Lecture Notes in Mathematics

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Representation Theory and Complex Analysis

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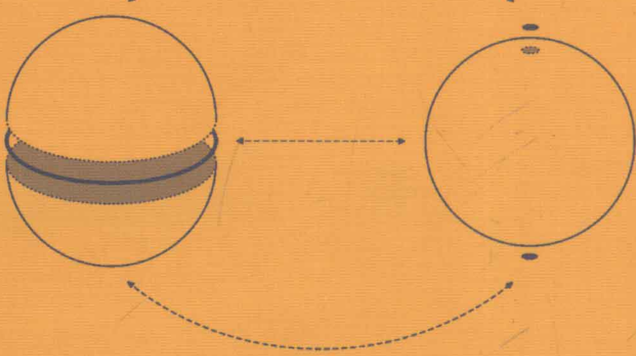

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Representation Theory and Complex Analysis

Lectures given at the
C.I.M.E. Summer School
held in Venice, Italy
June 10–17, 2004

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Preface

This volume collects the notes of six series of lectures given on the occasion of the CIME session *Representation Theory and Complex Analysis* held in Venice on July 10–17, 2004. We thank Venice International University for its hospitality at the beautiful venue of San Servolo island.

Our aim in organizing this meeting was to present the audience with a wide spectrum of recent results on the subject of the title, ranging from topics with an analytical flavor, to more algebraic or geometric oriented ones, without neglecting interactions with other domains, such as quantum computing.

Two papers present a general introduction to ideas and properties of analysis on semi-simple Lie groups and their unitary representations. MICHAEL COWLING presents a panorama of various interactions between representation theory and harmonic analysis on semisimple groups and symmetric spaces. Unexpected phenomena occur in this context, as for instance the Kunze–Stein property, that reveal a dramatic difference between these groups and group actions and the classical amenable group (an extension of abelian groups). Results of this type are strongly related to the vanishing of coefficients of unitary representations. Complementarily, ALAIN VALETTE recalls the notion of amenability and investigates its relations with vanishing of coefficients of unitary representations of semi-simple groups and with ergodic actions. He applies these ideas to show another surprising property of representations of semi-simple groups and their lattices, namely Margulis' super-rigidity.

Three papers deal in full detail with the hard analysis of semisimple group representations. Ideally, this analysis could be split into representations of real groups or complex groups, or of algebraic groups over local fields. A deep account of the interaction between the real and complex world is given by MASAKI KASHIWARA, whose paper studies the relations between the representation theory of real semisimple Lie groups and the (microlocal) geometry of the flag manifolds associated with the corresponding complex algebraic groups. These results, a considerable part of which are joint work with W. Schmid, were announced some years ago, and are published here in complete form for the first time. David Vogan expresses unitary representations of real or complex semi-simple groups using tools of complex analysis, such as minimal globalizations realized on Dolbeault cohomology with compact supports. Edward Frenkel describes the geometric Langlands correspondence for complex algebraic curves, concentrating on the ramified case where a finite number of regular singular points is allowed.

Finally, Nolan Wallach illustrates briefly a surprising application that could be relevant for the future of computing and its complexity: his paper studies how representation theory is related to quantum computing, focusing attention in particular on the study of qubit entanglement.

We wish to thank all the lecturers for the excellence of their live and written contributions, as well as the many participants from all age ranges and parts of the world, who created a very pleasant working atmosphere.

Roma and Venezia, November 2006

Enrico Casadio Tarabusi
Andrea D'Agnolo
Massimo Picardello

Contents

Applications of Representation Theory to Harmonic Analysis of Lie Groups (and Vice Versa)

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These notes began as lectures that I intended to deliver in Edinburgh in April, 1999. Unfortunately I was not able to leave Australia at the time. Since then there has been progress on many of the topics, some of which is reported here, and I have added another lecture, on uniformly bounded representations, so that these notes are expanded on the original version in several ways.

I have tried to make these notes an understandable introduction to the subject for mathematicians with little experience of analysis on Lie groups or Lie theory. I aimed to present a wide panorama of different aspects of harmonic analysis on semisimple groups and symmetric spaces, and to try to illuminate some of the links between these aspects; I may well not have succeeded in this aim. Many readers will find much of what is written here to be elementary, and others may well disagree with my perspective. I apologise in advance to both the neophytes for whom my outline is too sketchy and to the experts for whom these notes are worthless.

I had hoped to produce an extensive bibliography, but I have not found the time to do so. Consequently I must bear the responsibility for the many omissions of important references in the subject.

Whoever wishes to delve into this subject more deeply will need a more complete introduction. There are many possibilities; the books of S. Helgason [59, 60, 62] and of A.W. Knapp [71] come to mind immediately as essential reading.

1 Basic Facts of Harmonic Analysis on Semisimple Groups and Symmetric Spaces

I will deal with noncompact classical algebraic semisimple Lie groups, such as $\mathrm{SO}(p,q)$, $\mathrm{SU}(p,q)$, $\mathrm{Sp}(p,q)$, $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$, and $\mathrm{SL}(n,\mathbb{H})$. The definitions of these may be found in [59, pp. 444–447] or [71, pp. 3–6].

All noncompact algebraic semisimple Lie groups have various standard subgroups and decompositions. I begin by describing these, then describe families of unitary representations parametrised by representations of some of these subgroups. Finally, I discuss the Plancherel formula. The fact that most of the important representations are parametrised by representations of subgroups allows arguments involving induction on the rank of the group.

1.1 Structure of Semisimple Lie Algebras

First, fix a *Cartan involution* θ of the Lie algebra $\mathfrak{g}$ of the group G, and write $\mathfrak{k}$ and $\mathfrak{p}$ for the $+1$ and -1 eigenspaces of θ. Then $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$, and $\mathfrak{p}$ is a subspace; $[X,Y] \in \mathfrak{k}$ for all $X, Y \in \mathfrak{p}$. Since θ is an involution, we have the Cartan decomposition of the Lie algebra:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

In this and future formulae about the Lie algebra, $\oplus$ means "vector space direct sum". All Cartan involutions are conjugate in the group of Lie algebra automorphisms of $\mathfrak{g}$, which is a finite extension of the group generated by $\{\exp \operatorname{ad} X : X \in \mathfrak{g}\}$. The Cartan involution θ extends to an automorphism Θ of the group G, whose fixed point set is a maximal compact subgroup K of G.

Next choose a maximal subalgebra of $\mathfrak{p}$; this is abelian, and is denoted by $\mathfrak{a}$. All such subalgebras are conjugate under K. Let $\operatorname{ad}(X)$ denote the derivation $Y \mapsto [X,Y]$ of $\mathfrak{g}$. Then the Killing form B, given by

$$B(X,Y) = \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y)) \qquad \forall X, Y \in \mathfrak{g},$$

gives rise to an inner product on $\mathfrak{a}$:

$$(X,Y)_B = -B(X, \theta Y) \qquad \forall X, Y \in \mathfrak{g},$$

which gives rise to a dual inner product, denoted in the same way, on $\mathfrak{a}^*$, which in turn extends to a bilinear form on $\mathfrak{a}_{\mathbb{C}}$, also denoted in the same way.

The third step in the description and construction of the various special subalgebras of $\mathfrak{g}$ and corresponding subgroups of G is to decompose $\mathfrak{g}$ as a direct sum of *root spaces* $\mathfrak{g}_\alpha$ and a subalgebra $\mathfrak{g}_0$. Simultaneously diagonalise the operators $\operatorname{ad}(H)$, for H in $\mathfrak{a}$. For α in the real dual $\mathfrak{a}^*$ of $\mathfrak{a}$ (that is, $\mathfrak{a}^* = \operatorname{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{R})$), define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H,X] = \alpha(H)X \quad \forall H \in \mathfrak{a}\}.$$

For most α in $\mathfrak{a}^*$, $\mathfrak{g}_\alpha = \{0\}$, but when $\alpha = 0$, then $\mathfrak{a} \subseteq \mathfrak{g}_0$, so $\mathfrak{g}_0 \neq \{0\}$. There are finitely many nonzero α in $\mathfrak{a}^*$ for which $\mathfrak{g}_\alpha \neq \{0\}$; these α are called the *real roots* of $(\mathfrak{g}, \mathfrak{a})$, and the set thereof is written Σ. This set is a *root system*, a highly symmetric subset of $\mathfrak{a}^*$. Because $\mathfrak{g}_0$ is θ-stable,

$$\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus (\mathfrak{g}_0 \cap \mathfrak{p}) = \mathfrak{m} \oplus \mathfrak{a},$$

say, where $\mathfrak{m}$ is the subalgebra of $\mathfrak{k}$ of elements which commute with $\mathfrak{a}$. Using the fact that $\operatorname{ad}(H)$ is a derivation of $\mathfrak{g}$ for each H in $\mathfrak{a}$, it is easy to check that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}. \tag{1.1}$$

In particular, $\mathfrak{g}_0$ is a subalgebra, and $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ commute when $\mathfrak{g}_{\alpha+\beta} = \{0\}$. Clearly

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma}^{\oplus} \mathfrak{g}_\alpha.$$

Now order the roots. The hyperplanes $\{H \in \mathfrak{a} : \alpha(H) = 0\}$, for α in Σ, divide $\mathfrak{a}$ into finitely many connected open cones, known as Weyl chambers. Pick one of these (arbitrarily) and fix it; it is called the positive Weyl chamber, and written $\mathfrak{a}^+$. A root α is now said to be *positive* or *negative* as $\alpha(H) > 0$ or $\alpha(H) < 0$ for all H in $\mathfrak{a}^+$. Write Σ^+ for the set of positive roots; then $\Sigma = \Sigma^+ \cup -(\Sigma^+)$. For some roots α and real numbers t, $t\alpha$ is also a root; the possibilities are that $t = \pm 1$ (this always happens), $t = \pm 1/2$ or $t = \pm 2$ (these last four possibilities may or may not occur). If $(1/2)\alpha$ is not a root, then α is said to be *indivisible*; denote by Σ_0^+ the set of indivisible positive roots.

We can now define some more important subalgebras: let

$$\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \qquad \text{and} \qquad \bar{\mathfrak{n}} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}\,;$$

it is easy to deduce from formula (1.1) that $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are *nilpotent* subalgebras of $\mathfrak{g}$. Define ρ by the formula

$$\rho(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad}(H)|_{\mathfrak{n}}) \qquad \forall H \in \mathfrak{a};$$

then $\rho = (1/2)\sum_{\alpha\in\Sigma^+} \dim(\mathfrak{g}_\alpha)\,\alpha$. We now have the ingredients for two more decompositions of $\mathfrak{g}$: the Iwasawa decomposition and the Bruhat decomposition, written

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \qquad \text{and} \qquad \mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

The proof of the second (Bruhat) decomposition is immediate. For the first (Iwasawa) decomposition, note that if $X \in \mathfrak{g}_\alpha$, then $\theta X \in \mathfrak{g}_{-\alpha}$, so that, if $X \in \bar{\mathfrak{n}}$, then

$$X = (X + \theta X) - \theta X \in \mathfrak{k} \oplus \mathfrak{n}.$$

1.2 Decompositions of Semisimple Lie Groups

At the group level, there are similar decompositions (usually known as factorisations in undergraduate linear algebra courses). Let K, A, N and $\overline{N}$ denote the connected subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\overline{\mathfrak{n}}$, and let A^+ and $\overline{A}^+$ be the subsemigroup $\exp(\mathfrak{a}^+)$ of A and its closure. Let M and M' be the centraliser and normaliser of $\mathfrak{a}$ in K. Then both M and M' have $\mathfrak{m}$ as their Lie algebra. The group M' is never connected, while M is connected in some examples and is not in others. However, M'/M is always finite. In fact, the adjoint action Ad of M' on $\mathfrak{a}$ induces an isomorphism of M'/M with a finite group of orthogonal transformations of $\mathfrak{a}$, generated by reflections. This is the Weyl group, $W(\mathfrak{g}, \mathfrak{a})$. It acts simply transitively on the space of Weyl chambers, that is, every Weyl chamber is the image of $\mathfrak{a}^+$ under a unique element of the Weyl group. By duality, this group also acts on $\mathfrak{a}^*$, and permutes the roots amongst themselves. Take a representative s_w in M' of each w in the Weyl group.

At the group level, there are three important decompositions:

$$G = K\overline{A}^+K, \tag{1.2}$$

$$G = KAN, \tag{1.3}$$

$$G = \bigsqcup_{w\in W} MANs_wMAN \tag{1.4}$$

(this last formula involves a disjoint union). The Cartan decomposition (1.2) arises from the "polar decomposition" $G = K\exp(\mathfrak{p})$, in which the map $(k, X) \mapsto k\exp(X)$ is a diffeomorphism from $K \times \mathfrak{p}$ onto G; every element of $\mathfrak{p}$ is conjugate to an element of $\overline{\mathfrak{a}}^+$ by an element of K. In the Iwasawa decomposition (1.3), the map $(k, a, n) \mapsto kan$ is a diffeomorphism from $K\times A\times N$ onto G. In the Bruhat decomposition (1.4), each of the sets $MANs_wMAN$ is a submanifold of G, and the $|W|$ submanifolds are pairwise disjoint. There is a unique *longest element* $\overline{w}$ of the Weyl group, which maps $\mathfrak{a}^+$ to $-\mathfrak{a}^+$; the corresponding submanifold of G is open and its complement is a union of submanifolds of lower dimension. More precisely,

$$\begin{aligned} G &= \bigsqcup_{w\in W} s_{\overline{w}}MANs_wMAN \\ &= \bigsqcup_{w\in W} s_{\overline{w}}s_w s_w^{-1}NAMs_wMAN \\ &= \bigsqcup_{w\in W} s_{\overline{w}w}\overline{N}_wMAN, \end{aligned}$$

where $\overline{N}_w = s_w^{-1}NS_w \cap \overline{N}$; each $\overline{N}_w$ is a Lie subgroup of $\overline{N}$, of lower dimension unless $w = \overline{w}$, and the map $(\overline{n}, m, a, n) \mapsto \overline{n}man$ is a diffeomorphism from $\overline{N}_w \times M \times A \times N$ onto $\overline{N}_wMAN$.

For many purposes it is sufficient to think of the Bruhat decomposition in the following way: the map $(\overline{n}, m, a, n) \mapsto \overline{n}MAN$ of $\overline{N} \times M \times A \times N$ to G is a diffeomorphism of $\overline{N}MAN$ onto an open dense subset of G whose complement is a finite union of lower dimensional submanifolds. In particular, $\overline{N}MAN$ is of full measure in G/MAN, equipped with any of the natural measures. I will use the abusive notation $G \simeq \overline{N}MAN$ to indicate this sort of "quasi-decomposition".

There are integral formulae associated with these group decompositions. In particular, we will use the formula

$$\int_G u(x)\,dx = C \int_K \int_{\mathfrak{a}^+} \int_K u(k_1 \exp(H) k_2) \prod_{\alpha\in\Sigma} \sinh(\alpha(H))^{\dim(\mathfrak{g}_\alpha)}\,dk_1\,dH\,dk_2, \tag{1.5}$$

which relates the Haar measure on G with the Haar measure dk on K and a weighted variant of Lebesgue measure dH on $\mathfrak{a}^+$. For the formulae for the Iwasawa and Bruhat decompositions, see [60, Propositions I.5.1 and I.5.21].

1.3 Parabolic Subgroups

The subgroup MAN, often written P, is known as a *minimal parabolic subgroup*. Any subgroup P_1 of G containing MAN is known as a parabolic subgroup; such a group may be decomposed in the form

$$P_1 = M_1 A_1 N_1,$$

where $M_1 \supseteq M$, $A_1 \subseteq A$, and $N_1 \subseteq N$. The group M_1 is a semisimple subgroup of G, and has its own Iwasawa and Bruhat decompositions:

$$M_1 = K^1 A^1 N^1 \qquad \text{and} \qquad M_1 \simeq \overline{N}^1 M^1 A^1 N^1.$$

In these formulae, $K^1 \subseteq K$, $A^1 \subseteq A$, $N^1 \subseteq N$, $M^1 \supseteq M$, and $\overline{N}^1 \subseteq \overline{N}$; moreover, $\overline{N}^1 = \Theta N^1$. If $\mathfrak{a}_1$, $\mathfrak{a}^1$, $\mathfrak{n}_1$ and $\mathfrak{n}^1$ denote the subalgebras of $\mathfrak{a}$ and $\mathfrak{n}$ corresponding to A_1, A^1, N_1 and N^1, then $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}^1$ and $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}^1$. To each parabolic subgroup P_1, we associate ρ_1 on $\mathfrak{a}_1$, defined similarly to ρ; more precisely,

$$\rho_1(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad}(H)|_{\mathfrak{n}_1}) \qquad \forall H \in \mathfrak{a}_1$$

The point of this is mainly that the set of all subgroups P_1 of G containing P is well understood: it is a finite lattice with a well determined structure.

We conclude our discussion of the structure of G with one more definition. A parabolic subgroup P_1 of G is called *cuspidal* if M_1 has a compact Cartan subgroup, that is, if there is a compact abelian subgroup of K_1 which cannot be extended to a larger abelian subgroup of M_1. Since M is compact,

P is automatically cuspidal. It is a deep theorem of Harish-Chandra that the semisimple groups which have discrete series representations, that is, irreducible unitary representations which are subrepresentations of the regular representation, are precisely those with compact Cartan subgroups.

1.4 Spaces of Homogeneous Functions on G

For this section, fix a parabolic subgroup $M_1A_1N_1$ of G. Take an irreducible unitary representation μ of M_1 and λ in the complexification $\mathfrak{a}_{1\mathbb{C}}^*$ of $\mathfrak{a}_1^*$ (that is, $\mathfrak{a}_{1\mathbb{C}}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_1, \mathbb{C})$). Let $\mathcal{H}_\mu$ denote the Hilbert space on which the representation μ acts. Consider the vector space $\mathcal{V}^{\mu,\lambda}$ of all smooth (infinitely differentiable) $\mathcal{H}_\mu$-valued functions ξ on G with the property that

$$\xi(xman) = e^{(i\lambda-\rho_1)(\log a)}\mu(m)^{-1}\xi(x),$$

for all x in G, all m in M_1, all a in A_1 and all n in N_1. These functions may also be viewed as functions on G/N_1, since $\xi(xn) = \xi(x)$ for all x in G and n in N_1, or as sections of a vector bundle over G/P_1. I shall take the naive viewpoint that they are functions on G, even though there are often good geometric reasons for using vector bundle terminology. Write $\pi^{\mu,\lambda}$ for the left translation representation on $\mathcal{V}^{\mu,\lambda}$:

$$[\pi^{\mu,\lambda}(y)\xi](x) = \xi(y^{-1}x) \qquad \forall x, y \in G.$$

The inner product on $\mathcal{H}_\mu$ induces a pairing $\mathcal{V}^{\mu,\lambda'} \times \mathcal{V}^{\mu,\lambda} \to \mathcal{V}^{1,\lambda'-\overline{\lambda}+i\rho_1}$: indeed,

$$\begin{aligned}&\langle \xi(xman), \eta(xman)\rangle \\ &\qquad = \langle e^{(i\lambda'-\rho_1)(\log a)}\mu(m)^{-1}\xi(x),\, e^{(i\lambda-\rho_1)(\log a)}\mu(m)^{-1}\eta(x)\rangle \\ &\qquad = e^{(i\lambda'-i\overline{\lambda}-2\rho_1)(\log a)}\langle \xi(x), \eta(x)\rangle,\end{aligned}$$

so the complex-valued function $x \mapsto \langle \xi(x), \eta(x)\rangle$ indeed satisfies the covariance condition characterising $\mathcal{V}^{1,\lambda'-\overline{\lambda}+i\rho_1}$.

Lemma 1.1. *There is a G-invariant positive linear functional I_{P_1} on $\mathcal{V}^{1,i\rho_1}$, which is unique up to a constant. It may be defined as (a constant multiple of) the Haar measure on K,*

$$\xi \mapsto \int_K \xi(k)\, dk,$$

or as (a constant multiple of) the Haar measure on $\overline{N}_1$,

$$\xi \mapsto \int_{\overline{N}_1} \xi(\overline{n})\, d\overline{n}.$$