Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1260

Nicolae H. Pavel

Nonlinear Evolution Operators and Semigroups

Applications to Partial Differential Equations

Springer-Verlag

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1260

Nicolae H. Pavel

Nonlinear Evolution Operators and Semigroups

Applications to Partial Differential Equations

Springer-Verlag

Berlin Heidelberg New York London Paris Tokyo

Author

Nicolae H. Pavel
Universitatea Iaşi, Facultatea de Matematică
6600 Iaşi, Romania
and
The Ohio State University, Department of Mathematics
231 West 18th Avenue, Columbus, OH 43210, USA

Mathematics Subject Classification (1980): Primary: 35A07, 35A35, 35B45, 35C99, 47H09
Secondary: 34G20, 39A10, 65J15

ISBN 3-540-17974-7 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-17974-7 Springer-Verlag New York Berlin Heidelberg

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in other ways, and storage in data banks. Duplication of this publication or parts thereof is only permitted under the provisions of the German Copyright Law of September 9, 1965, in its version of June 24, 1985, and a copyright fee must always be paid. Violations fall under the prosecution act of the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1987
Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

Lecture Notes in Mathematics

For information about Vols. 1–1062 please contact your bookseller or Springer-Verlag.

Vol. 1063: Orienting Polymers. Proceedings, 1983. Edited by J. L. Ericksen. VII, 166 pages. 1984.

Vol. 1064: Probability Measures on Groups VII. Proceedings, 1983. Edited by H. Heyer. X, 588 pages. 1984.

Vol. 1065: A. Cuyt, Padé Approximants for Operators: Theory and Applications. IX, 138 pages. 1984.

Vol. 1066: Numerical Analysis. Proceedings, 1983. Edited by D. F. Griffiths. XI, 275 pages. 1984.

Vol. 1067: Yasuo Okuyama, Absolute Summability of Fourier Series and Orthogonal Series. VI, 118 pages. 1984.

Vol. 1068: Number Theory, Noordwijkerhout 1983. Proceedings. Edited by H. Jager. V, 296 pages. 1984.

Vol. 1069: M. Kreck, Bordism of Diffeomorphisms and Related Topics. III, 144 pages. 1984.

Vol. 1070: Interpolation Spaces and Allied Topics in Analysis. Proceedings, 1983. Edited by M. Cwikel and J. Peetre. III, 239 pages. 1984.

Vol. 1071: Padé Approximation and its Applications, Bad Honnef 1983. Prodeedings. Edited by H. Werner and H. J. Bünger. VI, 264 pages. 1984.

Vol. 1072: F. Rothe, Global Solutions of Reaction-Diffusion Systems. V, 216 pages. 1984.

Vol. 1073: Graph Theory, Singapore 1983. Proceedings. Edited by K. M. Koh and H. P. Yap. XIII, 335 pages. 1984.

Vol. 1074: E. W. Stredulinsky, Weighted Inequalities and Degenerate Elliptic Partial Differential Equations. III, 143 pages. 1984.

Vol. 1075: H. Majima, Asymptotic Analysis for Integrable Connections with Irregular Singular Points. IX, 159 pages. 1984.

Vol. 1076: Infinite-Dimensional Systems. Proceedings, 1983. Edited by F. Kappel and W. Schappacher. VII, 278 pages. 1984.

Vol. 1077: Lie Group Representations III. Proceedings, 1982–1983. Edited by R. Herb, R. Johnson, R. Lipsman, J. Rosenberg. XI, 454 pages. 1984.

Vol. 1078: A. J. E. M. Janssen, P. van der Steen, Integration Theory. V, 224 pages. 1984.

Vol. 1079: W. Ruppert. Compact Semitopological Semigroups: An Intrinsic Theory. V, 260 pages. 1984

Vol. 1080: Probability Theory on Vector Spaces III. Proceedings, 1983. Edited by D. Szynal and A. Weron. V, 373 pages. 1984.

Vol. 1081: D. Benson, Modular Representation Theory: New Trends and Methods. XI, 231 pages. 1984.

Vol. 1082: C.-G. Schmidt, Arithmetik Abelscher Varietäten mit komplexer Multiplikation. X, 96 Seiten. 1984.

Vol. 1083: D. Bump, Automorphic Forms on GL (3,IR). XI, 184 pages. 1984.

Vol. 1084: D. Kletzing, Structure and Representations of Q-Groups. VI, 290 pages. 1984.

Vol. 1085: G. K. Immink, Asymptotics of Analytic Difference Equations. V, 134 pages. 1984.

Vol. 1086: Sensitivity of Functionals with Applications to Engineering Sciences. Proceedings, 1983. Edited by V. Komkov. V, 130 pages. 1984

Vol. 1087: W. Narkiewicz, Uniform Distribution of Sequences of Integers in Residue Classes. VIII, 125 pages. 1984.

Vol. 1088: A. V. Kakosyan, L. B. Klebanov, J. A. Melamed, Characterization of Distributions by the Method of Intensively Monotone Operators. X, 175 pages. 1984.

Vol. 1089: Measure Theory, Oberwolfach 1983. Proceedings. Edited by D. Kölzow and D. Maharam-Stone. XIII, 327 pages. 1984.

Vol. 1090: Differential Geometry of Submanifolds. Proceedings, 1984. Edited by K. Kenmotsu. VI, 132 pages. 1984.

Vol. 1091: Multifunctions and Integrands. Proceedings, 1983. Edited by G. Salinetti. V, 234 pages. 1984.

Vol. 1092: Complete Intersections. Seminar, 1983. Edited by S. Greco and R. Strano. VII, 299 pages. 1984.

Vol. 1093: A. Prestel, Lectures on Formally Real Fields. XI, 125 pages. 1984.

Vol. 1094: Analyse Complexe. Proceedings, 1983. Edité par E. Amar, R. Gay et Nguyen Thanh Van. IX, 184 pages. 1984.

Vol. 1095: Stochastic Analysis and Applications. Proceedings, 1983. Edited by A. Truman and D. Williams. V, 199 pages. 1984.

Vol. 1096: Théorie du Potentiel. Proceedings, 1983. Edité par G. Mokobodzki et D. Pinchon. IX, 601 pages. 1984.

Vol. 1097: R. M. Dudley, H. Kunita, F. Ledrappier, École d'Éte de Probabilités de Saint-Flour XII – 1982. Edité par P. L. Hennequin. X, 396 pages. 1984.

Vol. 1098: Groups – Korea 1983. Proceedings. Edited by A. C. Kim and B. H. Neumann. VII, 183 pages. 1984.

Vol. 1099: C. M. Ringel, Tame Algebras and Integral Quadratic Forms. XIII, 376 pages. 1984.

Vol. 1100: V. Ivrii, Precise Spectral Asymptotics for Elliptic Operators Acting in Fiberings over Manifolds with Boundary. V, 237 pages. 1984.

Vol. 1101: V. Cossart, J. Giraud, U. Orbanz, Resolution of Surface Singularities. Seminar. VII, 132 pages. 1984.

Vol. 1102: A. Verona, Stratified Mappings – Structure and Triangulability. IX, 160 pages. 1984.

Vol. 1103: Models and Sets. Proceedings, Logic Colloquium, 1983, Part I. Edited by G. H. Müller and M. M. Richter. VIII, 484 pages. 1984.

Vol. 1104: Computation and Proof Theory. Proceedings, Logic Colloquium, 1983, Part II. Edited by M. M. Richter, E. Börger, W. Oberschelp, B. Schinzel and W. Thomas. VIII, 475 pages. 1984.

Vol. 1105: Rational Approximation and Interpolation. Proceedings, 1983. Edited by P. R. Graves-Morris, E. B. Saff and R. S. Varga. XII, 528 pages. 1984.

Vol. 1106: C. T. Chong, Techniques of Admissible Recursion Theory. IX, 214 pages. 1984.

Vol. 1107: Nonlinear Analysis and Optimization. Proceedings, 1982. Edited by C. Vinti. V, 224 pages. 1984.

Vol. 1108: Global Analysis – Studies and Applications I. Edited by Yu. G. Borisovich and Yu. E. Gliklikh. V, 301 pages. 1984.

Vol. 1109: Stochastic Aspects of Classical and Quantum Systems. Proceedings, 1983. Edited by S. Albeverio, P. Combe and M. Sirugue-Collin. IX, 227 pages. 1985.

Vol. 1110: R. Jajte, Strong Limit Theorems in Non-Commutative Probability. VI, 152 pages. 1985.

Vol. 1111: Arbeitstagung Bonn 1984. Proceedings. Edited by F. Hirzebruch, J. Schwermer and S. Suter. V, 481 pages. 1985.

Vol. 1112: Products of Conjugacy Classes in Groups. Edited by Z. Arad and M. Herzog. V, 244 pages. 1985.

Vol. 1113: P. Antosik, C. Swartz, Matrix Methods in Analysis. IV, 114 pages. 1985.

Vol. 1114: Zahlentheoretische Analysis. Seminar. Herausgegeben von E. Hlawka. V, 157 Seiten. 1985.

Vol. 1115: J. Moulin Ollagnier, Ergodic Theory and Statistical Mechanics. VI, 147 pages. 1985.

Vol. 1116: S. Stolz, Hochzusammenhängende Mannigfaltigkeiten und ihre Ränder. XXIII, 134 Seiten. 1985.

continued on page 289

Preface.

The first aim of this book is to present in a coherent way some of the fundamental results and recent research on nonlinear evolution operators and semigroups. The second aim is to show how to apply these abstract results to unify the treatment of several types of partial differential equations arising in physics (the heat equation, wave equation, Schrödinger equation, and so on).

The motivation of this theory is clearly pointed out in the following quotation from: Autumn Course on Semigroups, Theory and Applications, held at the International Centre For Theoretical Physics, Trieste (Italy), 12 November - 14 December 1984 (Brezis-Crandall- Kappel, Directors).

"The last two decades have witnessed a tremendous use of semigroups and evolution equations techniques in solving problems related to PDE and FDE. This allows the treatment of PDE and FDE as suitable ODE in infinite dimensional Banach spaces. This method has considerably simplified and clarified the the proofs, and has unified the treatment of several different classes of differential equations. It has solved many problems that had been left open by previously known methods, and has been very succesful in dealing with discontinuous data and regularity."

Chapter 1 deals with the construction and main properties of nonlinear evolution operator $U(t,s)$ associated with a class of nonlinear (possible multivalued) operators $A(t)$ with time dependent domain, satisfying Hypotheses $H(2.1)$ and $H(2.2)$ in Section 2. We also say that $U(t,s)$ is associated with the nonautonomous differential equation (inclusion) $x'(t) \in A(t)x(t)$. In the convergence of DS -approximate solutions (i.e., in the construction of $U(t,s)$) the fundamental estimate is given by (2.40), essentially due to Kobayashi, Kobayasi and Oharu. Among other general results, we mention Theorem 5.1 which gives a characterization of the compactness of evolution operators.

Note that $U(t,s)$ associated with the equation $x'(t) \in A(t)x(t)$ allows a unifying treatment of the existence, uniqueness and behaviour of the various types of solutions to the Cauchy problem for this equation.

Chapter 2 is devoted to nonlinear semigroups $S_A(t)$ which are generated by the DS -limit solutions associated with the dissipative operator A. In the case A - m-dissipative, $S_A(t)$ is given by the exponential formula of Crandall-Liggett. We say also that $S_A(t)$ is generated by A via the exponential formula. The semigroup approach is important in the study of the solutions of the autonomous differential equation $x' \in Ax$, which includes several different classes of PDE and FDE.

In order to avoid duplication and to reduce the length of this work, we have tried to make (as much as possible) the autonomous case as a special subcase of the time-dependent case (this was also a suggestion of the referee). Of course this is an economic way to present such a theory, but not the simplest one. For the sake of simplicity, the reader may start with the autonomous case.

In the theory of the generation of nonlinear semigroups, the fundamental estimate (given by (1.16)) due to Kobayashi, is derived from (2.40) in Chapter 1, i.e., from nonautonomous case.

In Chapter 3, one applies the results of Chapters 1 and 2, both to a class of multivalued evolution equations and to some partial differential equations modelling physical phenomena.

Most of the results here are presented for the first time in a book (e.g., Brezis' characterization of nonlinear compact semigroups in Chapter 2, the theory of nonlinear evolution operators in Chapter 1 and most of the material in Chapter 3. Some of the results are very recent and not yet published (e.g., the characterization of compactness of evolution operators given by the author, the characterization of compactness of a linear semigroup solely in terms of the resolvent of its infinitesimal generator due to Vrabie and so on).

此为试读，需要完整PDF请访问：www.ertongbook.com

The discussions (at the "Al.I.Cuza" University of Iasi - Romania) with my colleagues Prof. V. Barbu, C. Ursescu and I. I. Vrabie have contributed to the improvement of many sections in this book. I am expressing my thanks to all of them.

Part of this work has been written during my long stay at the International Centre for Theoretical Physics (ICTP) and SISSA, Trieste (Italy). I am very grateful to Professor Abdus Salam, Nobel Laureate, founder and the Director of the ICTP, for the pleasant hospitality and stimulating discussions.

I also wish to thank Professors A. Cellina, J. Eells, G. Vidossich and Dr. L. K. Shayo for stimulating my activity during my stay at the ICTP.

This book has been completed during my stay at The Ohio State University. I express my gratitude to Professors D. Burghelea, C. Corduneanu, J. Ferrar, T. Hallam, H. Moscovici, as well as to Ms. M. Howard, who have facilitated my activity after my arrival in the USA.

Finally, I wish to thank Springer-Verlag for their pleasant co-operation as well as Ms. Lidia Bogo and Terry England, for the professional typing of the manuscript.

Columbus, February 1987 Nicolae H. Pavel

CONTENTS

Preface

Chapter 1. NONLINEAR EVOLUTION OPERATORS

§1. Preliminaries. Discrete Schemes (DS) 1
§2. The convergence of DS-approximate solutions. 4
2.1. The time dependence of $A(t)$ 4
2.2. A remarkable estimate 7
§3. Convergence of DS-approximate solutions and generation of Evolution Operators. Integral solutions 14
3.1. Convergence of DS-approximate solutions 14
3.2. Integral solutions 17
3.3. Evolution Operator 23
3.4. Generation of evolution operator 28
3.5. Existence of DS-approximate solutions 29
3.6. The case $A(t)$ m-dissipative 32
3.7. Evolution operator and strong solutions 34
§4. Other properties of evolution operators 36
4.1. The estimate of the difference of two integral solutions 36
4.2. The relationship between $U(s+h,s)$ and $S(h)x$ 39
4.3. The quasi-autonomous case 41
§5. Compact evolution operators 43
5.1. Necessary conditions for compactness 43
5.2. The extension of Brezis' theorem 46
§6. Other types of t dependence of $A(t)$: conditions of Kato, Crandall-Pazy 49
§7. Range condition. Tangential condition 56

Chapter 2. NONLINEAR SEMIGROUPS

§1. Discrete schemes in autonomous case 61
§2. DS-limit solutions. Integral solutions Generation of nonlinear semigroups 64
§3. Other properties of nonlinear semigroups 68
3.1. Lipschitz continuity 68
3.2. The relationship of the semigroup $S_A(t)$ and $(I-tA)^{-1}$. The exponential formula of Crandall-Liggett 70
§4. A generalized domain for semigroup generators 74
§5. Strong solutions 79
5.1. A characterization of strong solution under "Range conditions" 79
5.2. Strong solutions on closed subsets in continuous case. Flow invariance 82
§6. Generation of compact semigroups 88
6.1. Nonlinear compact semigroups 88
6.2. Linear compact semigroups. Hille-Yosida Theorem 93
§7. Differential inclusions in Hilbert spaces. Smoothing effect of semigroups generated by Subdifferentials 103
§8. Some results of nonlinear perturbation theory 113
8.1. Perturbation of m-dissipativity 113
8.2. On the equation $y \in dx - Ax - Bx$ 116
8.3. Perturbation of maximal monotonicity 118
8.4. Continuous perturbation of compact semigroups generators. The case of subgradients 120
§9. Some results on the asymptotic behaviour of nonlinear semigroups 122

§10. Integral and strong solutions and smoothing effect in quasi-autonomous case 129
10.1. The general case $u' \in Au + f$ 129
10.2. The case $A = -\partial j$ 134
10.3. Solutions everywhere differentiable at the right 137
10.4. Some results on the asymptotic behaviour of solutions 142
§11. The correspondence $A \to S_A(t)$. The existence of accretive sets 148
Chapter 3. APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS
§1. A class of (nonlinear) evolution equations associated with time-dependent domain multivalued operatos 152
1.1. A general existence theory 152
1.2. The simplest proof of Peano's existence theorem 156
1.3. The behaviour of maximal (integral) solution as $t \uparrow t_{max}$ 156
§2. A class of partial differential equations with time-dependent domain, modelling long water waves of small amplitude. Evolution operators approach 159
2.1. Model equations for long waves in nonlinear dispersive systems 159
2.2. Weak solutions and classical solutions 167
§3. The semigroup approach to some partial differential equations in L^1 177
3.1. Phenomena that lead to Porous Medium Equation (PME) 177
3.1.1. Thermal conduction in ionized gases at height temperature 177
3.1.2. The flow of a gas in porous media 179
3.1.3. PME as a mathematical model for population dynamic 179
3.2. Some results on PME 182
§4. Other examples of m-dissipative operators and compact semigroups 188
§5. Partial differential Equations of Parabolic Type 202
5.1. The heat equation 202
5.2. Some nonlinear parabolic equations 208
5.3. A class of semilinear parabolic equations 213
§6. Wave Equation. Schrödinger Equation 227
6.1. Wave equation 227
6.2. The semilinear wave equations 229
6.3. Semilinear Schrödinger Equation 231
APPENDIX
§1. Duality mapping. Subdifferentials 234
§2. Dissipative Operators 246
§3. The regularization of a function 260
NOTES AND REFERENCES 266
REFERENCES 269
INDEX 284

Chapter 1

Nonlinear Evolution Operators

The aim of this chapter is to study the nonlinear evolution operators $U(t,s)$ associated with a class of nonlinear possible multivalued operators with time-dependent domain.

1. Preliminaries. Discrete Schemes (DS)

Let us consider the differential inclusion

$$u'(t) \in A(t)u(t), \quad s \leq t \leq T \tag{1.1}$$

with initial conditions

$$u(s) = x_o, \; x_o \in \overline{D(A(s))}, \tag{1.1}$$

where $A(t) : D(A(t)) \subset X \to 2^X$ is a time-dependent (possible multivalued) nonlinear operator acting in the real Banach space X with the time-dependent domain $D(A(t))$. (The equation (1.1) is said to be nonautonomous).

The conditions we shall impose on $A(t)$ (see (2.11)), allow even the closure $\overline{D(A(t))}$ of $D(A(t))$ to be time-dependent. We shall see that this is the case in some concrete situations (see Ch. 3, §2).

The key of the construction of $U(t,s)$ is the introduction of the "DS-approximate solution" u_n as the following step function

$$u_n(t) = \begin{cases} x_o^n, & \text{for } t = t_o^n = s \\ x_k^n, & \text{for } t \in]t_{k-1}^n, t_k^n] , \end{cases} \tag{1.2}$$

where n is a positive integer ($n \in N$), $k = o,1,\dots,N_n$ and $t_k^n \in [s,T]$ $x_k^n \in D(A(t_k^n))$ are defined in that follows. (Note that DS is the abbreviation of "Discrete Schemes").

Let $s,T \in R$ with $s < T$ and $x_o \in \overline{D(A(s))}$. Suppose that there is a partition P_n of $[s,T]$

$$P_n = \{ s = t_o^n, t_1^n, \ldots, t_{N_n-1}^n, t_{N_n}^n \}$$

with

$$s = t_o^n < t_1^n < \ldots < t_{N_n-1}^n < T = t_{N_n}^n \qquad (1.3)$$

and

$$d_n = \max_{1 \le k \le N_n} (t_k^n - t_{k-1}^n) \to o \text{ as } n \to \infty , \quad \omega d_n \le \tfrac{1}{2} \qquad (1.4)$$

(see (2.36))

Assume, in addition, that there are some elements $x_k^n \in D(A(t_k^n)) \subset X$ and $p_k^n \in X$ such that

$$y_k^n \equiv \frac{x_k^n - x_{k-1}^n}{t_k^n - t_{k-1}^n} - p_k^n \in A(t_k^n) x_k^n, \quad k = 1,2,\ldots,N_n \qquad (1.5)$$

$$x_o^n \in D(A(s)), \quad x_o^n \to x_o \quad \text{as } n \to \infty \qquad (1.6)$$

$$b_n = \sum_{k=1}^{N_n} (t_k^n - t_{k-1}^n) \, \|p_k^n\| \to o \text{ as } n \to \infty . \qquad (1.7)$$

In the situation $N_n = \infty$, $t_{N_n}^n \equiv T \equiv \lim_{k\to\infty} t_k^n$, and $u_n(T) = \lim_{k\to\infty} x_k^n$. We shall give conditions that guarantee the existence of such elements with the properties (1.3)-(1.7), having the additional property that the corresponding "DS-approximate solution" u_n is convergent to a continuous function $u = u(t;s,x_o)$ (called DS-limit solution to (1.1)+(1.1)').

Moreover, we will prove that u is well-defined (i.e. every DS-approximate solution u_n has the same limit u) and that the operator $U(t,s)$: $\overline{D(A(s))} \to \overline{D(A(t))}$ defined by

$$U(t,s)x_o \equiv u(t;s,x_o) = \lim_{n\to\infty} u_n(t), x_o \in \overline{D(A(s))}, s \le t \le T \qquad (1.8)$$

is an evolution operator (as in Section 3). Of course, we shall study the relationship of U with the "strong" solution to (1.1)+(1.1')(Section 3).

Various applications to some partial differential equations will be given in Chapter 3.

The DS-limit solution u

$$u(t;s,x_o) = \lim_{n \to \infty} u_n(t) \tag{1.9}$$

is also called "generalized solution", or "mild solution" (or still "weak solution") to (1.1)+(1.1').

The uniqueness of the mild solution is provable by means of the "Benilan uniqueness theorem" (Section 3).

Roughly speaking, the existence of "DS-approximate solutions" is guaranteed by the "Range condition"

$$R(I-hA(t+h)) \supset \overline{D(A(t))}, \quad o < h \leq h_o, \qquad s \leq t < T \tag{1.10}$$

for some small $h_o > 0$. In this case (1.5) holds with $p_k^n = 0 \in X$.

As we shall see in Section 7, a strictly more general condition than (1.10) is the following "tangential condition"

$$\lim_{h \downarrow o} \frac{1}{h} d[x;R(I-hA(t+h))] = o, \ \forall \ x \in \overline{D(A(t))}, \ s \leq t < T, \tag{1.11}$$

where $d[x;B]$ stands for the distance from $x \in X$ to the set $B \subset X$.

It is easy to check that

$$|d[x;B] - d[y;B]| \leq ||x-y|| , \quad x,y \in X, \tag{1.11'}$$

where $|r|$ is the absolute value of $r \in R$ and $||x||$ is the norm of $x \in X$.

Another important "tangential condition" with significant geometric intterpretation is the following one

$$\lim_{h \downarrow 0} \frac{1}{h} d[x+hA(t)x;D(A(t+h))] = 0, \ \forall \ x \in D(A(t)), \ s \leq t < T, \tag{1.12}$$

(where $A(t) : D(A(t)) \subset X \to X$ is now supposed to be single-valued).

The relationship between (1.11) and (1.12) will be pointed out later (see § 7). Now, we only mention that if $D(A(t))$ is closed and if $(t,x) \to A(t)x$ is continuous, then (1.12) implies (1.11).

此为试读,需要完整PDF请访问: www.ertongbook.com

The convergence of the sequence of DS-approximate solutions u_n is guaranteed by a condition on the t-dependence of A(t) (which implies that for each t, A(t) is dissipative). Such a condition is given in the next section. See (H.(2.1)).

Remark 1.1. The condition (1.11) (and respectively (1.12)) is important in applications. Thus, in the case of the (PDE) (2.1) in Chapter 3, (1.10) is not satisfied, but (1.11) holds. The condition (1.12) plays also a crucial role in the theory of the flow-invariance of a set with respect to a differential equation (Cf. Pavel [15]).

§2. The convergence of DS-approximate solutions.

2.1. The time dependence of A(t).

For the sake of selfcontainment we start with the introduction of the functions

$$\langle y,x\rangle_s = \lim_{h\downarrow o} \frac{\|x+hy\|^2 - \|x\|^2}{2h}, \quad x,y\in X \tag{2.1}$$

$$\langle y,x\rangle_+ = \lim_{h\downarrow o} \frac{\|x+hy\| - \|x\|}{h}, \quad x,y\in X \tag{2.2}$$

$$\langle y,x\rangle_i = \lim_{h\uparrow o} \frac{\|x+hy\|^2 - \|x\|^2}{2h}, \quad x,y\in X \tag{2.3}$$

$$\langle y,x\rangle_- = \lim_{h\uparrow o} \frac{\|x+hy\| - \|x\|}{h}, \quad x,y\in X. \tag{2.4}$$

These functions are well-defined since both $h \to \|x+hy\|^2$ and $h\to \|x+hy\|$ are real convex functions. For each $h \neq o$ and $x,y \in X$ set

$$\langle y,x\rangle_h = \frac{\|x+hy\| - \|x\|}{h}. \tag{2.5}$$

The following properties are obvious

$$\langle y,x\rangle_s = \|x\|\langle y,x\rangle_+, \quad \langle y,x\rangle_i = \|x\|\langle y,x\rangle_- \tag{2.6}$$

$$\langle y,x\rangle_+ \le \langle y,x\rangle_h \le \|y\|, \text{ if } h > o; \quad \langle y,-x\rangle_p = \langle -y,x\rangle_p \tag{2.7}$$

where $p = i$ or s,

$$\langle y,x\rangle_h \le \langle y,x\rangle_- , \text{ if } h < o, \quad \langle y,x\rangle_i \le \langle y,x\rangle_s \le \|x\|\ \|y\|. \tag{2.8}$$

Recall also the definition of the duality mapping $F:X \to X^*$ of X, i.e.,

$$F(x) = \{x^* \in X^* ;\ x^*(x) = \|x\|^2 = \|x^*\|^2\} , \quad x \in X, \qquad (2.9)$$

where X^* is the dual of X. The norm on X^* is denoted also by $\|\cdot\|$.

The result below is well-known.

Proposition 2.1. For each $x,y \in X$, there are $x_i^* \in F(x)$, $i = 1,2$, such that:

$$\begin{aligned} \langle y,x\rangle_s &= x_1^*(y) = \sup\{x^*(y);\ x^* \in F(x)\} , \\ \langle y,x\rangle_i &= x_2^*(y) = \inf\{x^*(y);\ x^* \in F(x)\} . \end{aligned} \qquad (2.10)$$

Here $x^*(y)$ denotes the value of $x^* \in X^*$ at $y \in X$. The proof of Proposition 2.1 is given in Appendix (Corollary 1.1 and Remark 1.1).

We are now prepared to introduce the basic hypothesis (H(2.1)) on the t-dependence of A(t).

(H(2.1)) - There exist $\omega \geq o$, a continuous function $f:]a,b[\to X$, and a bounded (on bounded subsets) function $L: R_+ \to R_+$ such that:

$$\langle y_1-y_2, x_1-x_2\rangle_i \leq \omega\|x_1-x_2\|^2 + \|f(t)-f(s)\| \ \|x_1-x_2\| L(\|x_2\|) \qquad (2.11)$$

for all $a < s \leq t \leq T < b$, $[x_1,y_1] \in A(t)$, $[x_2,y_2] \in A(s)$, $-\infty \leq a < b \leq +\infty$.

(H(2.2)) - The domain D(A(t)) of A(t) depends on $t \in [s,T]$ in the following sense:

If $t_n \uparrow t$ in $]s,T]$, $x_n \in D(A(t_n))$ and $x_n \to x$ in X, then $x \in \overline{D(A(t))}$.

Remark 2.1 If D(A(t)) is a closed set for each $t \in [s,T]$, then (H(2.2)) means that the mapping $t \to D(A(t))$ is closed.

Example 2.1. Hypotheses (H(2.1)) and (H(2.2)) do not imply, in general, that $\overline{D(A(t))}$ is independent of t. For example, let $X=R=]-\infty ,+\infty[$ and

$$A(t)x = \sqrt{x-t} + 1, \text{ with } D(A(t)) = [t,+\infty[\ = \overline{D(A(t))}, \ t \in R.$$

In this case (1.11) is equivalent with (1.12) which is satisfied because $R(I+hA(t)) \subset D(A(t+h)$ for all $h > o$. Clearly (2.11) holds since

$$(*) \qquad (A(t)x-A(s)y(x-y) \leq \sqrt{t-s}\,|x-y|, \quad x \geq t,\ y \geq s.$$

Examples in partial differential equations in which $\overline{D(A(t))}$ is also time-dependent are given in Chapter 3, § 2.

However, if $A(t)$ is m-dissipative for every $t\in[s,T]$, then $\overline{D(A(t))} = \overline{D}$ is necessarily independent on t (see Remark 4.2). Take for Example $\overline{D} = \overline{D(A(o))}$ (in this case). Obviously, the inequality (*) is stronger than (2.11) and corresponds to the case $\|f(t)-f(s)\| \leq \sqrt{t-s}$ (see (2.45)).

Remark 2.2. The notation $[x,y]\in A(t)$ means $x\in D(A(t))$ and $y\in A(t)x$. For $t=s$ Condition (2.11) implies the ω-dissipativity of $A(t)$, i.e.,

$$\langle y_1-y_2, x_1-y_2\rangle_i \leq \omega\|x_1-x_2\|^2, \quad [x_j,y_j]\in A(t), \tag{2.12}$$

$j = 1,2$, $t\in]a,b[$. Some details in this direction may be found in Appendix. Condition (H(2.1)) allows $\overline{D(A(t))}$ to be t-independent (see Example 2.1 above and Section 2 in Chapter 3).

In the theory of the convergence of DS-approximate solutions, the result below is essential.

Propostion 2.2. (1) the condition (2.11) is equivalent with

$$(1-\lambda\omega)\|x_1-x_2\| \leq \|x_1-x_2-\lambda(y_1-y_2)\|+\lambda\|f(t)-f(s)\|L(\|x_2\|) \tag{2.13}$$

for all $\lambda > o, a\leq s\leq t\leq T$, $[x_1,y_1]\in A(t)$, $[x_2,y_2]\in A(s)$.

(2) The inequality (2.13) implies

$$(\lambda+\mu-\lambda\mu\omega)\|x_1-x_2\| \leq \lambda\|x_2-\mu y_2-x_1\| + \mu\|x_1-\lambda y_1-x_2\| + $$
$$+ \lambda\mu\|f(t)-f(s)\|L(\|x_2\|) \tag{2.14}$$

for all $\lambda,\mu > o$, $a\leq s\leq t\leq T$, $[x_1,y_1]\in A(t)$, $[x_2,y_2]\in A(s)$, and (2.14) implies

$$(1-\lambda\omega)\|x_1-u\| \leq \|x_1-\lambda y_1-u\| + \lambda|A(s)u| + \lambda\|f(t)-f(s)\|L(\|u\|) \tag{2.15}$$

for all $\lambda > o$, $a < s \leq t \leq T$, $[x_1,y_1]\in A(t)$, $u\in D(A(s))$, where

$$|A(s)u| = \inf\{\|v\|\,;\, v\in A(s)u\}. \tag{2.16}$$

Remark 2.3. Inequality (2.14) is equivalent to

$$\langle y_1, x_1-x_2\rangle_i + \langle y_2, x_2-x_1\rangle_i \leq \omega\|x_1-x_2\|^2 + \|f(t)-f(s)\|\,\|x_1-x_2\|L(\|x_2\|) \tag{2.14'}$$

for all $a < s \leq t \leq T$, $[x_1, y_1] \in A(t)$, $[x_2, y_2] \in A(s)$ (see Appendix).

Proof of Proposition 2.2. (1) In view of Proposition 2.1 there is $x^* \in F(x_1 - x_2)$, such that

$$\langle y_1 - y_2,\ x_1 - x_2 \rangle_i = x^*(y_1 - y_2). \tag{2.17}$$

It is now easy to check that (2.11) implies (2.13). Indeed, by (2.11) and (2.15) we have

$$\|x_1 - x_2\|^2 = x^*(x_1 - x_2) = x^*(x_1 - x_2 - \lambda(y_1 - y_2)) + \lambda x^*(y_1 - y_2)$$
$$\leq \|x_1 - x_2\| \ \|x_1 - x_2 - \lambda(y_1 - y_2)\| + \lambda\omega \|x_1 - x_2\|^2 +$$
$$\lambda \|f(t) - f(s)\| L(\|x_2\|) \|x_1 - x_2\|$$

which yields (2.13). We now prove that (2.13) implies (2.11). To this goal, observe that (2.13) can be written in the form

$$\frac{\|x_1 - x_2 - \lambda(y_1 - y_2)\| - \|x_1 - x_2\|}{-\lambda} \leq \omega \|x_1 - x_2\| + \|f(t) - f(s)\| L(\|x_2\|). \tag{2.18}$$

In view of (2.6) we see that (2.18) implies (2.11). Similarly, we show that (2.11) implies (2.14), namely

$$(\lambda + \mu)\|x_1 - x_2\|^2 = \lambda x^*(x_1 - x_2) + \mu x^*(x_1 - x_2) =$$
$$= \mu x^*(x_1 - x_2 - \lambda y_1) - \lambda x^*(x_2 - x_1 - \mu y_2) + \lambda\mu x^*(y_1 - y_2). \tag{2.19}$$

Combining (2.11), (2.17) and (2.19), we get obviously (2.14). Finally, Triangular inequality, $x_2 = u$, $y_2 \in A(s)u$ and (2.14) imply clearly (subtracting $\lambda \|x_1 - x_2\|$ and then dividing by μ)

$$(1 - \lambda\omega)\|x_1 - u\| \leq \|x_1 - \lambda y_1 - u\| + \lambda \|y_2\| + \lambda \|f(t) - f(s)\| L(\|u\|),$$

$\forall y_2 \in A(s)u$, which yields (2.15). The proof is complete.

2.2. A remarkable estimate.

We now consider the discrete scheme $\{\hat{P}_m, \hat{x}_j^m, \hat{y}_j^m\} = DS$ corresponding to $\hat{s} \in [o, T]$ and $\hat{x}_o \in D(A(\hat{s}))$ is the sense of (1.2)-(1.7). Therefore

$$\hat{P}_m = \{\hat{s} = \hat{t}_o^m, \hat{t}_1^m, \ldots, \hat{t}_{N_{m-1}}^m, T\} \text{ with}$$

$$\hat{s} = \hat{t}^m_o < \hat{t}^m_1 < \ldots < \hat{t}^m_j < \ldots < \hat{t}^m_{N_{m-1}} < \hat{t}^m_{N_m} \equiv T\} , \quad m \in N$$

$$\hat{d}_m = \max_{1 \leq j \leq N_m} (\hat{t}^m_j - \hat{t}^m_{j-1}) \to o \text{ as } m \to \infty, \ \omega \hat{d}_m > \tfrac{1}{2} \qquad (2.19)'$$

$$\hat{y}^m_j = \frac{\hat{x}^m_j - \hat{x}^m_{j-1}}{\hat{t}^m_j - \hat{t}^m_{j-1}} - \hat{p}^m_j \in A(\hat{t}^m_j)\hat{x}^m_j , \quad j = 1,2,\ldots,\hat{N}_m \qquad (2.20)$$

$$\hat{x}^m_j \in D(A(\hat{t}^m_j)), \ j = o,1,\ldots, N_m, \quad \hat{x}^m_o \to \hat{x}_o \in \overline{D(A(\hat{s}))} \qquad (2.21)$$

$$\hat{b}_m = \sum_{j=1}^{N_m} (\hat{t}^m_j - \hat{t}^m_{j-1}) \|\hat{p}^m_j\| \to o \quad \text{as } m \to \infty . \qquad (2.22)$$

The DS-approximate solution $\hat{u}_m$ corresponding to the above discrete scheme is defined as u_n (see (1.2)), that is

$$u_m(t) = \begin{cases} \hat{x}^m_o , & t = \hat{t}^m_o = \hat{s} \\ \hat{x}^m_j , & t \in]\hat{t}^m_{j-1}, \hat{t}^m_j] . \end{cases} \qquad (2.23)$$

For simplicity of writing, set

$$h^n_k = t^n_k - t^n_{k-1}, \ \hat{h}^m_j = \hat{t}^m_j - \hat{t}^m_{j-1}, \ k=1,2,\ldots, N_n, \ j = 1,2,\ldots, \hat{N}_m. \qquad (2.24)$$

Then, by (1.6) and (2.20), we have

$$x^n_k - h^n_k y^n_k = x^n_{k-1} + h^n_k p^n_k, \quad \hat{x}^m_j - \hat{h}^m_j \hat{y}^m_j = \hat{x}^m_{j-1} + \hat{h}^m_j \hat{p}^m_j, \qquad (2.25)$$

with

$$y^n_k \in A(t^n_k)x^n_k, \ \hat{y}^m_j \in A(\hat{t}^m_j)\hat{x}^m_j, \ k = 1,2,\ldots, N_n, \ j=1,2,\ldots,\hat{N}_m.$$

It is also convenient to denote by

$$a_{k,j} = \|x^n_k - \hat{x}^m_j\|, \ d_{k,j} = \|f(t^n_k) - f(\hat{t}^m_j)\| \leq \rho(|t^n_k - \hat{t}^m_j|) \ \text{(see(2.38))}$$

$$\alpha_{k,j} = \hat{h}^m_j/(h^n_k + \hat{h}^m_j), \ \beta_{k,j} = h^n_k/(h^n_k + \hat{h}^m_j), \ \gamma_{k,j} = h^n_k \hat{h}^m_j/(h^n_k + \hat{h}^m_j) \qquad (2.26)$$

$$c_{k,j}(\eta) = [(t^n_k-\hat{t}^m_j-\eta)^2+d_n(t^n_k-s)+\hat{d}_m(\hat{t}^m_j-\hat{s})]^{1/2}, \quad 0 \le |\eta| < T. \tag{2.27}$$

The next simple lemma will play an essential role in the proof of the main estimates.

Lemma 2.1. The following inequality

$$\alpha_{k,j}\, c_{k-1,j}(\eta)+\beta_{k,j}c_{k,j-1}(\eta) \le c_{k,j}(\eta) \tag{2.28}$$

holds for $k = 1,2,\ldots,N_n$, $j=1,2,\ldots,\hat{N}_m$.

Proof. Since $\alpha_{k,j}+\beta_{k,j} = 1$ we have

$$\begin{aligned} I_1 &= \alpha_{k,j}c_{k-1,j}(\eta)+\beta_{k,j}c_{k,j-1}(\eta) \\ &\le (\alpha_{k,j}c^2_{k-1,j}(\eta)+\beta_{k,j}c^2_{k,j-1}(\eta))^{1/2} \end{aligned} \tag{2.29}$$

Here we have used the elementary inequality

$$a_1b_1+a_2b_2 \le (a_1^2+a_2^2)^{1/2}(b_1^2+b_2^2)^{1/2}$$

with $a_1 = (\alpha_{k,j})^{1/2}$, $a_2 = (\beta_{k,j})^{1/2}$, and so on.

For simplicity, and since there is no danger of confusion, in this proof we drop the indices m and n (i.e., we write $t^n_k = t_k$, $\hat{t}^m_j = \hat{t}_j$ and so on); Thus, according to the notations in (2.24) we have $t_{k-1}=t_k-h_k$, $\hat{t}_{j-1}=\hat{t}_j-\hat{h}_j$, and therefore

$$\begin{aligned} (t_{k-1}-\hat{t}_j-\eta)^2 &= (t_k-\hat{t}_j-\eta)^2-2h_k(t_k-\hat{t}_j-\eta)+h_k^2 \\ (t_k-\hat{t}_{j-1}-\eta)^2 &= (t_k-\hat{t}_j-\eta)^2+2\hat{h}_j(t_k-\hat{t}_j-\eta)+\hat{h}_j^2. \end{aligned} \tag{2.30}$$

Consequently

$$I_1^2 \le \frac{1}{h_k+\hat{h}_j}[\hat{h}_j(t_{k-1}-\hat{t}_k-\eta)^2+d_n(t_{k-1}-s)+d_m(\hat{t}_j-\hat{s}) +$$

$$\frac{1}{h_k+\hat{h}_j}\quad h_k(t_k-\hat{t}_{j-1}-\eta)^2+d_n(t_k-s)+\hat{d}_m(\hat{t}_{j-1}-\hat{s})] \quad =$$

$$(t_k-\hat{t}_j-\eta)^2+d_n(t_k-s)+\hat{d}_m(t_j-\hat{s})+ \frac{h_k\hat{h}_j}{h_k+\hat{h}_j}(h_k-d_n+\hat{h}_j-\hat{d}_m)$$

此为试读，需要完整PDF请访问: www.ertongbook.com