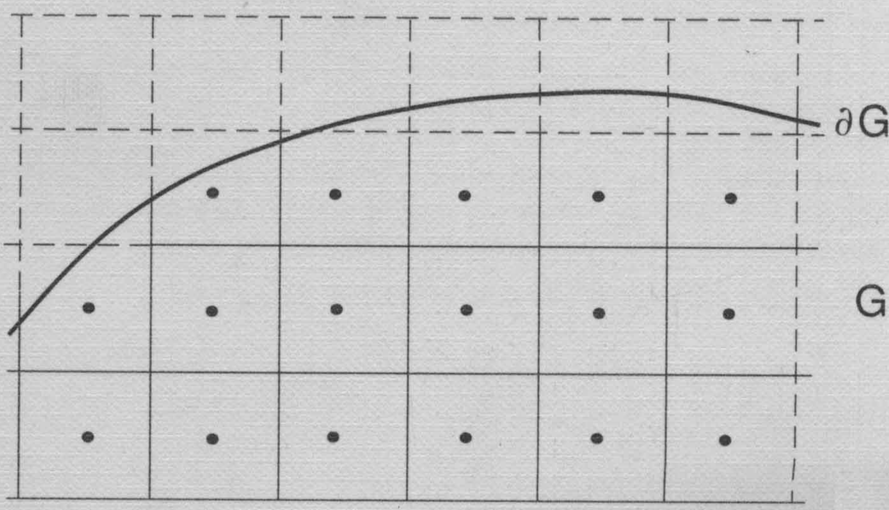
Lecture Notes in Mathematics

1549

Gennadi Vainikko

Multidimensional Weakly Singular Integral Equations



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Lecture Notes in Mathematics 1549

In memory
of Solomon Mikhlin

PREFACE

In these lecture notes we deal with the integral equation

$$u(x) = \int_G K(x,y)\,u(y)dy + f(x), \qquad x \in G, \tag{0.1}$$

where $G \subset \mathbb{R}^n$ is an open bounded region or, more generally, an open bounded set (possibly non-connected). The functions f and K are assumed to be smooth but K may have a weak singularity on the diagonal:

$$|K(x,y)| \le b(1 + |x-y|^{-\nu}), \qquad b = \text{const}, \quad \nu < n. \tag{0.2}$$

The main problems of interest to us are the following:

— the smoothness of the exact solution to equation (0.1);

— discretization methods for equation (0.1).

Usually, the derivatives of the solution to a weakly singular integral equation have singularities near the boundary ∂G of the domain of integration $G \subset \mathbb{R}^n$. A unified description of the singularities in all possible cases is complicated, and up to now this problem has not been solved fully. In Chapter 3 we give estimates which are sharp in many practically interesting cases. The behaviour of the tangential derivatives thereby turns out to be less singular than the behaviour of the normal derivatives. All this information is used designing approximate methods for integral equation (0.1). We restrict ourselves to collocation and related schemes, thoroughly examining simplest schemes based on the piecewise constant approximation of the solution and the superconvergence phenomenon at the collocation points (Chapters 5 and 6). In the case where $G \subset \mathbb{R}^n$ is a parallelepiped, higher order collocation methods on graded grids are also considered (Chapter 7); again the superconvergence at the collocation points is examined.

Technically, our convergence analysis is based on the discrete convergence theory outlined in Chapter 4 of the book. This short chapter can be used for a first acquaintance with the theory for linear equations $u = Tu + f$; for eigenvalue problems and nonlinear equations, the results are presented without proofs.

In Chapter 8, some of the main results of Chapters 3 and 5-7 are extended to nonlinear integral equations.

Examples of (linear) integral equations (0.1), (0.2) can be found in radiation transfer theory (see Section 1); some interior-exterior boundary value problems too have their most natural formulations as integral equations of type (0.1), (0.2). Perhaps some readers will be disappointed to find that our treatment concerns only integral equations on an open set $G \subset \mathbb{R}^n$. In practice, there is a great interest also in the boundary integral equations

$$u(x) = \int_{\partial G} K(x,y)\, u(y)\, dS_y + f(x), \qquad x \in \partial G. \tag{0.3}$$

Such equations arise, for instance, in solving the Dirichlet or Neumann problem for the Laplace equation (see e.g. Mikhlin (1970) or Atkinson (1990)). A natural question of whether the results of the lecture book can be extended or modified to boundary integral equations then arises. The answer is non-unique. If ∂G is smooth then the solution of the boundary integral equation is smooth too, and the results concerning the collocation and related methods can even be strengthened and the arguments can be simplified. On the other hand, if ∂G is non-smooth then the standard boundary integral operators, e.g. the ones corresponding to the Laplace equation, are non-compact, and our arguments fail fully. The case of an integral equation on a smooth (relative) region $\Gamma \subset \partial G$ with a smooth (relative) boundary $\partial \Gamma$ seems to be the most adequate case that can be treated by our arguments. But this assertion may be considered only as a conjecture not discussed anywhere.

We use only a minimum of references in the main text. Nevertheless, an extended commented bibliography is added. Young mathematicians looking for problems to work on will find a list of unsolved problems too. The lectures are based on the author's recent publications (see Vainikko (1990a,b), (1991a,b), (1992a,b), Vainikko and Pedas (1990)) but actually the results were elaborated during a much longer time lecturing at University of Tartu, the Technical University of Chemnitz and Colorado State University. A significant milestone for us was the booklet by Vainikko, Pedas and Uba (1984) concerning the one-dimensional case ($n=1$). In the present lectures, we always assume that $n \geq 2$.

CONTENTS

1. SOME PROBLEMS LEADING TO MULTIDIMENSIONAL WEAKLY SINGULAR INTEGRAL EQUATIONS

In this chapter we present two examples on problems of mathematical physics which can be reformulated as multidimensional weakly singular integral equations — an interior-exterior boundary value problem and a radiation transfer problem.

1.1. An interior-exterior problem. Let $G \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded set with piecewise smooth boundary ∂G and let a and f be given real or complex valued bounded continuous functions on G (we write $a, f \in BC(G)$). Consider the following problem: find a function $\varphi \in C^1(\mathbb{R}^n) \cap H^2_{loc}(\mathbb{R}^n)$ such that

$$\Delta\varphi(x) = a(x)\varphi(x) + f(x)\ , \quad x \in G, \tag{1.1}$$

$$\Delta\varphi(x) = 0, \quad x \in \mathbb{R}^n \backslash \overline{G} \tag{1.2}$$

whereby, in case $n \geq 3$,

$$\varphi(x) \to 0 \quad \text{as} \quad |x| \to \infty \tag{1.3}$$

or, in case $n = 2$,

$$|\varphi(x)| \text{ is bounded as } |x| \to \infty. \tag{1.3'}$$

Here the following standard notations are adopted: $C^1(\mathbb{R}^n)$ is the space of continuously differentiable functions on $\mathbb{R}^n$; $H^2_{loc}(\mathbb{R}^n)$ is the space of functions on $\mathbb{R}^n$ which have locally square-integrable (generalized) derivatives up to the second order; Δ is the Laplace operator, $\Delta\varphi = \partial^2\varphi/\partial x_1^2 + \ldots + \partial^2\varphi/\partial x_n^2$. Note that the condition $\varphi \in C^1(\mathbb{R}^n)$ contains a requirement that φ itself as well its first normal derivative have equal boundary values as x approaches ∂G from inside and outside of G.

1.2. A physical background (n = 2). Crouseix and Descloux (1988) describe a mathematical model of the electromagnetic casting process. When the ingot is sufficiently long, the electromagnetic part of the problem reduces to the search of a complex potential φ in $\mathbb{R}^2$, of class C^1, satisfying the conditions

$$\Delta\varphi + 2i\alpha^2(\varphi + c_k) = 0 \quad \text{in } G_k \ (k = 1, \ldots, l),$$

$$\Delta\varphi = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{G}, \quad G = \bigcup_{k=1}^{l} G_k.$$

Here $G_k \subset \mathbb{R}^2$, $1 \le k \le l$, are the cross-sections in the x_1, x_2 plane of cylindrical electric conductors in which a current with angular freguency ω runs; $2\alpha^2 = \mu_0 s\omega$ is a real constant where μ_0 is the magnetic permeability of the air and s the conductivity; $i = \sqrt{-1}$ is the imaginary unit and c_k's are given complex constants. Thus, we have a special case of problem (1.1), (1.2) with $a(x) = -2i\alpha^2$, $f(x) = -2i\alpha^2 c_k$ for $x \in G_k$ $(k = 1, \dots, l)$.

An instruction from this background is that we ought to avoid an assumption about the connectivity of $G \subset \mathbb{R}^n$ when problem (1.1), (1.2) will be discussed.

1.3. Integral equation formulation ($n \ge 3$). We look for a solution of problem (1.1) - (1.3) in the form of the Newton potential (see Bers et al. (1964))

$$\varphi(x) = -c_n \int_G |x-y|^{-(n-2)} u(y)\,dy, \qquad x \in \mathbb{R}^n, \tag{1.4}$$

where $c_n = 1/((n-2)\sigma_n)$, $\sigma_n = \Gamma(n/2)/(2\pi^{n/2})$ is the area of the unit sphere in $\mathbb{R}^n$ and $u \in BC(G)$ is the density which we have to determine. Condition (1.3) is automatically fulfilled. First derivatives of φ can be found differentiating (1.4) under the integral sign, the result is a weakly singular integral again, and it is easy to see that $\varphi \in C^1(\mathbb{R}^n)$. Further, it is well known that

$$\Delta\varphi(x) = \begin{cases} u(x), & x \in G, \\ 0, & x \in \mathbb{R}^n \setminus \overline{G} \end{cases}$$

(in the sense of distributions as well in the sense of pointwise equalities). We see that (1.2) is fulfilled, too. A consequence of $\Delta\varphi \in L^2(\mathbb{R}^n)$ is that $\varphi \in L^2_{loc}(\mathbb{R}^n)$ (together with (1.3) we have even $\varphi \in H^2(\mathbb{R}^n)$). Condition (1.1) takes the form of the following integral equation to determine u:

$$u(x) = -c_n a(x) \int_G |x-y|^{-(n-2)} u(y)\,dy + f(x), \qquad x \in G. \tag{1.5}$$

Thus, to solve problem (1.1) - (1.3), we have to solve integral equation (1.5) and then apply formula (1.4). Actually, (1.4) is needed only for $x \in \mathbb{R}^n \setminus \overline{G}$; for $x \in G$ we have from (1.4) and (1.5)

$$\varphi(x) = (u(x) - f(x))/a(x).$$

Let us make sure that we exhaust all solutions of problem (1.1) - (1.3) in this way. Indeed, let $\varphi \in C^1(\mathbb{R}^n) \cap H^2_{loc}(\mathbb{R}^n)$ be an arbitrary solution of (1.1) - (1.3). Denote

$$u(x) = \Delta\varphi(x) = a(x)\varphi(x) + f(x), \qquad x \in G,$$

$$\psi(x) = \varphi(x) + c_n \int_G |x-y|^{-(n-2)} u(y)\,dy, \quad x \in \mathbb{R}^n.$$

It is clear that $u \in BC(G)$, $\psi \in C^1(\mathbb{R}^n) \cap H^2_{loc}(\mathbb{R}^n)$. In addition, $\Delta\psi(x)=0$ for $x \in G$ and $x \in \mathbb{R}^n \setminus \overline{G}$, i.e. $\Delta\psi=0$ a.e. in $\mathbb{R}^n$. Together with the smoothness of ψ (see above), this means that $\Delta\psi=0$ in $\mathbb{R}^n$ in the sense of distributions. Now, using the hypoellipticity property of the Laplace operator (see e.g. Yosida (1965) or Lions and Magenes (1968)), we obtain that $\psi \in C^\infty(\mathbb{R}^n)$ and $\Delta\psi(x)=0$ for all $x \in \mathbb{R}^n$. Further, $\psi(x) \to 0$ as $|x| \to \infty$, hence $\psi(x)=0$ for all $x \in \mathbb{R}^n$, i.e. φ has a representation (1.4) with $u(x)=\Delta\varphi(x)$, $x \in G$, q.e.d.

Problem (1.1) - (1.3) is uniquely solvable if and only if integral equation (1.5) has a unique solution $u \in BC(G)$. This occurs if and only if the corresponding homogeneous integral equation $u = Tu$ has in $BC(G)$ only the trivial solution. Note that operator $T: BC(G) \to BC(G)$ is compact (a proof in a more general setting is given in Section 2.3)

1.4. Integral equation formulation (n=2). We look for a solution of problem (1.1), (1.2), (1.3') in the form

$$\varphi(x) = (2\pi)^{-1} \int_G \log|x-y|\, u(y)\,dy + \beta, \qquad x \in \mathbb{R}^2, \tag{1.6}$$

where we have to determine the density $u \in BC(G)$ and the constant β. Again, $\varphi \in C^1(\mathbb{R}^2)$, $\Delta\varphi(x) = u(x)$ for $x \in G$, $\Delta\varphi(x)=0$ for $x \in \mathbb{R}^2 \setminus \overline{G}$, (1.2) is fulfilled and (1.1) takes the form

$$u(x) = (2\pi)^{-1} a(x) \int_G \log|x-y|\, u(y)\,dy + \beta a(x) + f(x), \qquad x \in G. \tag{1.7}$$

Condition (1.3') is fulfilled if and only if

$$\int_G u(x)\,dx = 0. \tag{1.8}$$

Indeed, rewrite (1.6) in the form

$$\varphi(x) = (2\pi)^{-1} \int_G \log \frac{|x-y|}{|x|}\, u(y)dy + (2\pi)^{-1} \log|x| \int_G u(y)\,dy + \beta.$$

Here the first integral tends to 0 as $|x| \to \infty$ since $\log(|x-y|/|x|) \to 0$ uniformly with respect to $y \in G$. Hence $\varphi(x)$ is bounded as $|x| \to \infty$ if and only if (1.8) holds.

Thus, to solve (1.1), (1.2), (1.3'), we have to find a pair $u \in BC(G), \beta \in \mathbb{C}$ (or $\mathbb{R}$) from equations (1.7), (1.8) and then apply (1.6). For $x \in G$ we have $\varphi(x) = (u(x) - f(x))/a(x)$ again, thus actually (1.6) is needed for $x \in \mathbb{R}^2 \setminus \overline{G}$ only. It is easy to check again that we exhaust in this way all solutions of problem (1.1), (1.2), (1.3').

Problem (1.1), (1.2), (1.3') is uniquely solvable if and only if problem (1.7), (1.8) is uniquely solvable. Problem (1.7), (1.8) preserves the Fredholm property — for its unique solvability, it is necessary and sufficient that the corresponding homogeneous problem

$$u(x) = (2\pi)^{-1}a(x)\int_G \log|x-y|u(y)dy + \beta a(x), \quad x \in G,$$

$$\int_G u(x)dx = 0$$

has in $BC(G)\times\mathbb{C}$ only the trivial solution $u=0$, $\beta=0$. We state a simple sufficient condition for the unique solvability:

$$a, 1/a \in BC(G), \quad \mathrm{Im}\, a \neq 0 \text{ and is sign constant in } G \tag{1.9}$$

where $\mathrm{Im}\, a$ is the imaginary part of a. Indeed, let $u \in BC(G)$, $\beta \in \mathbb{C}$ be a solution of the homogeneous problem. Then we have the equalities

$$\frac{u(x)}{|a(x)|^2}\,\bar{a}(x) = \frac{1}{2\pi}\int_G \log|x-y|u(y)dy + \beta, \quad x \in G,$$

$$\int_G u(y)dy = 0$$

where $\bar{a}$ is the complex conjugate to a. Taking the scalar product of the first equality with u and using the second one we obtain

$$\int_G \frac{|u(x)|^2}{|a(x)|^2}\,\bar{a}(x)dx = (\Lambda u, u) \tag{1.10}$$

where the number

$$(\Lambda u, u) = (2\pi)^{-1}\int_G\int_G \log|x-y|u(y)\bar{u}(x)dy\,dx$$

is real due to the symmetry of the kernel $\log|x-y|$. Since $\mathrm{Im}\, a(x) > 0$ or $\mathrm{Im}\, a(x) < 0$ on G, (1.10) is possible in case $u=0$ only. Now we see that $\beta=0$, too, q.e.d.

Note that for the physical problem considered in Section 1.2, condition (1.9) is fulfilled.

A further sufficient condition for the unique solvability of the problem (1.7), (1.8) can be formulated as follows: (i) the homogeneous integral equation

$$u(x) = (2\pi)^{-1}a(x)\int_G \log|x-y|u(y)dy, \quad x \in G,$$

has only the trivial solution $u=0$; (ii) the solution u_a of the integral equation

$$u(x) = (2\pi)^{-1}a(x)\int_G \log|x-y|u(y)dy + a(x), \quad x \in G,$$

satisfies the condition $\int_G u_a(x)dx \neq 0$.

If conditions (i) and (ii) are fulfilled then the unique solution of problem (1.7), (1.8) is given by

$$\beta = -\int_G u_f(x)dx \Big/ \int_G u_a(x)dx\,, \quad u(x) = u_f(x) + \beta u_a(x)$$

where u_f is the solution of the integral equation

$$u(x) = (2\pi)^{-1}a(x)\int_G \log|x-y|\,u(y)dy + f(x), \quad x \in G.$$

Thus, problem (1.7), (1.8) can be reduced to two standard integral equations with the logarithmic kernel. But numerically one usually prefers to solve (1.7), (1.8) directly.

1.5. Radiation transfer problem. Let $G \subset \mathbb{R}^3$ be an open bounded convex region, ∂G its boundary and

$$S = \{s \in \mathbb{R}^3 : |s| = (s_1^2 + s_2^2 + s_3^2)^{1/2} = 1\}$$

the unit sphere in $\mathbb{R}^3$. For $x \in \partial G$, let us denote by

$$S_x = \{s \in S : \exists \lambda > 0 : x + \lambda s \in G\}$$

the set of directions falling into G; if x is a smoothness point of ∂G then we simply have

$$S_x = \{s \in S : s \cdot \nu(x) > 0\}$$

where $\nu(x)$ is the unit inner normal to ∂G at $x \in \partial G$.

A standard radiation transfer problem reads as follows: find a function $\varphi : \overline{G} \times S \to \mathbb{R}_+$ (the intensity of the radiation) such that

$$\sum_{j=1}^{3} s_j \frac{\partial \varphi(x,s)}{\partial x_j} + \sigma(x)\varphi(x,s) = \frac{\sigma_0(x)}{4\pi}\int_S g(x,s,s')\varphi(x,s')ds' + f(x,s) \quad (x \in G,\ s \in S), \tag{1.11}$$

$$\varphi(x,s) = \varphi_0(x,s) \quad (x \in \partial G,\ s \in S_x). \tag{1.12}$$

Here $\sigma : G \to \mathbb{R}_+$ (the extinction coefficient), $\sigma_0 : G \to \mathbb{R}_+$ (the scattering coefficient), $g : G \times S \times S \to \mathbb{R}_+$ (the phase function of scattering), $f : G \times S \to \mathbb{R}_+$ (the source function) and $\varphi_0 : \{(x,s) : x \in \partial G,\ s \in S_x\} \to \mathbb{R}_+$ (the inflow radiation intensity) are given functions whereby

$$\sigma_0(x) \le \sigma(x), \qquad g(x,s,s') = g(x,s',s), \qquad \frac{1}{4\pi}\int_S g(x,s,s')ds' = 1$$

(we shall refer to those as "physical conditions"). For a more detailed exposition we refer to Chandrasekhar (1950). We have adopted a terminology used in the atmospheric optics. In the theory of nuclear reactors, (1.11), (1.12) occurs as a main problem, too, but the terminology is slightly different (see e.g. Case and Zweifel (1967) or Marchuk and Lebedev (1984)).

1.6. Integral equation formulation of the radiation transfer problem. Let us split (1.11) into two equations:

$$\sum_{j=1}^{3} s_j \frac{\partial\varphi(x,s)}{\partial x_j} + \sigma(x)\varphi(x,s) = u(x,s) \qquad (x\in G,\ s\in S), \tag{1.11'}$$

$$u(x,s) = \frac{\sigma_0(x)}{4\pi}\int_S g(x,s,s')\varphi(x,s')ds' + f(x,s) \qquad (x\in G,\ s\in S). \tag{1.11''}$$

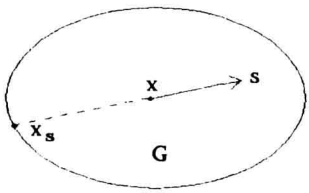

Fig. 1.1

One can solve (1.11'), (1.12) explicitly:

$$\varphi(x,s) = \varphi_0(x_s,s)\exp(-\tau(x,x_s)) + \int_{-|x-x_s|}^{0} u(x+\lambda s,s)\exp(-\tau(x,x+\lambda s))d\lambda \qquad (x\in G,\ s\in S) \tag{1.13}$$

where x_s is the point on ∂G which lies in the direction $-s$ from x (see Figure 1.1) and

$$\tau(x,y) = |x-y|\int_0^1 \sigma(tx+(1-t)y)dt \tag{1.14}$$

is the optical distance between points $x,y\in G$.

Indeed, let us write down that φ satisfies (1.11') at the points of the straight line $x+\lambda s$ $(-|x-x_s|<\lambda<0)$:

$$\sum_{j=1}^{3} s_j \frac{\partial\varphi(x+\lambda s,s)}{\partial x_j} + \sigma(x+\lambda s)\varphi(x+\lambda s,s) = u(x+\lambda s,s).$$

Since

$$\sum_{j=1}^{3} s_j \frac{\partial\varphi(x+\lambda s,s)}{\partial x_j} = \frac{d}{d\lambda}\varphi(x+\lambda s,s),$$

we obtain the linear ordinary differential equation of the first order

$$\frac{d}{d\lambda}\varphi(x+\lambda s,s)+\sigma(x+\lambda s)\varphi(x+\lambda s,s) = u(x+\lambda s,s), \qquad -|x-x_s|\le\lambda\le 0,$$