

LECTURE NOTES
IN PHYSICS

Y. Kosmann-Schwarzbach
B. Grammaticos
K. M. Tamizhmani
(Eds.)

Integrability of Nonlinear Systems



Springer

Y. Kosmann-Schwarzbach B. Grammaticos
K.M. Tamizhmani (Eds.)

Integrability of Nonlinear Systems



Springer

Editors

Yvette Kosmann-Schwarzbach
Centre de Mathématiques
École Polytechnique
91128 Palaiseau, France

K. M. Tamizhmani
Department of Mathematics
Pondicherry University
Kalapet
Pondicherry 605 014, India

Basil Grammaticos
GMPiB, Université Paris VII
Tour 24-14, 5^e étage, case 7021
2 place Jussieu
75251 Paris Cedex 05, France

Y. Kosmann-Schwarzbach, B. Grammaticos, K. M. Tamizhmani (eds.), *Integrability of Nonlinear Systems*, Lect. Notes Phys. 638 (Springer-Verlag Berlin Heidelberg 2004), DOI 10.1007/b94605

Library of Congress Cataloging-in-Publication Data.

Cataloging-in-Publication Data applied for

A catalog record for this book is available from the Library of Congress.

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie;
detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>

ISSN 0075-8450

ISBN 3-540-20630-2 Springer-Verlag Berlin Heidelberg New York

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

Springer-Verlag is a part of Springer Science+Business Media
springeronline.com

© Springer-Verlag Berlin Heidelberg 2004
Printed in Germany

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting: Camera-ready by the authors/editor
Data conversion: PTP-Berlin Protago-TeX-Production GmbH
Cover design: *design & production*, Heidelberg

Printed on acid-free paper
54/3141/du - 5 4 3 2 1 0

Lecture Notes in Physics

Editorial Board

R. Beig, Wien, Austria

B.-G. Englert, Singapore

U. Frisch, Nice, France

P. Hänggi, Augsburg, Germany

K. Hepp, Zürich, Switzerland

W. Hillebrandt, Garching, Germany

D. Imboden, Zürich, Switzerland

R. L. Jaffe, Cambridge, MA, USA

R. Lipowsky, Golm, Germany

H. v. Löhneysen, Karlsruhe, Germany

I. Ojima, Kyoto, Japan

D. Sornette, Nice, France, and Los Angeles, CA, USA

S. Theisen, Golm, Germany

W. Weise, Trento, Italy, and Garching, Germany

J. Wess, München, Germany

J. Zittartz, Köln, Germany

Springer

Berlin

Heidelberg

New York

Hong Kong

London

Milan

Paris

Tokyo

Physics and Astronomy



springeronline.com

The Editorial Policy for Edited Volumes

The series *Lecture Notes in Physics* (LNP), founded in 1969, reports new developments in physics research and teaching - quickly, informally but with a high degree of quality. Manuscripts to be considered for publication are topical volumes consisting of a limited number of contributions, carefully edited and closely related to each other. Each contribution should contain at least partly original and previously unpublished material, be written in a clear, pedagogical style and aimed at a broader readership, especially graduate students and nonspecialist researchers wishing to familiarize themselves with the topic concerned. For this reason, traditional proceedings cannot be considered for this series though volumes to appear in this series are often based on material presented at conferences, workshops and schools.

Acceptance

A project can only be accepted tentatively for publication, by both the editorial board and the publisher, following thorough examination of the material submitted. The book proposal sent to the publisher should consist at least of a preliminary table of contents outlining the structure of the book together with abstracts of all contributions to be included. Final acceptance is issued by the series editor in charge, in consultation with the publisher, only after receiving the complete manuscript. Final acceptance, possibly requiring minor corrections, usually follows the tentative acceptance unless the final manuscript differs significantly from expectations (project outline). In particular, the series editors are entitled to reject individual contributions if they do not meet the high quality standards of this series. The final manuscript must be ready to print, and should include both an informative introduction and a sufficiently detailed subject index.

Contractual Aspects

Publication in LNP is free of charge. There is no formal contract, no royalties are paid, and no bulk orders are required, although special discounts are offered in this case. The volume editors receive jointly 30 free copies for their personal use and are entitled, as are the contributing authors, to purchase Springer books at a reduced rate. The publisher secures the copyright for each volume. As a rule, no reprints of individual contributions can be supplied.

Manuscript Submission

The manuscript in its final and approved version must be submitted in ready to print form. The corresponding electronic source files are also required for the production process, in particular the online version. Technical assistance in compiling the final manuscript can be provided by the publisher's production editor(s), especially with regard to the publisher's own \LaTeX macro package which has been specially designed for this series.

LNP Homepage (springerlink.com)

On the LNP homepage you will find:

- The LNP online archive. It contains the full texts (PDF) of all volumes published since 2000. Abstracts, table of contents and prefaces are accessible free of charge to everyone. Information about the availability of printed volumes can be obtained.
- The subscription information. The online archive is free of charge to all subscribers of the printed volumes.
- The editorial contacts, with respect to both scientific and technical matters.
- The author's / editor's instructions.

Preface

This second edition of *Integrability of Nonlinear Systems* is both streamlined and revised. The eight courses that compose this volume present a comprehensive survey of the various aspects of integrable dynamical systems. Another expository article in the first edition dealt with chaos: for this reason, as well as for technical reasons, it is not reprinted here. Several texts have been revised and others have been corrected or have had their bibliography brought up to date. The present edition will be a valuable tool for graduate students and researchers.

The first edition of this book, which appeared in 1997 as Lecture Notes in Physics 495, was the development of the lectures delivered at the International School on Nonlinear Systems which was held in Pondicherry (India) in January 1996, organized by CIMPA-Centre International de Mathématiques Pures et Appliquées/International Center for Pure and Applied Mathematics and Pondicherry University. In February 2003, another International School was held in Pondicherry, sponsored by CIMPA, UNESCO and the Pondicherry Government, dealing with *Discrete Integrable Systems*. The lectures of that school are now being edited as a volume in the Lecture Notes in Physics series by B. Grammaticos, Y. Kosmann-Schwarzbach and Thamizharasi Tamizhmani, and will constitute a companion volume to the essays presented here.

We are very grateful to the scientific editors of Springer-Verlag, Prof. Wolf Beiglböck and Dr. Christian Caron, who invited us to prepare a new edition. We acknowledge with thanks the renewed editorial advice of Dr. Bertram E. Schwarzbach, and we thank Miss Sandra Thoms for her expert help in the production of the book.

Paris, September 2003

The Editors

List of Contributors

Mark J. Ablowitz

Department of Applied Mathematics,
Campus Box 526,
University of Colorado at Boulder,
Boulder Colorado 80309-0526, USA
markjab@newton.colorado.edu

Paolo Casati

Dipartimento di Matematica
e Applicazioni,
Università di Milano-Bicocca,
Via degli Arcimboldi 8,
20126 Milano, Italy
casati@matapp.unimib.it

Gregorio Falqui

SISSA,
Via Beirut 2/4,
34014 Trieste, Italy
falqui@sissa.it

Basil Grammaticos

GMPiB,
Université Paris VII,
Tour 24-14, 5^e étage, case 7021,
75251 Paris, France
grammati@paris7.jussieu.fr

Rod Halburd

Department of Mathematical Sciences,
Loughborough University,
Loughborough,
Leicestershire LE11 3TU,
United Kingdom
r.g.halburd@lboro.ac.uk

Jarmo Hietarinta

Department of Physics,
University of Turku,
20014 Turku, Finland
hietarin@utu.fi

Nalini Joshi

School of Mathematics
and Statistics F07,
University of Sydney,
NSW 2006, Australia
nalini@maths.usyd.edu.au

Yvette Kosmann-Schwarzbach

Centre de Mathématiques,
École Polytechnique,
91128 Palaiseau, France
yks@math.polytechnique.fr

Martin D. Kruskal

Department of Mathematics,
Rutgers University,
New Brunswick NJ 08903, USA
kruskal@math.rutgers.edu

Franco Magri

Dipartimento di Matematica
e Applicazioni,
Università di Milano-Bicocca,
Via degli Arcimboldi 8,
20126 Milano, Italy
magri@matapp.unimib.it

Marco Pedroni

Dipartimento di Matematica,
Università di Genova,
Via Dodecaneso 35,
16146 Genova, Italy
pedroni@dima.unige.it

Alfred Ramani

CPT, École Polytechnique,
CNRS, UMR 7644,
91128 Palaiseau, France
ramani@cpht.polytechnique.fr

Junkichi Satsuma

Graduate School
of Mathematical Sciences,
University of Tokyo,
Komaba, Meguro-ku,
Tokyo 153-8914, Japan
satsuma@ms.u-tokyo.ac.jp

Michael Semenov-Tian-Shansky

Laboratoire Gevrey
de Mathématique physique,
Université de Bourgogne, BP 47870,
21078 Dijon Cedex, France
semenov@mail.u-bourgogne.fr

K.M. Tamizhmani

Department of Mathematics,
Pondicherry University,
Kalapet, Pondicherry-605 014, India
tamizh@yahoo.com

Contents

Introduction

<i>The Editors</i>	1
1 Analytic Methods	2
2 Painlevé Analysis	2
3 τ -functions, Bilinear and Trilinear Forms	2
4 Lie-Algebraic and Group-Theoretical Methods	3
5 Bihamiltonian Structures	3

Nonlinear Waves, Solitons, and IST

<i>M.J. Ablowitz</i>	5
1 Fundamentals of Waves	5
2 IST for Nonlinear Equations in 1+1 Dimensions	9
3 Scattering and the Inverse Scattering Transform	11
4 IST for 2+1 Equations	19
5 Remarks on Related Problems	24

Integrability – and How to Detect It

<i>B. Grammaticos, A. Ramani</i>	31
1 General Introduction: Who Cares about Integrability?	31
2 Historical Presentation: From Newton to Kruskal	33
3 Towards a Working Definition of Integrability	41
3.1 Complete Integrability	43
3.2 Partial and Constrained Integrability	47
4 Integrability and How to Detect It	48
4.1 Fixed and Movable Singularities	49
4.2 The Ablowitz-Ramani-Segur Algorithm	50
5 Implementing Singularity Analysis: From Painlevé to ARS and Beyond	54
6 Applications to Finite and Infinite Dimensional Systems	66
6.1 Integrable Differential Systems	66
6.2 Integrable Two-Dimensional Hamiltonian Systems	68
6.3 Infinite-Dimensional Systems	71
7 Integrable Discrete Systems Do Exist!	73
8 Singularity Confinement: The Discrete Painlevé Property	77

9	Applying the Confinement Method:	
	Discrete Painlevé Equations and Other Systems	79
9.1	The Discrete Painlevé Equations	79
9.2	Multidimensional Lattices and Their Similarity Reductions	86
9.3	Linearizable Mappings	87
10	Discrete/Continuous Systems:	
	Blending Confinement with Singularity Analysis	87
10.1	Integrodifferential Equations of the Benjamin-Ono Type	89
10.2	Multidimensional Discrete/Continuous Systems	90
10.3	Delay-Differential Equations	90
11	Conclusion	90

Introduction to the Hirota Bilinear Method

	<i>J. Hietarinta</i>	95
1	Why the Bilinear Form?	95
2	From Nonlinear to Bilinear	95
	2.1 Bilinearization of the KdV Equation	96
	2.2 Another Example: The Sasa-Satsuma Equation	97
	2.3 Comments	98
3	Constructing Multi-soliton Solutions	99
	3.1 The Vacuum, and the One-Soliton Solution	99
	3.2 The Two-Soliton Solution	100
	3.3 Multi-soliton Solutions	101
4	Searching for Integrable Evolution Equations	101
	4.1 KdV	102
	4.2 mKdV and sG	102
	4.3 nlS	103

Lie Bialgebras, Poisson Lie Groups, and Dressing Transformations

	<i>Y. Kosmann-Schwarzbach</i>	107
	Introduction	107
1	Lie Bialgebras	110
	1.1 An Example: $\mathfrak{sl}(2, \mathbb{C})$	110
	1.2 Lie-Algebra Cohomology	111
	1.3 Definition of Lie Bialgebras	113
	1.4 The Coadjoint Representation	114
	1.5 The Dual of a Lie Bialgebra	114
	1.6 The Double of a Lie Bialgebra. Manin Triples	115
	1.7 Examples	116
	1.8 Bibliographical Note	119
2	Classical Yang-Baxter Equation and r -Matrices	119
	2.1 When Does δr Define a Lie-Bialgebra Structure on \mathfrak{g} ?	119
	2.2 The Classical Yang-Baxter Equation	122
	2.3 Tensor Notation	126

2.4 *R*-Matrices and Double Lie Algebras 128

2.5 The Double of a Lie Bialgebra Is a Factorizable Lie Bialgebra ... 131

2.6 Bibliographical Note 132

3 Poisson Manifolds. The Dual of a Lie Algebra. Lax Equations 133

3.1 Poisson Manifolds..... 133

3.2 The Dual of a Lie Algebra 135

3.3 The First Russian Formula 136

3.4 The Traces of Powers of Lax Matrices Are in Involution 138

3.5 Symplectic Leaves and Coadjoint Orbits 139

3.6 Double Lie Algebras and Lax Equations 142

3.7 Solution by Factorization 145

3.8 Bibliographical Note 146

4 Poisson Lie Groups 146

4.1 Multiplicative Tensor Fields on Lie Groups..... 147

4.2 Poisson Lie Groups and Lie Bialgebras 149

4.3 The Second Russian Formula (Quadratic Brackets)..... 151

4.4 Examples 151

4.5 The Dual of a Poisson Lie Group 153

4.6 The Double of a Poisson Lie Group 155

4.7 Poisson Actions 155

4.8 Momentum Mapping 158

4.9 Dressing Transformations 158

4.10 Bibliographical Note 162

Appendix 1. The ‘Big Bracket’ and Its Applications 162

Appendix 2. The Poisson Calculus and Its Applications 165

**Analytic and Asymptotic Methods
for Nonlinear Singularity Analysis:**

A Review and Extensions of Tests for the Painlevé Property

M.D. Kruskal, N. Joshi, R. Halburd 175

1 Introduction 175

2 Nonlinear-Regular-Singular Analysis 180

2.1 The Painlevé Property 182

2.2 The α -Method 185

2.3 The Painlevé Test 187

2.4 Necessary versus Sufficient Conditions for the Painlevé Property . 191

2.5 A Direct Proof of the Painlevé Property for ODEs 192

2.6 Rigorous Results for PDEs 194

3 Nonlinear-Irregular-Singular Point Analysis 195

3.1 The Chazy Equation 196

3.2 The Bureau Equation 199

4 Coalescence Limits 201

Eight Lectures on Integrable Systems

F. Magri, P. Casati, G. Falqui, M. Pedroni 209

1st Lecture: Bihamiltonian Manifolds 210

2nd Lecture: Marsden–Ratiu Reduction 215

3rd Lecture: Generalized Casimir Functions 220

4th Lecture: Gel’fand–Dickey Manifolds 225

5th Lecture: Gel’fand–Dickey Equations 229

6th Lecture: KP Equations 235

7th Lecture: Poisson–Nijenhuis Manifolds 241

8th Lecture: The Calogero System 247

Bilinear Formalism in Soliton Theory

J. Satsuma 251

1 Introduction 251

2 Hirota’s Method 252

3 Algebraic Identities 255

4 Extensions 259

 4.1 q -Discrete Toda Equation 259

 4.2 Trilinear Formalism 260

 4.3 Ultra-discrete Systems 263

5 Concluding Remarks 266

Quantum and Classical Integrable Systems

M.A. Semenov-Tian-Shansky 269

1 Introduction 269

2 Generalities 271

 2.1 Basic Theorem: Linear Case 273

 2.2 Two Examples 283

3 Quadratic Case 291

 3.1 Abstract Case: Poisson Lie Groups
 and Factorizable Lie Bialgebras 293

 3.2 Duality Theory for Poisson Lie Groups
 and Twisted Spectral Invariants 296

 3.3 Sklyanin Bracket on $G(z)$ 304

4 Quantization 305

 4.1 Linear Case 305

 4.2 Quadratic Case. Quasitriangular Hopf Algebras 319

Introduction

The Editors

Nonlinear systems model all but the simplest physical phenomena. In the classical theory, the tools of Poisson geometry appear in an essential way, while for quantum systems, the representation theory of Lie groups and algebras, and of the infinite-dimensional loop and Kac-Moody algebras are basic. There is a class of nonlinear systems which are integrable, and the methods of solution for these systems draw on many fields of mathematics. They are the subject of the lectures in this book.

There is both a continuous and a discrete version of the theory of integrable systems. In the continuous case, one has to study either systems of ordinary differential equations, in which case the tools are those of finite-dimensional differential geometry, Lie algebras and the Painlevé test – the prototypical example is that of the Toda system –, or partial differential equations, in which case the tools are those of infinite-dimensional differential geometry, loop algebras and the generalized Painlevé test – the prototypical examples are the Korteweg-de Vries equation (KdV), the Kadomtsev-Petviashvili equation (KP) and the nonlinear Schrödinger equation (NLS). In the discrete case there appear discretized operators, which are either differential-difference operators, or difference operators, and the tools for studying them are those of q -analysis.

At the center of the theory of integrable systems lies the notion of a Lax pair, describing the isospectral deformation of a linear operator, a matrix differential operator, usually depending on a parameter, so that the Lax operator takes values in a loop algebra or a loop group. A Lax pair (L, M) is such that the time evolution of the Lax operator, $\dot{L} = [L, M]$, is equivalent to the given nonlinear system. The study of the associated linear problem $L\psi = \lambda\psi$ can be carried out by various methods.

In another approach to integrable equations, a given nonlinear system is written as a Hamiltonian dynamical system with respect to some Hamiltonian structure on the underlying phase-space. (For finite-dimensional manifolds, the term “Poisson structure” is usually preferred, that of “Hamiltonian structure” being more frequently applied to the infinite-dimensional case.) For finite-dimensional Hamiltonian systems on a symplectic manifold (a Poisson manifold with a nondegenerate Poisson tensor) of dimension $2n$, integrability in the sense of Liouville (1855) and Arnold (1974) is defined by the requirement that there exist n conserved quantities that are functionally independent on a dense open set and in involution, *i. e.*, whose pairwise Poisson brackets vanish. Geometric methods are then applied in various ways.

1 Analytic Methods

The inverse scattering method (ISM), using the inverse scattering transform (IST), is closely related to the Riemann-Hilbert factorization problem and to the $\bar{\partial}$ method. This is the subject of M.J. Ablowitz's survey, "Nonlinear waves, solitons and IST", which treats IST for equations both in one space variable, $(1 + 1)$ -dimensional problems, and in 2 space variables, $(2 + 1)$ -dimensional problems, and whose last section contains a review of recent work on the self-dual Yang-Mills equations (SDYM) and their reductions to integrable systems.

2 Painlevé Analysis

In the Painlevé test for an ordinary differential equation, the time variable is complexified. If all movable critical points of the solutions are poles, the equation passes the test. It contributes to the determination of the integrability or non-integrability of nonlinear equations, defined in terms of their solvability by means of an associated linear problem. In the Ablowitz-Ramani-Segur method for the detection of integrability, the various ordinary differential equations that arise as reductions of a given nonlinear partial differential equation are tested for the Painlevé property.

In their survey, "Analytic and asymptotic methods for nonlinear singularity analysis", M.D. Kruskal, N. Joshi and R. Halburd review the Painlevé property and its generalizations, the various methods of singularity analysis, and recent developments concerning irregular singularities and the preservation of the Painlevé property under asymptotic limits.

The review by B. Grammaticos and A. Ramani, "Integrability", describes the various definitions of integrability, their comparison and implementation for both finite- and infinite-dimensional systems, and for both continuous and discrete systems, including some recent results obtained in collaboration with K.M. Tamizhmani. The method of singularity confinement, a discrete equivalent of the Painlevé method, is explained and applied to the discrete analogues of the Painlevé equations.

3 τ -functions, Bilinear and Trilinear Forms

Hirota's method is the most efficient known for the determination of soliton and multi-soliton solutions of integrable equations. Once the equation is written in bilinear form in terms of a new dependent variable, the τ -function, and of Hirota's bilinear differential operators, multi-soliton solutions of the original nonlinear equation are obtained by combining soliton solutions. J. Hietarinta's "Introduction to the Hirota bilinear method" is an outline of the method with examples, while J. Satsuma's "Bilinear formalism in soliton theory" develops the theory further, treats the bilinear identities satisfied by the τ -functions, and shows how the method can be generalized to a trilinear formalism valid

for multi-dimensional extensions of the soliton equations, to the q -discrete and ultra-discrete cases and how it can be applied to the study of cellular automata.

4 Lie-Algebraic and Group-Theoretical Methods

When the Poisson brackets of the matrix elements of the Lax matrix, viewed as linear functions on a Lie algebra of matrices, can be expressed in terms of a so-called “ r -matrix”, the traces of powers of the Lax matrix are in involution, and in many cases the integrability of the original nonlinear system follows. It turns out that a Lie algebra equipped with an “ r -matrix” defining a Poisson bracket, e.g., satisfying the classical or modified Yang-Baxter equation (CYBE, MYBE) is a special case of a Lie bialgebra, the infinitesimal object associated with a Lie group equipped with a Poisson structure compatible with the group multiplication, called a Poisson Lie group. Poisson Lie groups play a role in the solution of equations on a 1-dimensional lattice, and they are the ingredients of the geometric theory of the dressing transformations for wave functions satisfying a zero-curvature equation under elements of the “hidden symmetry group”. The quantum version of these objects, quantum R -matrices satisfying the quantum Yang-Baxter equation (QYBE), and quantum groups are the ingredients of the quantum inverse scattering method (QISM), while the Bethe Ansatz, constructing eigenvectors for a quantum Hamiltonian by applying creation operators to the vacuum, can be interpreted in terms of the representation theory of quantum groups associated with Kac-Moody algebras.

The lectures by Y. Kosmann-Schwarzbach, “Lie bialgebras, Poisson Lie groups and dressing transformations”, are an exposition, including the proofs of all the main results, of the theory of Lie bialgebras, classical r -matrices, Poisson Lie groups and Poisson actions.

The survey by M.A. Semenov-Tian-Shansky, “Quantum and classical integrable systems”, treats the relation between the Hamiltonians of a quantum system solvable by the quantum inverse scattering method and the Casimir elements of the underlying hidden symmetry algebra, itself the universal enveloping algebra of a Kac-Moody algebra or a q -deformation of such an algebra, leading to deep results on the spectrum and the eigenfunctions of the quantum system. This study is preceded by that of the analogous classical situation which serves as a guide to the quantum case and utilizes the full machinery of classical r -matrices and Poisson Lie groups, and the comparison between the classical and the quantum cases is explicitly carried out.

5 Bihamiltonian Structures

When a dynamical system can be written in Hamiltonian form with respect to two Hamiltonian structures, which are compatible, in the sense that the sum of the corresponding Poisson brackets is also a Poisson bracket, this dynamical system possesses conserved quantities in involution with respect to both Poisson

brackets. This fundamental idea, due to F. Magri, is the basis of “Eight lectures on integrable systems”, by F. Magri, P. Casati, G. Falqui, and M. Pedroni, where they develop the geometry of bihamiltonian manifolds and various reduction theorems in Poisson geometry, before applying the results to the theory of both infinite- and finite-dimensional soliton equations. They show which reductions yield the Gelfand-Dickey and the Kadomtsev-Petviashvili equations, and they derive the bihamiltonian structure of the Calogero system.

The surveys included in this volume treat many aspects of the theory of nonlinear systems, they are different in spirit but not unrelated. For example, there is a parallel, which deserves further explanation, between the role of q -analysis in the theory of discrete integrable systems and that of q -deformations of algebras of functions on Lie groups and of universal enveloping algebras of Lie algebras in the theory of quantum integrable systems, while the r -matrix method for classical integrable systems on loop algebras, which seems to be purely algebraic, is in fact an infinitesimal version of the Riemann-Hilbert factorization problem.

The theory of nonlinear systems, and in particular of integrable systems, is related to several very active fields of theoretical physics. For instance, the role played in the theory of integrable systems by infinite Grassmannians (on which the τ -function “lives”), the boson-fermion correspondence, the representation theory of W -algebras, the Virasoro algebra in particular, all show links with conformal field theory.

We hope that this book will permit the reader to study some of the many facets of the theory of nonlinear systems and their integrability, and to follow their future developments, both in mathematics and in theoretical physics.

Nonlinear Waves, Solitons, and IST

M.J. Ablowitz

Department of Applied Mathematics, Campus Box 526, University of Colorado at Boulder, Boulder Colorado 80309-0526, USA
markjab@newton.colorado.edu

Abstract. These lectures are written for a wide audience with diverse backgrounds. The subject is approached from a general perspective and overly detailed discussions are avoided. Many of the topics require only a standard background in applied mathematics.

The lectures deal with the following topics: fundamentals of linear and nonlinear wave motion; isospectral flows with associated compatible linear systems including PDE's in 1+1 and 2+1 dimensions, with remarks on differential-difference and partial difference equations; the Inverse Scattering Transform (IST) for decaying initial data on the infinite line for problems in 1+1 dimensions; IST for 2+1 dimensional problems; remarks on self-dual Yang-Mills equations and their reductions. The first topic is extremely broad, but a brief review provides motivation for the other subjects covered in these lectures.

1 Fundamentals of Waves

Water waves are an interesting physical model and a natural way for us to begin our discussion. Consequently let us consider the equations of water waves for an irrotational, incompressible, inviscid fluid:

$$\nabla^2 \phi = 0 \quad \text{in} \quad -h < z < \eta \quad (1.1)$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -h \quad (1.2)$$

$$\frac{\partial \eta}{\partial t} + \nabla \phi \cdot \nabla \eta = \frac{\partial \phi}{\partial z} \quad \text{on} \quad z = \eta \quad (1.3)$$

$$\frac{\partial \phi}{\partial t} + g\eta + \frac{1}{2} |\nabla \phi|^2 = 0 \quad \text{on} \quad z = \eta, \quad (1.4)$$

where η denotes the free surface, and, since the fluid is ideal, the velocity is derivable from a potential, $\bar{u} = \nabla \phi$. For simplicity, we shall assume waves in one dimension, $\eta = \eta(x, t)$, $\bar{u} = (u, w) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial z} \right)$, $\phi = \phi(x, z, t)$. It will be convenient for us to consider the linearized equations whereby we expand the free surface conditions (1.3), the kinematic equation of a free surface, and (1.4), the Bernoulli equation, around $z = 0$:

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{on} \quad z = 0, \quad (1.3a)$$