constructible subset $\tilde{M}$ of the real spectrum Sper $R[T_1,\ldots,T_n]$ of the polynomial ring $R[T_1,\ldots,T_n]$ (cf. [BCR, Chap. 7]). $\tilde{M}$ turns out to have only finitely many connected components, and $\tilde{M}$ is quasicompact. Thus in some sense the difficulties described above are overcome. The subspace topology of M in $\tilde{M}$ is the strong topology we started with.

One could also pass from M to the subspace $\tilde{M}^{\max}$ of closed points of $\tilde{M}$, which still contains M as a dense subset and is a compact Hausdorff space with only finitely many connected components. But although this compactification $\tilde{M}^{\max}$ of M has its merits (cf. [B_1]), the more interesting and more useful space is $\tilde{M}$ itself. The main reason for this is that $\tilde{M}$ is a spectral space, as defined by Hochster [Ho], and that the constructible subsets Y of $\tilde{M}$ correspond bijectively with the semialgebraic subsets N of M via the relation $Y = \tilde{N}$. A very nice consequence of this is that the "semialgebraic structure" of M is encoded to a large extent in the topology of $\tilde{M}$, since the lattice $\mathfrak{K}(\tilde{M})$ of constructible subsets of $\tilde{M}$ is by definition the boolean lattice generated by the lattice $\mathring{\mathfrak{K}}(\tilde{M})$ of quasicompact open subsets of $\tilde{M}$, and thus

$\mathfrak{K}(\tilde{M})$ is completely determined by the topology of $\tilde{M}$ (cf. [Ho]). We call the space $\tilde{M}$ the *abstraction* of the semialgebraic set M.

The wisdom of passing back and forth between the semialgebraic sets and their abstractions has been displayed well in the book [BCR] by Bochnak, Coste and Roy. Curiously another very important and fascinating aspect of the abstract approach is scarcely touched on in that book: One can study the constructible subsets of the real spectrum $\operatorname{Sper} A$ of *any* commutative ring A. Thus the abstract approach opens the door for an "abstract" semialgebraic topology where no base field (real closed or not) needs to be present. Coste and Roy were certainly well aware of this aspect at an early stage (cf. for example Roy's paper on abstract Nash functions [R]) but chose not to give much space to this in their book with Bochnak.

The geometric approach (cf. [DK]) relies on the following two ideas, the first one being very simple.

1° Don't consider any subset of a semialgebraic set $M \subset R^n$ which is not semialgebraic or any map $f: M \to N$ between semialgebraic sets which is not semialgebraic!

In this context a *map* f is called *semialgebraic* if the graph of f is a semialgebraic set and f is continuous with respect to the strong topologies of M and N.

2° Install on M a Grothendieck topology such that the semialgebraic functions, i.e., semialgebraic maps to R on the open semialgebraic subsets U of M (open with respect to the strong topology) form a sheaf $\mathcal{O}_M$ of R-algebras! Instead of studying M as a semialgebraic subset of R^n study the ringed space $(M, \mathcal{O}_M)$!

Let me give some comments and explanations on these ideas.

Ad 1°: The reason that this idea makes sense is Tarski's principle. It guarantees that many of the usual constructions of new sets and maps from given ones give us semialgebraic sets and maps if we start with such sets and maps. In particular, if $f: M \to N$ is a semialgebraic map between semialgebraic sets then the image $f(A)$ of a semialgebraic subset A of M is semialgebraic and the preimage $f^{-1}(B)$ of a semialgebraic subset B of N is semialgebraic. Continuity of f is not necessary for this but is appropriate since we want to do "topology".

Ad 2°: The Grothendieck topology on M is defined as follows. The underlying category is the category $\mathring{\mathfrak{S}}(M)$ of open semialgebraic subsets of M (i.e. semialgebraic subsets which are open in the strong topology), the morphisms being the inclusion mappings. An admissible open covering $(U_i | i \in I)$ of some $U \in \mathring{\mathfrak{S}}(M)$ is a family $(U_i | i \in I)$ in $\mathring{\mathfrak{S}}(M)$ with $U = \bigcup_{i \in I} U_i$, such that there exists a finite subset J of I with $U = \bigcup_{i \in J} U_i$. {Thus a property similar to quasicompactness is forced to hold.} Then the semialgebraic functions on the sets $U \in \mathring{\mathfrak{S}}(M)$ indeed form a sheaf $\mathcal{O}_M$. It turns out that a morphism from $(M, \mathcal{O}_M)$ to $(N, \mathcal{O}_N)$ is determined by the underlying map f from M to N, and that these maps f are just the semialgebraic maps from M to N as introduced above (cf. [DK, §7], by definition the morphism has to respect the R-algebra structures of the structure sheaves).

Replacing a semialgebraic set $M \subset R^n$ by the ringed space $(M, \mathcal{O}_M)$ allows us to forget the embedding $M \hookrightarrow R^n$. We call any ringed space of R-algebras which is isomorphic to such a space $(M, \mathcal{O}_M)$ an *affine semialgebraic space* over R. By abuse of notations we do not distinguish between a semialgebraic set M and the corresponding ringed space $(M, \mathcal{O}_M)$.

A *semialgebraic path* in M is a semialgebraic map from the unit interval [0,1] (which is a semialgebraic subset of R^1) to M. Having this notion of paths at hand one defines the path components of M in the obvious way. It turns out, that M has only finitely many path components $M_1, \dots, M_r$ and that these are semialgebraic in M and closed, hence also open in the strong topology, cf. [DK]. Every M_i is "semialgebraically connected", i.e. M_i is not the union of two disjoint non empty open semialgebraic subsets, since this holds for [0,1], as is easily seen. Thus we have dealt with the first difficulty mentioned above, exploiting only idea N° 1. By the way, the abstractions $\tilde{M}_1, \dots, \tilde{M}_r$ are the connected components of the topological space $\tilde{M}$.

In order to cope with the second difficulty one also needs idea N° 2. The category of affine semialgebraic spaces over R has fiber products. Thus we can define proper morphisms as in algebraic geometry. We call a semialgebraic map $f: M \to N$ *closed*, if the image $f(A)$ of a closed semialgebraic subset A of M is again closed. We call f *proper* if f is universally closed, i.e. for any semialgebraic map $g: N' \to N$ the cartesian square

$$\begin{array}{ccc} M \times_N N' & \xrightarrow{f'} & N' \\ g' \downarrow & & \downarrow g \\ M & \xrightarrow[f]{} & N \end{array}$$

gives us a closed semialgebraic map f'. We call an affine semialgebraic space M *complete* if the map from M to the one-point space is proper. Even more than in algebraic geometry over an algebraically closed field, it is true for many purposes, that complete spaces are the right substitute for compact spaces in topology. For example, a semialgebraic function on a complete space attains its maximum and minimum.

It turns out that there exist in abundance relevant complete affine semialgebraic spaces. Namely, the following analogue of the Heine-Borel theorem holds: A semialgebraic subset M of R^n is a complete space iff M is closed and bounded in R^n.

§3 The state of art in 1981

I give a rough sketch of the technical progress up till 1981. This is just to give an impression of the state of art at the first Rennes conference. It is not meant, of course, as a complete account of everything done up to that time.

In the geometric theory we have the following list.

1) Connected components
2) Complete affine semialgebraic spaces and the semialgebraic Heine-Borel theorem
3) Dimension theory
4) Existence of triangulations
5) Hardt's theorem
6) Semialgebraic homology

Here are some comments on these.
N° 1 and N° 2 have been described above. One may add to N° 2 that in 1981 we also had a good insight into the nature of proper maps between affine semialgebraic spaces [DK, §9 and §12].

Ad 3: The dimension $\dim M$ of a semialgebraic set can be defined as the maximal integer d such that M contains a subspace N which is isomorphic to the unit ball in R^d {[DK, §8], there a different but equivalent definition had been given}. This notion of dimension behaves very well, better than in classical topology. For example, if a partition of M into finitely many semialgebraic subsets $A_1, \dots, A_r$ is given, then $\dim M$ is the maximum of the numbers $\dim A_1, \dots, \dim A_r$.

Ad 4: If M is an affine semialgebraic space and $A_1, \dots, A_r$ are finitely many semialgebraic subsets of M then there exists a finite simplicial complex X over R and an isomorphism of spaces $\varphi: X \xrightarrow{\sim} M$ such that, for every $i \in \{1, \dots, r\}$, the set $\varphi^{-1}(A_i)$ is a subcomplex of X [DK, §2]. Here the word "simplicial complex" is used in a non classical meaning: X is the union of finitely many open simplices $\sigma_1, \dots, \sigma_t$ in some R^N such that the intersection $\bar{\sigma}_i \cap \bar{\sigma}_j$ of the closures of any two simplices σ_i, σ_j is either a common face of them or empty. Thus the closure $\bar{X}$ of X is a classical finite simplicial complex ($\approx$ finite polyhedron), and X is obtained from $\bar{X}$ by omitting some open faces. Also "subcomplex" means just the union of some of the sets $\sigma_1, \dots, \sigma_t$. Clearly $X = \bar{X}$ iff M is complete.

In the case $R = \mathbf{R}$ the triangulation theorem has been well known since the sixties, even for semianalytic sets [L, Gi].

Ad 5: Hardt's theorem states that for every semialgebraic map $f: M \to N$ there exists a partition of N into finitely many semialgebraic subsets $N_1, \ldots, N_r$ such that f is trivial over each N_j, i.e. $f^{-1}(N_j)$ is isomorphic over N_j to a direct (= cartesian) product $N_j \times F_j$, cf [DK_1, §6]. The theorem had been proved for $R = \mathbb{R}$ by R. Hardt around 1978 [Ha].

Ad 6: In his thesis [D] Delfs constructed homology and cohomology groups with arbitrary constant coefficients for affine semialgebraic spaces over any real closed field R. In the case $R = \mathbb{R}$ these groups coincide with the singular groups known from classical topology.

Certainly Delfs' homology theory was the most profound achievement in semialgebraic topology up till 1981. But the proofs of the triangulation theorem and of Hardt's theorem also needed new ideas beyond the known proofs for $R = \mathbb{R}$.

The triangulation theorem is the main technical tool in developing semialgebraic homology (and also semialgebraic homotopy theory, cf. §10 below). Hardt's theorem is very useful if one wants to profit from semialgebraic homology. For a good example, cf. [DK_1, §7]. I will say more about semialgebraic homology in the next section §4.

Remark. Only recently (1989) I learned from Gert-Martin Greuel about the unpublished dissertation of Helmut Brakhage [Bra] (Heidelberg 1954, thesis advisor F.K. Schmidt). Here Brakhage studies semialgebraic topology over an arbitrary real closed field. He exploits idea N° 1 of the geometric theory (cf. §2) to an enormous extent and obtains many of the results we had found up to 1981, in particular the triangulation theorem. The introduction to Brakhage's thesis reads very much like the talks Delfs and I used to give around 1980. He would have saved us a lot of work if we only would have known about his thesis. Brakhage is now a professor at Kaiserslautern, working mostly in applied mathematics.

It is difficult to give a good picture of the state of art in semialgebraic topology in 1981 on the abstract side, since in the abstract theory the main bias was towards algebraic problems. Topology seems to have been studied mainly as an aid for solving algebraic problems of current interest. I give the following list.

1) Connected components
2) Compactness of constructible sets
3) Specialization theory
4) Dimension theory
5) Abstract Nash functions
6) Separation of connected components by global quadratic forms

Here only N° 1 - 4 truly belong to semialgebraic topology, but N° 5 and 6 use topology in an essential way, and have also turned out to be stimulating for semialgebraic topology since 1981.

N^o 1 has been discussed above, N^o 2 alludes to the easily accessible but extremely important fact, that the real spectrum Sper A of any commutative ring A is compact in the constructible topology. This means that, if X is a constructible subset and $(Y_i|i \in I)$ is a family of constructible subsets of Sper A with $X \subset \bigcup_{i\in I} Y_i$, then there exists a finite subset J of I with $X \subset \bigcup_{i\in J} Y_i$. The quasicompactness of $\tilde{M}$ stated above is a rather special consequence of this.

Ad 3: If x and y are points of a topological space X then we say that y is a *specialization* of x (and x is generalization of y) if y lies in the closure of the set $\{x\}$. We write $x \succ y$ for this. N^o 3 alludes to some – again simple but important – facts about specializations in a real spectrum Sper A, cf. [CR$_2$], [BCR, 7.1], [KS, III §3 and §7]. In particular, the specializations of a given point x in Sper A form a chain, i.e. if $x \succ y$ and $x \succ z$ then $y \succ z$ or $z \succ y$. Moreover if neither $x \succ y$ nor $y \succ x$ then there exist disjoint open subsets U, V in Sper A with $x \in U$ and $y \in V$.

Ad 4: The *dimension* $\dim X$ of a constructible subset X of Sper A is defined as the supremum of the lengths of the specialization chains in X. {Up till now it has been adequate to put $\dim X = \infty$ if the lengths do not have a finite bound.} The main result is that, if M is a semialgebraic set over some real closed field, then this "combinatorial" dimension $\dim \tilde{M}$ of the abstraction $\tilde{M}$ coincides with the semialgebraic dimension $\dim M$ from above, cf. [CR$_2$], [BCR].

Ad 5 and 6: One of the most important achievements in the early work of Coste and Roy is the construction of a sheaf of "abstract Nash functions" $\mathfrak{N}_A$ on the real spectrum of an arbitrary commutative ring A [R], which generalizes the sheaf of classical Nash functions for algebraic manifolds over **R**. Indeed, right from the beginning they had the idea of constructing the real spectrum as a ringed space $(\text{Sper}\, A, \mathfrak{N}_A)$ [CR], [CR$_1$], thus bringing semialgebraic geometry close to the spirit of abstract algebraic geometry in the sense of Grothendieck. The sheaf $\mathfrak{N}_A$ is more algebraic in nature than the sheaf of semialgebraic functions discussed in §2. It does not belong to semialgebraic topology, but nevertheless relies on the topological fact that every etale morphism $A \xrightarrow{\varphi} B$ induces a local homeomorphism $\text{Sper}\,\varphi\colon \text{Sper}\, B \to \text{Sper}\, A$.

Building on this, Mahé was able to solve one of the main open problems of quadratic form theory from the seventies [K$_1$, Problem 16] affirmatively, namely the separation by global quadratic forms of the connected components of the set $V(\mathbf{R})$ of real points of an affine algebraic variety V, and later, together with Houdebine, also of a projective algebraic variety V over **R** [M], [HM]. In fact, they prove such a theorem over any real closed field R, and also for the real spectrum of any commutative ring.

Mahé's theorem in [M] is probably the first result which signaled to the outside world

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that something new in principle had happened in real algebraic geometry around 1980.

§4 Sheaves and homology

After 1981 semialgebraic topology has been dominated by two major new trends: A strong interaction between the geometric and the abstract theory, and the employment of new spaces. An important instance of the first trend is sheaf theory.

Let M be a semialgebraic set over some real closed field R. Then a (set valued) sheaf over M is essentially the same object as a sheaf over the abstraction $\tilde{M}$. Indeed, as was already known before 1981 [CR$_2$], [D], [De], a semialgebraic subset U of the affine semialgebraic space M is open iff the abstraction $\tilde{U}$ is open in $\tilde{M}$. Moreover, a family $(U_i | i \in I)$ of open semialgebraic subsets of M is an admissible open covering of U iff $(\tilde{U}_i | i \in I)$ is an open covering of $\tilde{U}$. The reason for this is the definition of the Grothendieck topology on M on the one hand, and the quasicompactness of $\tilde{U}$ on the other. Since the quasicompact open subsets of $\tilde{M}$ form a basis of the topology of $\tilde{M}$, all of this gives us a canonical isomorphism $\mathcal{F} \mapsto \tilde{\mathcal{F}}$ from the category of sheaves on M to the category of sheaves on $\tilde{M}$, via the rule $\mathcal{F}(U) = \tilde{\mathcal{F}}(\tilde{U})$.

Henceforth we only consider sheaves of abelian groups. Recall that M is dense in $\tilde{M}$. For $x \in M$ the stalks $\mathcal{F}_x$ and $\tilde{\mathcal{F}}_x$ are equal. It may well happen that all stalks $\mathcal{F}_x$, $x \in M$, are zero, but $\mathcal{F}$ is not zero. {An example is given in [D$_2$, I.1.7].} This is by no means astonishing: Of course, $\mathcal{F} \neq 0$ iff $\tilde{\mathcal{F}} \neq 0$. Then, since $\tilde{M}$ is an honest topological space, there exists some $\alpha \in \tilde{M}$ with $\tilde{\mathcal{F}}_\alpha \neq 0$. But it may happen that none of these points α lies in M.

This discussion makes it clear that most often sheaf theoretic techniques work better in the abstract setting than the geometric one. Only there one can argue "stalk by stalk" without further justification.

Now is a good moment to say something about the semialgebraic homology theory of Hans Delfs, since he has been able to simplify his theory greatly by using sheaves and the interplay back and forth between semialgebraic sets and their abstractions [D$_1$].

I first describe the main problem in defining homology groups $H_q(M, G)$ for a semialgebraic set M over some real closed field R and some abelian group of coefficients G. Let us assume for simplicity that M is complete. We choose a triangulation $\varphi: X \xrightarrow{\sim} M$. Here X is a finite simplicial complex in the classical sense but over R; X may be regarded as the realization $|K|_R$ over R of an abstract finite simplicial complex K, a purely combinatorial object (cf. [Spa, 3.1]; the realization is defined exactly as in the case $R = \mathbf{R}$).

It is intuitively clear that $H_q(M, G)$ should coincide, up to isomorphism, with the combinatorial homology group $H_q(K, G)$ from classical topology. To make an honest definition out of this, one has to verify that (up to natural isomorphism) the group $H_q(K, G)$ does not depend on the choice of the triangulation. The now traditional way to prove this is to define a complex $C.(M, G)$ of singular chains and to verify the seven Eilenberg-Steenrod axioms for the homology groups [ES, I §3]. Then one obtains, in a well known manner, that $H_q(C.(M, G)) \cong H_q(K, G)$ for the triangulation φ above. {One also has to consider noncomplete spaces M and the relative chain complex $C.(M, A; G)$ for A a semialgebraic subset of M. I omit these technicalities.}

We can indeed define singular chain groups $C_q(M, G)$ along classical lines, decreeing that a singular simplex is a semialgebraic map from the q-dimensional standard simplex Δ_q to M. Six of the seven Eilenberg-Steenrod axioms can be proved as in the classical theory, always using semialgebraic maps instead of continuous maps. But the excision axiom is difficult. The classical way to prove it is to make a given singular cycle Z "small" with respect to a given covering of M by two open (semialgebraic) sets U_1, U_2, by iterated barycentric subdivision of Z. This means that every singular simplex occuring in the subdivided cycle has its image either in U_1 or U_2. But if the base field R is not archimedian then this procedure completely breaks down, since then usually a given bounded semialgebraic set cannot be covered by finitely many semialgebraic sets all whose diameters are smaller than a given $\epsilon > 0$.

In his thesis [D] Delfs found the following way out of this difficulty. He defined cohomology groups $H^q(M, G)$ as the sheaf cohomology groups of the constant sheaf G_M on M, and similarly relative cohomology groups $H^q(M, A; G)$ as the sheaf cohomology groups of a suitable sheaf $G_{M,A}$ on M. {Recall that M is equipped with a Grothendieck topology.} For these groups $H^q(M, A; G)$ Delfs succeeded in verifying the Eilenberg-Steenrod axioms. Then he knew that $H^q(M, G)$ is isomorphic to the combinatorial group $H^q(K, G)$. Thus $H^q(K, G)$ is independent of the choice of the triangulation, up to natural isomorphism. From this Delfs concluded that also $H_q(K, G)$ is independent of the choice of triangulation [D].

The verification of six of the seven Eilenberg-Steenrod axioms for the groups $H^q(M, A, G)$ is straightforward, but this time the homotopy axiom causes difficulties. Delfs surmounted these difficulties in [D] by a complicated geometric procedure.

Later Delfs found an easier way [D_1]. He realized that the homotopy axiom follows from the statement that, for any sheaf $\mathcal{F}$ on M, the adjunction homomorphism $\mathcal{F} \to \pi_* \pi^* \mathcal{F}$, with π the projection from $M \times [0,1]$ to M, is an isomorphism and $R^q \pi_*(\pi^* \mathcal{F}) = 0$ for $q \geq 1$. [D_1, Prop. 4.2 and 4.4]. This then could be deduced via a stalk by stalk argument from the fact that $H^q([0,1], G) = 0$ for $q \geq 1$ and any abelian group G, which in turn can be verified in an easy geometric way. The crucial point is that one needs the fact $H^q([0,1], G) = 0$ not just over R but over the residue class fields $k(x)$ of all points $x \in \tilde{M}$. Roughly one can summarize that Delfs reduced the verification of the homotopy axiom to an easy special case using sheaf theory, at the expense of enlarging the real closed base field in many ways.

§5 Locally semialgebraic spaces

Delfs and I had already introduced *"semialgebraic spaces"* over a real closed field R before 1981 by gluing together finitely many affine semialgebraic spaces over R along open subspaces [DK, §7]. What then was still missing was a handy criterion for a semialgebraic space $M = (M, \mathcal{O}_M)$ to be again affine. Such a criterion would allow the building of semialgebraic spaces M from semialgebraic sets in an "abstract" manner, i.e. without explicitly looking at polynomials, such that M eventually turns out to be an affine space, in other words, a semialgebraic set.

In 1982 R. Robson proved his imbedding theorem [Ro] which gives such a criterion. The theorem says that a semialgebraic space M over R is affine iff M is *regular*, i.e. a point x and a closed semialgebraic subset A of M with $x \notin A$ can be separated by open semialgebraic neighbourhoods. {A subset A of M is called closed semialgebraic if the complement $M - A$ is an open semialgebraic, i.e., an admissible open subset of M.}

Robson's theorem really paved the way for the trend of employing new spaces in the geometric theory. Before I go into details about this I should say some words about covering maps.

Having semialgebraic paths at hand we may define the fundamental group $\pi_1(M, x_0)$ for M a semialgebraic space over R and $x_0 \in M$, as in the classical theory, by considering homotopy classes of semialgebraic loops with base point x_0. Of course, homotopies also have to be defined in the semialgebraic sense, starting from the unit interval [0,1] in R, cf. §10 below. It turns out that for affine M the group $\pi_1(M, x_0)$ is very respectable. It is finitely presented and coincides with the topological fundamental group in the case $R = \mathbb{R}$. {These are consequences of the two comparison theorems on homotopy sets [DK_2, III §3 and §5], to be discussed in §10 below.}

Assume since now that M is affine and path connected. The question arises whether the subgroups of $\pi_1(M, x_0)$ classify "semialgebraic covering spaces" of M, as one might expect from classical topology.

It seems clear what a semialgebraic covering map $\pi: N \to M$ has to be: N should be a semialgebraic space and π a semialgebraic map. Further there should exist an admissible open covering $(U_i | i \in I)$ of M such that π is trivial over each U_i with discrete fibers, i.e. $\pi^{-1}(U_i) \cong U_i \times F_i$ over U_i for a discrete semialgebraic space F_i. But what does it mean for a semialgebraic space F to be discrete? Reasonable answers, one can think of, are: $\dim F = 0$; the path components of F are one-point sets; the one-point sets in M are open in F; the one-point sets in M form an admissible open covering of M. – All of these properties mean the same thing, namely that the space F consists of finitely many points. We conclude that every semialgebraic covering map $\pi: N \to M$ has finite degree.

Working with path lifting techniques one verifies that the semialgebraic coverings $\pi: N \to M$ of M are indeed classified by the conjugacy classes of subgroups of $\pi_1(M, x_0)$ of finite index [K_3]. Using Robson's embedding theorem one also sees that N is again affine.

Having verified this in 1982 [DK_3], Delfs and I realized that the category of semialgebraic spaces is not broad enough. There should exist some sort of covering space N of M corresponding to any given subgroup H of $\pi_1(M, x_0)$, in particular a "universal covering", corresponding to $H = \{1\}$. This led us to introduce locally semialgebraic spaces. A *locally semialgebraic space* M *over* R is obtained by gluing together (maybe infinitely many) affine semialgebraic spaces over R along open semialgebraic subspaces. Of course, the gluing is meant in the sense of ringed spaces with Grothendieck topologies, cf. [DK_2, I §1].

The nice locally semialgebraic spaces are those which are *regular* (defined in the same way as above) and *paracompact*, as defined in [DK_2, I §4]. The category $LSA(R)$ of regular paracompact locally semialgebraic spaces over R contains the category of affine semialgebraic spaces over R as a full subcategory. In $LSA(R)$ we have a fully satisfactory theory of covering spaces. In particular every space $M \in LSA(R)$ has a universal covering (cf. [DK_3, §5]; a full treatment of this topic still awaits publication [K_3]).

In $LSA(R)$ there exist fibre products. There is also a good notion of subspaces. Namely, if M is a locally semialgebraic space and $(M_i | i \in I)$ is an admissible open covering of M, such that every M_i is an affine semialgebraic space, then a subset A of M is called a *subspace* if $A \cap M_i$ is semialgebraic in M_i for every $i \in I$. Indeed, collecting the affine semialgebraic spaces $A \cap M_i$ we obtain on A the structure of a locally semialgebraic space over R, which is independent of the choice of the covering $(M_i | i \in I)$. This space A is regular and paracompact if M has these properties [DK_2, I, §3 and §4].

Up to now $LSA(R)$ has proved to be the appropriate basic category for all geometric studies over R, as long as one does not pass to abstract spaces. In particular the triangulation theorem for semialgebraic sets (cf. §3 above) extends to a triangulation theorem of equal strength for these spaces (simultaneous triangulation of M and a locally finite family of subspaces of M, cf. [DK_2, Chap.II]). Also the homology theory of Delfs discussed above extends to these spaces [DK_2, Chap.III]. And we have a fairly good homotopy theory in $LSA(R)$ at hand, to be discussed below.

§6 Abstract semialgebraic functions and real closed spaces

We come back to the relationships between a semialgebraic set M over R and its abstraction $\tilde{M}$. Recall from §4 that the sheaves on the affine semialgebraic space M correspond uniquely with the sheaves on $\tilde{M}$. In particular we have a sheaf of rings

$\tilde{\mathcal{O}}_M$ on $\tilde{M}$ which corresponds to the sheaf $\mathcal{O}_M$ of semialgebraic functions on M. The question arises whether $\tilde{\mathcal{O}}_M$ generalizes in a natural way to a sheaf of rings $\mathcal{O}_X$ on any constructible subset X of any real spectrum Sper A, which then can be regarded as a sheaf of "abstract" semialgebraic functions on X.

This is indeed the case. Around 1983 G. Brumfiel [B4] and N. Schwartz [S] gave two solutions of this problem. A (slightly "corrected", cf. [D, 1.7], [S, Example 58]) version of Brumfiel's definition runs as follows. Let p: Sper $A[T] \to$ Sper A be the natural map from the real spectrum of the polynomial ring $A[T]$ in one variable over A to Sper A, induced by the inclusion $A \hookrightarrow A[T]$. For any quasicompact open subset U of the space X the elements of $\mathcal{O}_X(U)$ are the continuous sections s of $p|p^{-1}(U): p^{-1}(U) \to U$ such that $s(U)$ is a closed constructible subset of $p^{-1}(U)$.

What does this mean? For any $x \in A$ we may identify $p^{-1}(x)$ with the real spectrum Sper $k(x)[T]$, where $k(x)$ denotes the residue class field of Sper A at x, a real closed field. This real spectrum is the abstraction of the real affine line over $k(x)$. Thus $k(x)$ injects into $p^{-1}(x)$ as a dense subset (cf. §2). For a section s as above, $s(x)$ lies in this subset and hence corresponds to an element $f(x)$ of $k(x)$, which should be regarded as the value of the abstract semialgebraic function f given by s. The section s is completely determined by the values $f(x)$ and should be regarded as the graph of f.

N. Schwartz defined an abstract semialgebraic function f on U directly as a family $(f(x)|x \in U) \in \prod_{x \in U} k(x)$ with compatibility relations between the values $f(x)$ coming from canonical valuations $\lambda_{x,y}: k(x) \to \kappa(x,y) \cup \infty$. For any pair (x,y) with $x \in U$ and y a specialization of x in U, $\kappa(x,y)$ is an overfield of $k(y)$, and $\lambda_{x,y}$ has to map $f(x)$ to $f(y) \in k(y) \subset \kappa(x,y)$. The definition of Schwartz has the advantage that here it is immediately clear that $\mathcal{O}_X(U)$ is a ring, while in Brumfiel's definition one has to work for this.

Then Delfs proved that the definitions of Brumfiel and Schwartz give the same sheaf $\mathcal{O}_X$ [D1, §1]. The stalks of $\mathcal{O}_X$ are local rings. In the geometric case, i.e., if $A = R[T_1, \ldots, T_n]$ and $X = \tilde{M}$ with $M \subset R^n$ a semialgebraic set, we indeed have $\mathcal{O}_X = \tilde{\mathcal{O}}_M$. From now on we call the ringed space $(\tilde{M}, \tilde{\mathcal{O}}_M)$ – instead of just the topological space $\tilde{M}$ – the abstraction of the affine semialgebraic space $(M, \mathcal{O}_M)$.

In the paper [B4] cited above Brumfiel introduced abstract semialgebraic functions as a tool to prove a vast generalization of Mahé's theorem on the separation of connected components by global quadratic forms. For every commutative ring A there is a natural homomorphism from the Witt ring $W(A)$ to the orthogonal K-group $KO_0(\text{Sper } A)$ of the real spectrum of A. Brumfiel proves that both the kernel and the cokernel of this homomorphism are 2-primary torsion groups. Thus, from our viewpoint, the localization $2^{-\infty}W(A)$ of $W(A)$ at the prime 2 is a purely topological object.

Brumfiel's paper is a bold step into the realm of abstract semialgebraic topology. A full understanding of it is a challenge even today, since some arguments are only sketched. For a discussion cf. [K, §6], and for a treatment in the geometric case cf. [BCR, 15.3].

N. Schwartz studied in [S] the spaces $(X, \mathcal{O}_X)$, with X a constructible subset of some real spectrum Sper A, for their own sake. The ring $\mathcal{O}(X)$ of global sections of $\mathcal{O}_X$ is a sort of "real closure" of the ring A. Schwartz describes how to obtain $\mathcal{O}(X)$ from the ring A in a constructive way. He further makes the important discovery that the natural map from X to Sper $\mathcal{O}(X)$ is an embedding which makes X a dense subspace of Sper $\mathcal{O}(X)$. Even more is true: the closed points and also the minimal points of Sper $\mathcal{O}(X)$ all lie in X. In the special case that X is convex in Sper A with respect to specialization, it turns out that the ringed spaces $(X, \mathcal{O}_X)$ and Spec $\mathcal{O}(X)$ are equal. In the geometric case $X = \tilde{M}$ this happens iff the semialgebraic set $M \subset R^n$ is locally closed in R^n.

Later Schwartz realized that all we have said above about $(X, \mathcal{O}_X)$ remains true if X is a *proconstructible* subset of Sper A, i.e., the intersection of an arbitrary family of constructible subsets of X [S_1], [S_2]. He called any ringed space isomorphic to such a space $(X, \mathcal{O}_X)$ an *affine real closed space*. He then introduced the category $\mathcal{R}$ of *real closed spaces* as a full subcategory of the category of all locally ringed spaces. The definition of a real closed space is simple: a ringed space $(X, \mathcal{O}_X)$ – always with X a genuine topological space, no Grothendieck topology – is called real closed if every point $x \in X$ has an open neighbourhood U such that $(U, \mathcal{O}_X|U)$ is an affine real closed space.

The books [S_1], [S_2] are both versions of Schwartz's Habilitationsschrift [S]. For the insiders they constitute a sort of bible of abstract semialgebraic topology - an incomplete bible, I should add, since more can and should be written down with the methods developed there. The shorter version [S_1] is easier to read, while [S_2] is closer to the original Habilitationsschrift and contains much more material.

In [S] Schwartz defined real closed spaces using as building blocks only constructible subsets of real spectra, instead of proconstructible ones. I will call these more special ringed spaces here *"abstract locally semialgebraic spaces"* and denote their category by $\mathcal{R}_0$. The analogy with locally semialgebraic spaces over a real closed field R is striking. But there is more than analogy. One can attach to any locally semialgebraic space $(M, \mathcal{O}_M)$ over R an abstract locally semialgebraic space $(\tilde{M}, \tilde{\mathcal{O}}_M)$ in a rather obvious way, starting from the abstractions of affine semialgebraic spaces discussed above. Schwartz proves that this gives an embedding of $LSA(R)$ into the category $\mathcal{R}_0$, making $LSA(R)$ a full subcategory of the category of abstract locally semialgebraic spaces over Sper R [S], [S_1], [S_2]. A good thing about $\mathcal{R}$ is that here more constructions - in particular more quotients - are possible than in $LSA(R)$.

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