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(Eds.)

Integrability of Nonlinear Systems

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Preface

This second edition of *Integrability of Nonlinear Systems* is both streamlined and revised. The eight courses that compose this volume present a comprehensive survey of the various aspects of integrable dynamical systems. Another expository article in the first edition dealt with chaos: for this reason, as well as for technical reasons, it is not reprinted here. Several texts have been revised and others have been corrected or have had their bibliography brought up to date. The present edition will be a valuable tool for graduate students and researchers.

The first edition of this book, which appeared in 1997 as Lecture Notes in Physics 495, was the development of the lectures delivered at the International School on Nonlinear Systems which was held in Pondicherry (India) in January 1996, organized by CIMPA-Centre International de Mathématiques Pures et Appliquées/International Center for Pure and Applied Mathematics and Pondicherry University. In February 2003, another International School was held in Pondicherry, sponsored by CIMPA, UNESCO and the Pondicherry Government, dealing with *Discrete Integrable Systems*. The lectures of that school are now being edited as a volume in the Lecture Notes in Physics series by B. Grammaticos, Y. Kosmann-Schwarzbach and Thamizharasi Tamizhmani, and will constitute a companion volume to the essays presented here.

We are very grateful to the scientific editors of Springer-Verlag, Prof. Wolf Beiglböck and Dr. Christian Caron, who invited us to prepare a new edition. We acknowledge with thanks the renewed editorial advice of Dr. Bertram E. Schwarzbach, and we thank Miss Sandra Thoms for her expert help in the production of the book.

Paris, September 2003 The Editors

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Contents

Introduction

The Editors

Nonlinear systems model all but the simplest physical phenomena. In the classical theory, the tools of Poisson geometry appear in an essential way, while for quantum systems, the representation theory of Lie groups and algebras, and of the infinite-dimensional loop and Kac-Moody algebras are basic. There is a class of nonlinear systems which are integrable, and the methods of solution for these systems draw on many fields of mathematics. They are the subject of the lectures in this book.

There is both a continuous and a discrete version of the theory of integrable systems. In the continuous case, one has to study either systems of ordinary differential equations, in which case the tools are those of finite-dimensional differential geometry, Lie algebras and the Painlevé test – the prototypical example is that of the Toda system –, or partial differential equations, in which case the tools are those of infinite-dimensional differential geometry, loop algebras and the generalized Painlevé test – the prototypical examples are the Korteweg-de Vries equation (KdV), the Kadomtsev-Petviashvilii equation (KP) and the nonlinear Schrödinger equation (NLS). In the discrete case there appear discretized operators, which are either differential-difference operators, or difference operators, and the tools for studying them are those of q-analysis.

At the center of the theory of integrable systems lies the notion of a Lax pair, describing the isospectral deformation of a linear operator, a matrix differential operator, usually depending on a parameter, so that the Lax operator takes values in a loop algebra or a loop group. A Lax pair (L, M) is such that the time evolution of the Lax operator, $\dot{L} = [L, M]$, is equivalent to the given nonlinear system. The study of the associated linear problem $L\psi = \lambda\psi$ can be carried out by various methods.

In another approach to integrable equations, a given nonlinear system is written as a Hamiltonian dynamical system with respect to some Hamiltonian structure on the underlying phase-space. (For finite-dimensional manifolds, the term "Poisson structure" is usually preferred, that of "Hamiltonian structure" being more frequently applied to the infinite-dimensional case.) For finite-dimensional Hamiltonian systems on a symplectic manifold (a Poisson manifold with a nondegenerate Poisson tensor) of dimension $2n$, integrability in the sense of Liouville (1855) and Arnold (1974) is defined by the requirement that there exist n conserved quantities that are functionnally independent on a dense open set and in involution, *i. e.*, whose pairwise Poisson brackets vanish. Geometric methods are then applied in various ways.

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1 Analytic Methods

The inverse scattering method (ISM), using the inverse scattering transform (IST), is closely related to the Riemann-Hilbert factorization problem and to the $\bar{\partial}$ method. This is the subject of M.J. Ablowitz's survey, "Nonlinear waves, solitons and IST", which treats IST for equations both in one space variable, $(1 + 1)$-dimensional problems, and in 2 space variables, $(2 + 1)$-dimensional problems, and whose last section contains a review of recent work on the self-dual Yang-Mills equations (SDYM) and their reductions to integrable systems.

2 Painlevé Analysis

In the Painlevé test for an ordinary differential equation, the time variable is complexified. If all movable critical points of the solutions are poles, the equation passes the test. It contributes to the determination of the integrability or non-integrability of nonlinear equations, defined in terms of their solvability by means of an associated linear problem. In the Ablowitz-Ramani-Segur method for the detection of integrability, the various ordinary differential equations that arise as reductions of a given nonlinear partial differential equation are tested for the Painlevé property.

In their survey, "Analytic and asymptotic methods for nonlinear singularity analysis", M.D. Kruskal, N. Joshi and R. Halburd review the Painlevé property and its generalizations, the various methods of singularity analysis, and recent developments concerning irregular singularities and the preservation of the Painlevé property under asymptotic limits.

The review by B. Grammaticos and A. Ramani, "Integrability", describes the various definitions of integrability, their comparison and implementation for both finite- and infinite-dimensional systems, and for both continuous and discrete systems, including some recent results obtained in collaboration with K.M. Tamizhmani. The method of singularity confinement, a discrete equivalent of the Painlevé method, is explained and applied to the discrete analogues of the Painlevé equations.

3 τ-functions, Bilinear and Trilinear Forms

Hirota's method is the most efficient known for the determination of soliton and multi-soliton solutions of integrable equations. Once the equation is written in bilinear form in terms of a new dependent variable, the τ-function, and of Hirota's bilinear differential operators, multi-soliton solutions of the original nonlinear equation are obtained by combining soliton solutions. J. Hietarinta's "Introduction to the Hirota bilinear method" is an outline of the method with examples, while J. Satsuma's "Bilinear formalism in soliton theory" develops the theory further, treats the bilinear identities satisfied by the τ-functions, and shows how the method can be generalized to a trilinear formalism valid for multi-dimensional extensions of the soliton equations, to the q-discrete and ultra-discrete cases and how it can be applied to the study of cellular automata.

4 Lie-Algebraic and Group-Theoretical Methods

When the Poisson brackets of the matrix elements of the Lax matrix, viewed as linear functions on a Lie algebra of matrices, can be expressed in terms of a so-called "r-matrix", the traces of powers of the Lax matrix are in involution, and in many cases the integrability of the original nonlinear system follows. It turns out that a Lie algebra equipped with an "r-matrix" defining a Poisson bracket, e.g., satisfying the classical or modified Yang-Baxter equation (CYBE, MYBE) is a special case of a Lie bialgebra, the infinitesimal object associated with a Lie group equipped with a Poisson structure compatible with the group multiplication, called a Poisson Lie group. Poisson Lie groups play a role in the solution of equations on a 1-dimensional lattice, and they are the ingredients of the geometric theory of the dressing transformations for wave functions satisfying a zero-curvature equation under elements of the "hidden symmetry group". The quantum version of these objects, quantum R-matrices satisfying the quantum Yang-Baxter equation (QYBE), and quantum groups are the ingredients of the quantum inverse scattering method (QISM), while the Bethe Ansatz, constructing eigenvectors for a quantum Hamiltonian by applying creation operators to the vacuum, can be interpreted in terms of the representation theory of quantum groups associated with Kac-Moody algebras.

The lectures by Y. Kosmann-Schwarzbach, "Lie bialgebras, Poisson Lie groups and dressing transformations", are an exposition, including the proofs of all the main results, of the theory of Lie bialgebras, classical r-matrices, Poisson Lie groups and Poisson actions.

The survey by M.A. Semenov-Tian-Shansky, "Quantum and classical integrable systems", treats the relation between the Hamiltonians of a quantum system solvable by the quantum inverse scattering method and the Casimir elements of the underlying hidden symmetry algebra, itself the universal enveloping algebra of a Kac-Moody algebra or a q-deformation of such an algebra, leading to deep results on the spectrum and the eigenfunctions of the quantum system. This study is preceded by that of the analogous classical situation which serves as a guide to the quantum case and utilizes the full machinery of classical r-matrices and Poisson Lie groups, and the comparison between the classical and the quantum cases is explicitly carried out.

5 Bihamiltonian Structures

When a dynamical system can be written in Hamiltonian form with respect to two Hamiltonian structures, which are compatible, in the sense that the sum of the corresponding Poisson brackets is also a Poisson bracket, this dynamical system possesses conserved quantities in involution with respect to both Poisson brackets. This fundamental idea, due to F. Magri, is the basis of "Eight lectures on integrable systems", by F. Magri, P. Casati, G. Falqui, and M. Pedroni, where they develop the geometry of bihamiltonian manifolds and various reduction theorems in Poisson geometry, before applying the results to the theory of both infinite- and finite-dimensional soliton equations. They show which reductions yield the Gelfand-Dickey and the Kadomtsev-Petviashvilii equations, and they derive the bihamiltonian structure of the Calogero system.

The surveys included in this volume treat many aspects of the theory of nonlinear systems, they are different in spirit but not unrelated. For example, there is a parallel, which deserves further explanation, between the role of q-analysis in the theory of discrete integrable sytems and that of q-deformations of algebras of functions on Lie groups and of universal enveloping algebras of Lie algebras in the theory of quantum integrable systems, while the r-matrix method for classical integrable systems on loop algebras, which seems to be purely algebraic, is in fact an infinitesimal version of the Riemann-Hilbert factorization problem.

The theory of nonlinear systems, and in particular of integrable systems, is related to several very active fields of theoretical physics. For instance, the role played in the theory of integrable systems by infinite Grassmannians (on which the τ-function "lives"), the boson-fermion correspondence, the representation theory of W-algebras, the Virasoro algebra in particular, all show links with conformal field theory.

We hope that this book will permit the reader to study some of the many facets of the theory of nonlinear systems and their integrability, and to follow their future developments, both in mathematics and in theoretical physics.

Nonlinear Waves, Solitons, and IST

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Abstract. These lectures are written for a wide audience with diverse backgrounds. The subject is approached from a general perspective and overly detailed discussions are avoided. Many of the topics require only a standard background in applied mathematics.

The lectures deal with the following topics: fundamentals of linear and nonlinear wave motion; isospectral flows with associated compatible linear systems including PDE's in 1+1 and 2+1 dimensions, with remarks on differential-difference and partial difference equations; the Inverse Scattering Transform (IST) for decaying initial data on the infinite line for problems in 1+1 dimensions; IST for 2+1 dimensional problems; remarks on self-dual Yang-Mills equations and their reductions. The first topic is extremely broad, but a brief review provides motivation for the other subjects covered in these lectures.

1 Fundamentals of Waves

Water waves are an interesting physical model and a natural way for us to begin our discussion. Consequently let us consider the equations of water waves for an irrotational, incompressible, inviscid fluid:

$$\nabla^2\phi = 0 \quad \text{in} \quad -h < z < \eta \tag{1.1}$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = -h \tag{1.2}$$

$$\frac{\partial\eta}{\partial t} + \nabla\phi\cdot\nabla\eta = \frac{\partial\phi}{\partial z} \quad \text{on} \quad z = \eta \tag{1.3}$$

$$\frac{\partial\phi}{\partial t} + g\eta + \frac{1}{2}|\nabla\phi|^2 = 0 \quad \text{on} \quad z = \eta\ , \tag{1.4}$$

where η denotes the free surface, and, since the fluid is ideal, the velocity is derivable from a potential, $\bar{u} = \nabla\phi$. For simplicity, we shall assume waves in one dimension, $\eta = \eta(x,t)$, $\bar{u} = (u,w) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial z}\right)$, $\phi = \phi(x,z,t)$. It will be convenient for us to consider the linearized equations whereby we expand the free surface conditions (1.3), the kinematic equation of a free surface, and (1.4), the Bernoulli equation, around $z = 0$:

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z} \quad \text{on} \quad z = 0\ , \tag{1.3a}$$

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