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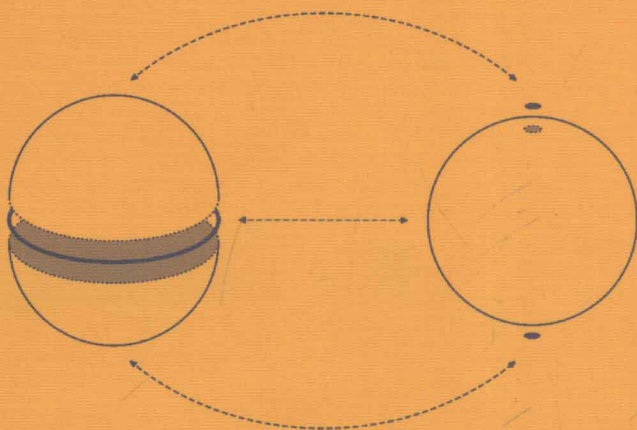
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Representation Theory and Complex Analysis

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Michael Cowling · Edward Frenkel
Masaki Kashiwara · Alain Valette
David A. Vogan, Jr. · Nolan R. Wallach

Representation Theory and Complex Analysis

Lectures given at the
C.I.M.E. Summer School
held in Venice, Italy
June 10–17, 2004

Editors: Enrico Casadio Tarabusi
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Preface

This volume collects the notes of six series of lectures given on the occasion of the CIME session *Representation Theory and Complex Analysis* held in Venice on July 10–17, 2004. We thank Venice International University for its hospitality at the beautiful venue of San Servolo island.

Our aim in organizing this meeting was to present the audience with a wide spectrum of recent results on the subject of the title, ranging from topics with an analytical flavor, to more algebraic or geometric oriented ones, without neglecting interactions with other domains, such as quantum computing.

Two papers present a general introduction to ideas and properties of analysis on semi-simple Lie groups and their unitary representations. MICHAEL COWLING presents a panorama of various interactions between representation theory and harmonic analysis on semisimple groups and symmetric spaces. Unexpected phenomena occur in this context, as for instance the Kunze–Stein property, that reveal a dramatic difference between these groups and group actions and the classical amenable group (an extension of abelian groups). Results of this type are strongly related to the vanishing of coefficients of unitary representations. Complementarily, ALAIN VALETTE recalls the notion of amenability and investigates its relations with vanishing of coefficients of unitary representations of semi-simple groups and with ergodic actions. He applies these ideas to show another surprising property of representations of semi-simple groups and their lattices, namely Margulis’ super-rigidity.

Three papers deal in full detail with the hard analysis of semisimple group representations. Ideally, this analysis could be split into representations of real groups or complex groups, or of algebraic groups over local fields. A deep account of the interaction between the real and complex world is given by MASAKI KASHIWARA, whose paper studies the relations between the representation theory of real semisimple Lie groups and the (microlocal) geometry of the flag manifolds associated with the corresponding complex algebraic groups. These results, a considerable part of which are joint work with W. Schmid, were announced some years ago, and are published here in

complete form for the first time. DAVID VOGAN expresses unitary representations of real or complex semi-simple groups using tools of complex analysis, such as minimal globalizations realized on Dolbeault cohomology with compact supports. EDWARD FRENKEL describes the geometric Langlands correspondence for complex algebraic curves, concentrating on the ramified case where a finite number of regular singular points is allowed.

Finally, NOLAN WALLACH illustrates briefly a surprising application that could be relevant for the future of computing and its complexity: his paper studies how representation theory is related to quantum computing, focusing attention in particular on the study of qubit entanglement.

We wish to thank all the lecturers for the excellence of their live and written contributions, as well as the many participants from all age ranges and parts of the world, who created a very pleasant working atmosphere.

Roma and Venezia, November 2006

Enrico Casadio Tarabusi
Andrea D'Agnolo
Massimo Picardello

Contents

Applications of Representation Theory to Harmonic Analysis of Lie Groups (and Vice Versa)

<i>Michael Cowling</i>	1
1 Basic Facts of Harmonic Analysis on Semisimple Groups and Symmetric Spaces	2
1.1 Structure of Semisimple Lie Algebras	2
1.2 Decompositions of Semisimple Lie Groups	4
1.3 Parabolic Subgroups	5
1.4 Spaces of Homogeneous Functions on G	6
1.5 The Plancherel Formula	8
2 The Equations of Mathematical Physics on Symmetric Spaces	10
2.1 Spherical Analysis on Symmetric Spaces	10
2.2 Harmonic Analysis on Semisimple Groups and Symmetric Spaces	12
2.3 Regularity of the Laplace–Beltrami Operator	16
2.4 Approaches to the Heat Equation	18
2.5 Estimates for the Heat and Laplace Equations	18
2.6 Approaches to the Wave and Schrödinger Equations	20
2.7 Further Results	21
3 The Vanishing of Matrix Coefficients	22
3.1 Some Examples in Representation Theory	22
3.2 Matrix Coefficients of Representations of Semisimple Groups	24
3.3 The Kunze–Stein Phenomenon	27
3.4 Property T	28
3.5 The Generalised Ramanujan–Selberg Property	29
4 More General Semisimple Groups	31
4.1 Graph Theory and its Riemannian Connection	31
4.2 Cayley Graphs	32
4.3 An Example Involving Cayley Graphs	33
4.4 The Field of p -adic Numbers	34
4.5 Lattices in Vector Spaces over Local Fields	35
4.6 Adèles	36

4.7 Further Results	37
5 Carnot–Carathéodory Geometry and Group Representations	38
5.1 A Decomposition for Real Rank One Groups	38
5.2 The Conformal Group of the Sphere in \mathbb{R}^n	38
5.3 The Groups $SU(1, n + 1)$ and $Sp(1, n + 1)$	41
References	46

Ramifications of the Geometric Langlands Program

<i>Edward Frenkel</i>	51
Introduction	51
1 The Unramified Global Langlands Correspondence	56
2 Classical Local Langlands Correspondence	61
2.1 Langlands Parameters	61
2.2 The Local Langlands Correspondence for GL_n	62
2.3 Generalization to Other Reductive Groups	63
3 Geometric Local Langlands Correspondence over \mathbb{C}	64
3.1 Geometric Langlands Parameters	64
3.2 Representations of the Loop Group	65
3.3 From Functions to Sheaves	66
3.4 A Toy Model	68
3.5 Back to Loop Groups	70
4 Center and Opers	71
4.1 Center of an Abelian Category	71
4.2 Opers	73
4.3 Canonical Representatives	75
4.4 Description of the Center	76
5 Opers vs. Local Systems	77
6 Harish–Chandra Categories	81
6.1 Spaces of K -Invariant Vectors	81
6.2 Equivariant Modules	82
6.3 Categorical Hecke Algebras	83
7 Local Langlands Correspondence: Unramified Case	85
7.1 Unramified Representations of $G(F)$	85
7.2 Unramified Categories $\widehat{\mathfrak{g}}_{\kappa_c}$ -Modules	87
7.3 Categories of $G[[t]]$ -Equivariant Modules	88
7.4 The Action of the Spherical Hecke Algebra	90
7.5 Categories of Representations and \mathcal{D} -Modules	92
7.6 Equivalences Between Categories of Modules	96
7.7 Generalization to other Dominant Integral Weights	98
8 Local Langlands Correspondence: Tamely Ramified Case	99
8.1 Tamely Ramified Representations	99
8.2 Categories Admitting $(\widehat{\mathfrak{g}}_{\kappa_c}, I)$ Harish-Chandra Modules	103
8.3 Conjectural Description of the Categories of $(\widehat{\mathfrak{g}}_{\kappa_c}, I)$ Harish-Chandra Modules	105
8.4 Connection between the Classical and the Geometric Settings ..	109

8.5	Evidence for the Conjecture	115
9	Ramified Global Langlands Correspondence	117
9.1	The Classical Setting	117
9.2	The Unramified Case, Revisited	120
9.3	Classical Langlands Correspondence with Ramification	122
9.4	Geometric Langlands Correspondence in the Tamely Ramified Case	122
9.5	Connections with Regular Singularities	126
9.6	Irregular Connections	130
	References	132

Equivariant Derived Category and Representation of Real Semisimple Lie Groups

	<i>Masaki Kashiwara</i>	137
1	Introduction	137
1.1	Harish-Chandra Correspondence	138
1.2	Beilinson-Bernstein Correspondence	140
1.3	Riemann-Hilbert Correspondence	141
1.4	Matsuki Correspondence	142
1.5	Construction of Representations of $G_{\mathbb{R}}$	143
1.6	Integral Transforms	146
1.7	Commutativity of Fig. 1	147
1.8	Example	148
1.9	Organization of the Note	151
2	Derived Categories of Quasi-abelian Categories	152
2.1	Quasi-abelian Categories	152
2.2	Derived Categories	154
2.3	t -Structure	156
3	Quasi-equivariant D -Modules	158
3.1	Definition	158
3.2	Derived Categories	162
3.3	Sumihiro's Result	163
3.4	Pull-back Functors	167
3.5	Push-forward Functors	168
3.6	External and Internal Tensor Products	170
3.7	Semi-outer Hom	171
3.8	Relations of Push-forward and Pull-back Functors	172
3.9	Flag Manifold Case	175
4	Equivariant Derived Category	176
4.1	Introduction	176
4.2	Sheaf Case	176
4.3	Induction Functor	179
4.4	Constructible Sheaves	179
4.5	D -module Case	180
4.6	Equivariant Riemann-Hilbert Correspondence	181

5	Holomorphic Solution Spaces	182
5.1	Introduction	182
5.2	Countable Sheaves	183
5.3	C^∞ -Solutions	185
5.4	Definition of $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}$	186
5.5	DFN Version	189
5.6	Functorial Properties of $\mathbf{R}\mathrm{Hom}^{\mathrm{top}}$	190
5.7	Relation with the de Rham Functor	192
6	Whitney Functor	194
6.1	Whitney Functor	194
6.2	The Functor $\mathbf{R}\mathrm{Hom}_{\mathcal{D}_X}^{\mathrm{top}}(\bullet, \bullet \otimes^{\mathrm{w}} \mathcal{O}_{X^{\mathrm{an}}})$	195
6.3	Elliptic Case	196
7	Twisted Sheaves	197
7.1	Twisting Data	197
7.2	Twisted Sheaf	198
7.3	Morphism of Twisting Data	199
7.4	Tensor Product	200
7.5	Inverse and Direct Images	200
7.6	Twisted Modules	201
7.7	Equivariant Twisting Data	201
7.8	Character Local System	202
7.9	Twisted Equivariance	202
7.10	Twisting Data Associated with Principal Bundles	203
7.11	Twisting (D -module Case)	204
7.12	Ring of Twisted Differential Operators	205
7.13	Equivariance of Twisted Sheaves and Twisted D -modules	207
7.14	Riemann-Hilbert Correspondence	207
8	Integral Transforms	208
8.1	Convolutions	208
8.2	Integral Transform Formula	209
9	Application to the Representation Theory	210
9.1	Notations	210
9.2	Beilinson-Bernstein Correspondence	212
9.3	Quasi-equivariant D -modules on the Symmetric Space	214
9.4	Matsuki Correspondence	216
9.5	Construction of Representations	217
9.6	Integral Transformation Formula	219
10	Vanishing Theorems	221
10.1	Preliminary	221
10.2	Calculation (I)	222
10.3	Calculation (II)	224
10.4	Vanishing Theorem	226
	References	229
	List of Notations	231
	Index	233

Amenability and Margulis Super-Rigidity

<i>Alain Valette</i>	235
1 Introduction	235
2 Amenability for Locally Compact Groups	236
2.1 Definition, Examples, and First Characterizations	236
2.2 Stability Properties	239
2.3 Lattices in Locally Compact Groups	240
2.4 Reiter's Property (P_1)	241
2.5 Reiter's Property (P_2)	242
2.6 Amenability in Riemannian Geometry	244
3 Measurable Ergodic Theory	244
3.1 Definitions and Examples	244
3.2 Moore's Ergodicity Theorem	247
3.3 The Howe-Moore Vanishing Theorem	249
4 Margulis' Super-rigidity Theorem	252
4.1 Statement	252
4.2 Mostow Rigidity	252
4.3 Ideas to Prove Super-rigidity, $k = \mathbf{R}$	253
4.4 Proof of Furstenberg's Proposition 4.1 - Use of Amenability	255
4.5 Margulis' Arithmeticity Theorem	256
References	257

Unitary Representations and Complex Analysis

<i>David A. Vogan, Jr</i>	259
1 Introduction	259
2 Compact Groups and the Borel-Weil Theorem	264
3 Examples for $SL(2, \mathbb{R})$	272
4 Harish-Chandra Modules and Globalization	274
5 Real Parabolic Induction and the Globalization Functors	284
6 Examples of Complex Homogeneous Spaces	294
7 Dolbeault Cohomology and Maximal Globalizations	302
8 Compact Supports and Minimal Globalizations	318
9 Invariant Bilinear Forms and Maps between Representations	327
10 Open Questions	341
References	343

Quantum Computing and Entanglement for Mathematicians

<i>Nolan R. Wallach</i>	345
1 The Basics	346
1.1 Basic Quantum Mechanics	346
1.2 Bits	348
1.3 Qubits	349
References	350
2 Quantum Algorithms	351
2.1 Quantum Parallelism	351

2.2 The Tensor Product Structure of n -qubit Space 352

2.3 Grover’s Algorithm 353

2.4 The Quantum Fourier Transform 354

References 355

3 Factorization and Error Correction 355

3.1 The Complexity of the Quantum Fourier Transform 356

3.2 Reduction of Factorization to Period Search 359

3.3 Error Correction 360

References 362

4 Entanglement 362

4.1 Measures of Entanglement 363

4.2 Three Qubits 365

4.3 Measures of Entanglement for Two and Three Qubits 367

References 368

5 Four and More Qubits 369

5.1 Four Qubits 369

5.2 Some Hilbert Series of Measures of Entanglement 374

5.3 A Measure of Entanglement for n Qubits 374

References 376

Applications of Representation Theory to Harmonic Analysis of Lie Groups (and Vice Versa)

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These notes began as lectures that I intended to deliver in Edinburgh in April, 1999. Unfortunately I was not able to leave Australia at the time. Since then there has been progress on many of the topics, some of which is reported here, and I have added another lecture, on uniformly bounded representations, so that these notes are expanded on the original version in several ways.

I have tried to make these notes an understandable introduction to the subject for mathematicians with little experience of analysis on Lie groups or Lie theory. I aimed to present a wide panorama of different aspects of harmonic analysis on semisimple groups and symmetric spaces, and to try to illuminate some of the links between these aspects; I may well not have succeeded in this aim. Many readers will find much of what is written here to be elementary, and others may well disagree with my perspective. I apologise in advance to both the neophytes for whom my outline is too sketchy and to the experts for whom these notes are worthless.

I had hoped to produce an extensive bibliography, but I have not found the time to do so. Consequently I must bear the responsibility for the many omissions of important references in the subject.

Whoever wishes to delve into this subject more deeply will need a more complete introduction. There are many possibilities; the books of S. Helgason [59, 60, 62] and of A.W. Knap [71] come to mind immediately as essential reading.

1 Basic Facts of Harmonic Analysis on Semisimple Groups and Symmetric Spaces

I will deal with noncompact classical algebraic semisimple Lie groups, such as $SO(p, q)$, $SU(p, q)$, $Sp(p, q)$, $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, and $SL(n, \mathbb{H})$. The definitions of these may be found in [59, pp. 444–447] or [71, pp. 3–6].

All noncompact algebraic semisimple Lie groups have various standard subgroups and decompositions. I begin by describing these, then describe families of unitary representations parametrised by representations of some of these subgroups. Finally, I discuss the Plancherel formula. The fact that most of the important representations are parametrised by representations of subgroups allows arguments involving induction on the rank of the group.

1.1 Structure of Semisimple Lie Algebras

First, fix a *Cartan involution* θ of the Lie algebra \mathfrak{g} of the group G , and write \mathfrak{k} and \mathfrak{p} for the $+1$ and -1 eigenspaces of θ . Then \mathfrak{k} is a maximal compact subalgebra of \mathfrak{g} , and \mathfrak{p} is a subspace; $[X, Y] \in \mathfrak{k}$ for all $X, Y \in \mathfrak{p}$. Since θ is an involution, we have the Cartan decomposition of the Lie algebra:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

In this and future formulae about the Lie algebra, \oplus means “vector space direct sum”. All Cartan involutions are conjugate in the group of Lie algebra automorphisms of \mathfrak{g} , which is a finite extension of the group generated by $\{\exp \text{ad } X : X \in \mathfrak{g}\}$. The Cartan involution θ extends to an automorphism Θ of the group G , whose fixed point set is a maximal compact subgroup K of G .

Next choose a maximal subalgebra of \mathfrak{p} ; this is abelian, and is denoted by \mathfrak{a} . All such subalgebras are conjugate under K . Let $\text{ad}(X)$ denote the derivation $Y \mapsto [X, Y]$ of \mathfrak{g} . Then the Killing form B , given by

$$B(X, Y) = \text{tr}(\text{ad}(X) \text{ad}(Y)) \quad \forall X, Y \in \mathfrak{g},$$

gives rise to an inner product on \mathfrak{a} :

$$(X, Y)_B = -B(X, \theta Y) \quad \forall X, Y \in \mathfrak{g},$$

which gives rise to a dual inner product, denoted in the same way, on \mathfrak{a}^* , which in turn extends to a bilinear form on $\mathfrak{a}_{\mathbb{C}}$, also denoted in the same way.

The third step in the description and construction of the various special subalgebras of \mathfrak{g} and corresponding subgroups of G is to decompose \mathfrak{g} as a direct sum of *root spaces* \mathfrak{g}_{α} and a subalgebra \mathfrak{g}_0 . Simultaneously diagonalise the operators $\text{ad}(H)$, for H in \mathfrak{a} . For α in the real dual \mathfrak{a}^* of \mathfrak{a} (that is, $\mathfrak{a}^* = \text{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{R})$), define

$$\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \quad \forall H \in \mathfrak{a}\}.$$

For most α in \mathfrak{a}^* , $\mathfrak{g}_\alpha = \{0\}$, but when $\alpha = 0$, then $\mathfrak{a} \subseteq \mathfrak{g}_0$, so $\mathfrak{g}_0 \neq \{0\}$. There are finitely many nonzero α in \mathfrak{a}^* for which $\mathfrak{g}_\alpha \neq \{0\}$; these α are called the *real roots* of $(\mathfrak{g}, \mathfrak{a})$, and the set thereof is written Σ . This set is a *root system*, a highly symmetric subset of \mathfrak{a}^* . Because \mathfrak{g}_0 is θ -stable,

$$\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus (\mathfrak{g}_0 \cap \mathfrak{p}) = \mathfrak{m} \oplus \mathfrak{a},$$

say, where \mathfrak{m} is the subalgebra of \mathfrak{k} of elements which commute with \mathfrak{a} . Using the fact that $\text{ad}(H)$ is a derivation of \mathfrak{g} for each H in \mathfrak{a} , it is easy to check that

$$(1.1) \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}.$$

In particular, \mathfrak{g}_0 is a subalgebra, and \mathfrak{g}_α and \mathfrak{g}_β commute when $\mathfrak{g}_{\alpha+\beta} = \{0\}$. Clearly

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma}^{\oplus} \mathfrak{g}_\alpha.$$

Now order the roots. The hyperplanes $\{H \in \mathfrak{a} : \alpha(H) = 0\}$, for α in Σ , divide \mathfrak{a} into finitely many connected open cones, known as Weyl chambers. Pick one of these (arbitrarily) and fix it; it is called the positive Weyl chamber, and written \mathfrak{a}^+ . A root α is now said to be *positive* or *negative* as $\alpha(H) > 0$ or $\alpha(H) < 0$ for all H in \mathfrak{a}^+ . Write Σ^+ for the set of positive roots; then $\Sigma = \Sigma^+ \cup -(\Sigma^+)$. For some roots α and real numbers t , $t\alpha$ is also a root; the possibilities are that $t = \pm 1$ (this always happens), $t = \pm 1/2$ or $t = \pm 2$ (these last four possibilities may or may not occur). If $(1/2)\alpha$ is not a root, then α is said to be *indivisible*; denote by Σ_0^+ the set of indivisible positive roots.

We can now define some more important subalgebras: let

$$\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \quad \text{and} \quad \bar{\mathfrak{n}} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha};$$

it is easy to deduce from formula (1.1) that \mathfrak{n} and $\bar{\mathfrak{n}}$ are *nilpotent* subalgebras of \mathfrak{g} . Define ρ by the formula

$$\rho(H) = \frac{1}{2} \text{tr}(\text{ad}(H)|_{\mathfrak{n}}) \quad \forall H \in \mathfrak{a};$$

then $\rho = (1/2) \sum_{\alpha \in \Sigma^+} \dim(\mathfrak{g}_\alpha) \alpha$. We now have the ingredients for two more decompositions of \mathfrak{g} : the Iwasawa decomposition and the Bruhat decomposition, written

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \quad \text{and} \quad \mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

The proof of the second (Bruhat) decomposition is immediate. For the first (Iwasawa) decomposition, note that if $X \in \mathfrak{g}_\alpha$, then $\theta X \in \mathfrak{g}_{-\alpha}$, so that, if $X \in \bar{\mathfrak{n}}$, then

$$X = (X + \theta X) - \theta X \in \mathfrak{k} \oplus \mathfrak{n}.$$

1.2 Decompositions of Semisimple Lie Groups

At the group level, there are similar decompositions (usually known as factorisations in undergraduate linear algebra courses). Let K , A , N and \overline{N} denote the connected subgroups of G with Lie algebras \mathfrak{k} , \mathfrak{a} , \mathfrak{n} and $\overline{\mathfrak{n}}$, and let A^+ and \overline{A}^+ be the subsemigroup $\exp(\mathfrak{a}^+)$ of A and its closure. Let M and M' be the centraliser and normaliser of \mathfrak{a} in K . Then both M and M' have \mathfrak{m} as their Lie algebra. The group M' is never connected, while M is connected in some examples and is not in others. However, M'/M is always finite. In fact, the adjoint action Ad of M' on \mathfrak{a} induces an isomorphism of M'/M with a finite group of orthogonal transformations of \mathfrak{a} , generated by reflections. This is the Weyl group, $W(\mathfrak{g}, \mathfrak{a})$. It acts simply transitively on the space of Weyl chambers, that is, every Weyl chamber is the image of \mathfrak{a}^+ under a unique element of the Weyl group. By duality, this group also acts on \mathfrak{a}^* , and permutes the roots amongst themselves. Take a representative s_w in M' of each w in the Weyl group.

At the group level, there are three important decompositions:

$$(1.2) \quad G = K\overline{A}^+K,$$

$$(1.3) \quad G = KAN,$$

$$(1.4) \quad G = \bigsqcup_{w \in W} MANs_wMAN$$

(this last formula involves a disjoint union). The Cartan decomposition (1.2) arises from the “polar decomposition” $G = K\exp(\mathfrak{p})$, in which the map $(k, X) \mapsto k\exp(X)$ is a diffeomorphism from $K \times \mathfrak{p}$ onto G ; every element of \mathfrak{p} is conjugate to an element of $\overline{\mathfrak{a}}^+$ by an element of K . In the Iwasawa decomposition (1.3), the map $(k, a, n) \mapsto kan$ is a diffeomorphism from $K \times A \times N$ onto G . In the Bruhat decomposition (1.4), each of the sets $MANs_wMAN$ is a submanifold of G , and the $|W|$ submanifolds are pairwise disjoint. There is a unique *longest element* \overline{w} of the Weyl group, which maps \mathfrak{a}^+ to $-\mathfrak{a}^+$; the corresponding submanifold of G is open and its complement is a union of submanifolds of lower dimension. More precisely,

$$\begin{aligned} G &= \bigsqcup_{w \in W} s_{\overline{w}}MANs_wMAN \\ &= \bigsqcup_{w \in W} s_{\overline{w}}s_ws_w^{-1}NAMs_wMAN \\ &= \bigsqcup_{w \in W} s_{\overline{w}w}\overline{N}_wMAN, \end{aligned}$$

where $\overline{N}_w = s_w^{-1}NS_w \cap \overline{N}$; each \overline{N}_w is a Lie subgroup of \overline{N} , of lower dimension unless $w = \overline{w}$, and the map $(\overline{n}, m, a, n) \mapsto \overline{n}man$ is a diffeomorphism from $\overline{N}_w \times M \times A \times N$ onto \overline{N}_wMAN .

For many purposes it is sufficient to think of the Bruhat decomposition in the following way: the map $(\bar{n}, m, a, n) \mapsto \bar{n}MAN$ of $\bar{N} \times M \times A \times N$ to G is a diffeomorphism of $\bar{N}MAN$ onto an open dense subset of G whose complement is a finite union of lower dimensional submanifolds. In particular, $\bar{N}MAN$ is of full measure in G/MAN , equipped with any of the natural measures. I will use the abusive notation $G \simeq \bar{N}MAN$ to indicate this sort of “quasi-decomposition”.

There are integral formulae associated with these group decompositions. In particular, we will use the formula

$$(1.5) \quad \int_G u(x) dx = C \int_K \int_{\bar{a}^+} \int_K u(k_1 \exp(H) k_2) \prod_{\alpha \in \Sigma} \sinh(\alpha(H))^{\dim(\mathfrak{g}_\alpha)} dk_1 dH dk_2,$$

which relates the Haar measure on G with the Haar measure dk on K and a weighted variant of Lebesgue measure dH on \mathfrak{a}^+ . For the formulae for the Iwasawa and Bruhat decompositions, see [60, Propositions I.5.1 and I.5.21].

1.3 Parabolic Subgroups

The subgroup MAN , often written P , is known as a *minimal parabolic subgroup*. Any subgroup P_1 of G containing MAN is known as a parabolic subgroup; such a group may be decomposed in the form

$$P_1 = M_1 A_1 N_1,$$

where $M_1 \supseteq M$, $A_1 \subseteq A$, and $N_1 \subseteq N$. The group M_1 is a semisimple subgroup of G , and has its own Iwasawa and Bruhat decompositions:

$$M_1 = K^1 A^1 N^1 \quad \text{and} \quad M_1 \simeq \bar{N}^1 M^1 A^1 N^1.$$

In these formulae, $K^1 \subseteq K$, $A^1 \subseteq A$, $N^1 \subseteq N$, $M^1 \supseteq M$, and $\bar{N}^1 \subseteq \bar{N}$; moreover, $\bar{N}^1 = \Theta N^1$. If \mathfrak{a}_1 , \mathfrak{a}^1 , \mathfrak{n}_1 and \mathfrak{n}^1 denote the subalgebras of \mathfrak{a} and \mathfrak{n} corresponding to A_1 , A^1 , N_1 and N^1 , then $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}^1$ and $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}^1$. To each parabolic subgroup P_1 , we associate ρ_1 on \mathfrak{a}_1 , defined similarly to ρ ; more precisely,

$$\rho_1(H) = \frac{1}{2} \operatorname{tr}(\operatorname{ad}(H)|_{\mathfrak{n}_1}) \quad \forall H \in \mathfrak{a}_1$$

The point of this is mainly that the set of all subgroups P_1 of G containing P is well understood: it is a finite lattice with a well determined structure.

We conclude our discussion of the structure of G with one more definition. A parabolic subgroup P_1 of G is called *cuspidal* if M_1 has a compact Cartan subgroup, that is, if there is a compact abelian subgroup of K_1 which cannot be extended to a larger abelian subgroup of M_1 . Since M is compact,

P is automatically cuspidal. It is a deep theorem of Harish-Chandra that the semisimple groups which have discrete series representations, that is, irreducible unitary representations which are subrepresentations of the regular representation, are precisely those with compact Cartan subgroups.

1.4 Spaces of Homogeneous Functions on G

For this section, fix a parabolic subgroup $M_1 A_1 N_1$ of G . Take an irreducible unitary representation μ of M_1 and λ in the complexification $\mathfrak{a}_{1\mathbb{C}}^*$ of \mathfrak{a}_1^* (that is, $\mathfrak{a}_{1\mathbb{C}}^* = \text{Hom}_{\mathbb{R}}(\mathfrak{a}_1, \mathbb{C})$). Let \mathcal{H}_μ denote the Hilbert space on which the representation μ acts. Consider the vector space $\mathcal{V}^{\mu, \lambda}$ of all smooth (infinitely differentiable) \mathcal{H}_μ -valued functions ξ on G with the property that

$$\xi(xman) = e^{(i\lambda - \rho_1)(\log a)} \mu(m)^{-1} \xi(x),$$

for all x in G , all m in M_1 , all a in A_1 and all n in N_1 . These functions may also be viewed as functions on G/N_1 , since $\xi(xn) = \xi(x)$ for all x in G and n in N_1 , or as sections of a vector bundle over G/P_1 . I shall take the naive viewpoint that they are functions on G , even though there are often good geometric reasons for using vector bundle terminology. Write $\pi^{\mu, \lambda}$ for the left translation representation on $\mathcal{V}^{\mu, \lambda}$:

$$[\pi^{\mu, \lambda}(y)\xi](x) = \xi(y^{-1}x) \quad \forall x, y \in G.$$

The inner product on \mathcal{H}_μ induces a pairing $\mathcal{V}^{\mu, \lambda'} \times \mathcal{V}^{\mu, \lambda} \rightarrow \mathbb{C}$: indeed,

$$\begin{aligned} \langle \xi(xman), \eta(xman) \rangle &= \langle e^{(i\lambda' - \rho_1)(\log a)} \mu(m)^{-1} \xi(x), e^{(i\lambda - \rho_1)(\log a)} \mu(m)^{-1} \eta(x) \rangle \\ &= e^{(i\lambda' - i\bar{\lambda} - 2\rho_1)(\log a)} \langle \xi(x), \eta(x) \rangle, \end{aligned}$$

so the complex-valued function $x \mapsto \langle \xi(x), \eta(x) \rangle$ indeed satisfies the covariance condition characterising $\mathcal{V}^{1, \lambda' - \bar{\lambda} + i\rho_1}$.

Lemma 1.1. *There is a G -invariant positive linear functional I_{P_1} on $\mathcal{V}^{1, i\rho_1}$, which is unique up to a constant. It may be defined as (a constant multiple of) the Haar measure on K ,*

$$\xi \mapsto \int_K \xi(k) dk,$$

or as (a constant multiple of) the Haar measure on \overline{N}_1 ,

$$\xi \mapsto \int_{\overline{N}_1} \xi(\bar{n}) d\bar{n}.$$