SET THEORY

WITH AN INTRODUCTION TO DESCRIPTIVE SET THEORY

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PREFACE TO THE FIRST EDITION

The creation of set theory can be traced back to the work of XIXth century mathematicians who tried to find a firm foundation for calculus. While the early contributors to the subject (Bolzano, Du Bois Reymond, Dedekind) were concerned with sets of numbers or of functions, the proper founder of set theory, Georg Cantor, made a decisive step and started an investigation of sets with arbitrary elements. The series of articles published by him in the years 1871–1883 contains an almost modern exposition of the theory of cardinals and of ordered and well-ordered sets. That step toward generalizations which Cantor made was a difficult one was witnessed by various contradictions (antinomies of set theory) discovered in set theory by various authors around 1900. The crisis created by these antinomies was overcome by Zermelo who formulated in 1904–1908 the first system of axioms of set theory. His axioms were sufficient to obtain all mathematically important results of set theory and at the same time did not allow the reconstruction of any known antinomy. Close ties between set theory and philosophy of mathematics date back to discussions concerning the nature of antinomies and the axiomatization of set theory. The fundamental problems of philosophy of mathematics such as the meaning of existence in mathematics, axiomatics versus description of reality, the need of consistency proofs and means admissible in such proofs were never better illustrated than in these discussions.

After an initial period of distrust the newly created set theory made a triumphal inroad in all fields of mathematics. Its influence on mathematics of the present century is clearly visible in the choice of modern problems and in the way these problems are solved. Applications of set theory are thus immense. But set theory developed also problems of its own. These problems and their solutions represent what is known as abstract set theory. Its achievements are rather modest in comparison to the applications of set theoretical methods in other branches of mathematics, some of which owe their very existence to set theory. Still, abstract set theory is a well-established part of mathematics and the knowledge of its basic notions is required from every mathematician.

Recent years saw a stormy advance in foundations of set theory. After breaking through discoveries of Gödel in 1940 who showed relative consistency of various set-theoretical hypotheses the recent works of Cohen allowed him and his successors to solve most problems of independence of these hypotheses while at the same time the works of Tarski showed how deeply can we delve in the domain of inaccessible cardinals whose magnitude surpasses all imagination. These recent works will certainly influence the future thinking on the philosophical foundations of mathematics.

The present book arose from a mimeographed text of Kuratowski from 1921 and from an enlarged edition prepared jointly by the two authors in 1951. As a glance on the list of contents will show, we intended to present the basic results of abstract set theory in the traditional order which goes back still to Cantor: algebra of sets, theory of cardinals, ordering and well-ordering of sets. We lay more stress on applications than it is usually done in texts of abstract set theory. The main field in which we illustrate set-theoretical methods is general topology. We also included a chapter on Borel, analytical and projective sets. The exposition is based on axioms which are essentially the ones of Zermelo–Fraenkel. We tried to present the proofs of all theorems even of the very trivial ones in such a way that the reader feels convinced that they are entirely based on the axioms. This accounts for some pedantry in notation and in the actual writing of several formulae which could be dispensed with if we did not wish to put the finger on axioms which we use in proofs. In some examples we use notions which are commonly known but which were not defined in our book by means of the primitive terms of our system. These examples are marked by the sign #.

In order to illustrate the role of the axiom of choice we marked by a small circle ° all theorems in which this axiom is used. There is in the book a brief account of the continuum hypothesis and a chapter on inaccessible cardinals. These topics deserve a more thorough presentation which however we could not include because of lack of space.

Also the last chapter which deals with the descriptive set theory is meant to be just an introduction to the subject.

Several colleagues helped us with the preparation of the text. Dr M. Mączyński translated the main part of the book and Mr R. Kowalsky collaborated with him in this difficult task. Professor J. Łoś wrote a penetrating appraisal of the manuscript of the 1951 edition as well as of the present one. His remarks and criticism allowed us to eliminate many errors and inaccuracies. Mr W. Marek and Mr K. Wiśniewski read the manuscript and the galley proofs and helped us in improving our text. To all these persons we express our deep gratitude.

KAZIMIERZ KURATOWSKI
ANDRZEJ MOSTOWSKI

PREFACE TO THE SECOND EDITION

The second edition of our *Set Theory* differs essentially from the first—which was translated from the Polish edition (by Professor M. Mączyński)—by the extension of its content. Our aim was to introduce the reader to some chapters of set theory which actually seem to be especially attractive and are cultivated by a large and still growing number of mathematicians.

The major changes introduced in the new edition are:

1. We wrote a new chapter on trees containing also a short introduction to the partition calculus and we completely rewrote Chapter 9 of the old edition dealing with inaccessible cardinals.

2. We introduced four chapters on descriptive set theory which replace Chapter 10 of the old edition. These four chapters (which were written by K. Kuratowski) contain

a. A short survey of the theory of Borel sets and Borel-measurable functions, preceded by a fairly general theory of L-measurable functions (where L is an arbitrary σ-lattice).

b. An insight into the theory of Souslin (analytic) sets and—more generally—of projective sets.

c. Some results on measurable selectors, mostly found within the last few years.

Some results presented in Chapters 11–14 are new as far as we know, whereas the first 10 chapters contain only results which are known from the literature.

We consistently tried to remain within the framework of the classical set theory. For this reason we did not include into our book any of the exciting recent results in whose proofs one uses model theoretical methods or notions borrowed from advanced parts of mathematical logic. See Mostowski [1].

We welcome this opportunity to express our gratitude—in addition to persons mentioned in the Preface to the first edition—to our younger

colleagues, J. Kaniewski, W. Marek, R. Pol, and P. Zbierski who read the manuscript, engaged in numerous discussions and provided many suggestions and corrections.

It is also our pleasure to express our thanks to Dr B. S. Niven from the White Agricultural Research Institute, South Australia, for correcting our English and to Mrs D. Wojciechowska for her help in preparing our manuscript.

Finally our thanks go to the North-Holland Publishing Company, as well as to the Polish Scientific Publishers and personally to Mrs Z. Osek, Mr W. Muszyński and Mr J. Panz for their assistance in the publication of this book.

KAZIMIERZ KURATOWSKI
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CONTENTS

CHAPTER I

ALGEBRA OF SETS

§ 1. Propositional calculus

Mathematical reasoning in set theory may be presented in a very clear form by making use of logical symbols and by basing arguments on the laws of logic formulated in terms of such symbols. In this section we shall present some basic principles of logic in order to refer to them later in this chapter and in the remainder of the book.

We shall designate arbitrary sentences by the letters $p, q, r, \ldots$ We assume that all of the sentences to be considered are either true or false. Since we consider only sentences of mathematics, we shall be dealing with sentences for which the above assumption is applicable.

From two arbitrary sentences, p and q, we can form a new sentence by applying to p and to q any one of the connectives:

$$\text{and,} \qquad \text{or,} \qquad \text{if} \ldots \text{then} \ldots, \qquad \text{if and only if.}$$

The sentence *p and q* we write in symbols $p \wedge q$. The sentence $p \wedge q$ is called the *conjunction* or the logical *product* of the sentences p and q which are the *components* of the conjunction. The conjunction $p \wedge q$ is true when both components are true. On the other hand, if any one of the components is false then the conjunction is false.

The sentence *p or q*, which we write symbolically $p \vee q$, is called the *disjunction* or the logical *sum* of the sentences p and q (the components of the disjunction). The disjunction is true if either of the components is true and is false only when both components are false.

The sentence *if p then q* is called the *implication* of q by p, where p is called the *antecedent* and q the *consequent* of the implication. Instead of writing *if p then q* we write $p \to q$. An implication is false if the consequent is false and the antecedent true. In all other cases the implication is true.

If the implication $p \to q$ is true we say that q *follows* from p; if we know that the sentence p is true we may conclude that the sentence q is also true.

In ordinary language the sense of the expression "if ..., then ..." does not entirely coincide with the meaning given above. However, in mathematics the use of such a definition as we have given is useful.

The sentence *p if and only if q* is called the *equivalence* of the two component sentences p and q and is written $p \equiv q$. This sentence is true provided p and q have the same logical value; that is, either both are true or both are false. If p is true and q false, or if p is false and q true, then the equivalence $p \equiv q$ is false.

The equivalence $p \equiv q$ can also be defined by the conjunction

$$(p \to q) \wedge (q \to p).$$

The sentence *it is not true that p* we call the *negation* of p and we write $\neg p$. The negation $\neg p$ is true when p is false and false when p is true. Hence $\neg p$ has the logical value opposite to that of p.

We shall denote an arbitrary true sentence by V and an arbitrary false sentence by F; for instance, we may choose for V the sentence $2 \cdot 2 = 4$, and for F the sentence $2 \cdot 2 = 5$.

Using the symbols F and V, we can write the definitions of truth and falsity for conjunction, disjunction, implication, equivalence and negation in the form of the following true equivalences:

(1) $F \wedge F \equiv F, \quad F \wedge V \equiv F, \quad V \wedge F \equiv F, \quad V \wedge V \equiv V,$

(2) $F \vee F \equiv F, \quad F \vee V \equiv V, \quad V \vee F \equiv V, \quad V \vee V \equiv V,$

(3) $(F \to F) \equiv V, \quad (F \to V) \equiv V, \quad (V \to F) \equiv F, \quad (V \to V) \equiv V,$

(4) $(F \equiv F) \equiv V, \quad (F \equiv V) \equiv F, \quad (V \equiv F) \equiv F, \quad (V \equiv V) \equiv V,$

(5) $\neg F \equiv V, \quad \neg V \equiv F.$

Logical *laws* or *tautologies* are those expressions built up from the letters $p, q, r, \ldots$ and the connectives $\wedge$, $\vee$, $\to$, $\equiv$, $\neg$ which have the property that no matter how we replace the letters $p, q, r, \ldots$ by arbitrary sentences (true or false) the entire expression itself is always true.

The truth or falsity of a sentence built up by means of connectives from the sentences $p, q, r, \ldots$ does not depend upon the meaning of the sentences $p, q, r, \ldots$ but only upon their logical values. Thus we can test whether an expression is a logical law by applying the following method: in place of the letters $p, q, r, \ldots$ we substitute the values F and V in every possible manner. Then using equations (1)–(5) we calculate the logical value of the expression for each one of these substitutions. If this value is always true, then the expression is a tautology.

Example. The expression $(p \wedge q) \rightarrow (p \vee r)$ is a tautology. It contains three variables p, q and r. Thus we must make a total of eight substitutions, since for each variable we may substitute either F or V. If, for example, for each letter we subtitute F, then we obtain $(F \wedge F) \rightarrow (F \vee F)$, and by (1) and (2) we obtain $F \rightarrow F$, namely V. Similarly, the value of the expression $(p \wedge q) \rightarrow (p \vee r)$ is true in each of the remaining seven cases.

Below we give several of the most important logical laws together with names for them. Checking that they are indeed logical laws is an exercise which may be left to the reader.

$(p \vee q) \equiv (q \vee p)$	*law of commutativity of disjunction,*
$[(p \vee q) \vee r] \equiv [p \vee (q \vee r)]$	*law of associativity of disjunction,*
$(p \wedge q) \equiv (q \wedge p)$	*law of commutativity of conjunction,*
$[p \wedge (q \wedge r)] \equiv [(p \wedge q) \wedge r]$	*law of associativity of conjunction,*
$[p \wedge (q \vee r)] \equiv [(p \wedge q) \vee (p \wedge r)]$	*first distributive law,*
$[p \vee (q \wedge r)] \equiv [(p \vee q) \wedge (p \vee r)]$	*second distributive law,*
$(p \vee p) \equiv p, \quad (p \wedge p) \equiv p$	*laws of tautology,*
$(p \wedge F) \equiv F, \quad (p \wedge V) \equiv p$ $(p \vee F) \equiv p, \quad (p \vee V) \equiv V$	*laws of absorption.*

In these laws the far reaching analogy between propositional calculus and ordinary arithmetic is made apparent. The major differences occur in the second distributive law and in the laws of tautology and absorption. In particular, the laws of tautology show that in the propositional calculus with logical addition and multiplication we need use neither coefficients nor exponents.

$[(p \to q) \wedge (q \to r)] \to (p \to r)$	*law of the hypothetical syllogism,*
$(p \vee \neg p) \equiv V$	*law of excluded middle,*
$(p \wedge \neg p) \equiv F$	*law of contradiction,*
$p \equiv \neg\neg p$	*law of double negation,*
$\neg(p \vee q) \equiv (\neg p \wedge \neg q)$ $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$	*de Morgan's laws,*
$(p \to q) \equiv (\neg q \to \neg p)$	*law of contraposition,*
$(p \to q) \equiv (\neg p \vee q),$	
$F \to p, \quad p \to p, \quad p \to V.$	

Throughout this book whenever we shall write an expression using logical symbols, we shall tacitly state that the expression is true. Remarks either preceding or following such an expression will always refer to a proof of its validity.

§2. Sets and operations on sets

The basic notion of set theory is the concept of *set*. This basic concept is, in turn, a product of historical evolution. Originally the theory of sets made use of an intuitive concept of set, characteristic of the so-called "naive" set theory. At that time the word "set" had the same imprecisely defined meaning as in everyday language. Such, in particular, was the concept of set held by Cantor,[1]) the creator of set theory.

Such a view was untenable, as in certain cases the intuitive concept proved to be unreliable. In Chapter II, §2 we shall deal with the antinomies of set theory, i.e. with the apparent contradictions which appeared at a certain stage in the development of the theory and

[1]) Georg Cantor (1845–1918) to whom we owe the creation of set theory was a German mathematician, professor at the University of Halle. He published his basic papers on set theory in "Mathematische Annalen" during the years 1879–1893. These papers were reprinted in Cantor [7]; this volume contains also a biography of Cantor written by E. Zermelo.

were due to the vagueness of intuition associated with the concept of set in certain more complicated cases. In the course of the polemic which arose over the antinomies it became obvious that different mathematicians had different concept of sets. As a result it became impossible to base set theory on intuition.

In the present book we shall present set theory as an axiomatic system. In geometry we do not examine directly the meaning of the terms "point", "line", "plane" or other "primitive terms", but from a well-defined system of axioms we deduce all the theorems of geometry without resorting to the intuitive meaning of the primitive terms. Similarly, we shall base set theory on a system of axioms from which we shall obtain theorems by deduction. Although the axioms have their source in the intuitive concept of sets, the use of the axiomatic method ensures that the intuitive content of the word "set" plays no part in proofs of theorems or in definitions of set theoretical concepts.

Sometimes we shall illustrate set theory with examples furnished by other branches of mathematics. This illustrative material involving axioms not belonging to the axiom system of set theory will be distinguished by the sign # placed at the beginning and at the end of the text.

The primitive notions of set theory are "set" and the relation "to be an element of". Instead of *x is a set* we shall write $Z(x)$, and instead of *x is an element of y* we shall write $x \in y$.[1]) The negation of the formula $x \in y$ will be written as x non $\in y$, or $x \notin y$ or $\neg(x \in y)$. To simplify the notation we shall use capital letters to denote sets; thus if a formula involves a capital letter, say A, then it is tacitly assumed that A is a set. Later on we shall introduce yet another primitive notion: $x\mathrm{TR}y$ (*x is the relational type of y*). We shall discuss it in Chapter II.

For the present we assume four axioms:

I. AXIOM OF EXTENSIONALITY: *If the sets A and B have the same elements then they are identical.*

[1]) The symbol $\in$ is derived from the Greek letter *epsilon*. The use of this letter for the elementhood relation was introduced by Peano [2] who selected it as the abbreviation of the Greek word "to be" (ἐστί). Many other mathematical and logical symbols also originated with Peano.