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Dedicated to the memory of

A. Jankowski

and

W. Pulikowski

PREFACE

The Symposium on Transformation Groups supported by the Adam Mickiewicz University in Poznań was held in Poznań, July 5-9, 1985. The symposium was dedicated to the memory of two of our teachers and friends, Andrzej Jankowski and Wojtek Pulikowski on the tenth anniversary of their deaths.

These proceedings contain papers presented at the symposium and also papers by mathematicians who were invited to the meeting but were unable to attend. All papers have been refereed and are in their final forms. We would like to express our gratitude to the authors and the many referees.

The participants and in particular the lecturers contributed to the success of the symposium and we are most grateful to all of them. Special thanks are due to our colleagues Ewa Marchow, Wojtek Gajda, Andrzej Gaszak, and Adam Neugebauer for their help with the organizational work and to Barbara Wilczyńska who handled the administrative and secretarial duties.

The second editor thanks Sonderforschungsbereich 170 in Göttingen for its hospitality which was very helpful in the preparation of the present volume. Finally, we would like to thank Marrie Powell and Christiane Gieseking for their excellent typing.

Stefan Jackowski

Krzysztof Pawałowski

Poznań/Warszawa, 20.06.1986

ANDRZEJ JANKOWSKI (1938-1975) WOJCIECH PULIKOWSKI (1947-1975)

Andrzej graduated in 1960 from the Nicolaus Copernicus University in Toruń. Topology was his passion and his interests were very broad. Andrzej worked on algebraic and differential topology, his main papers being concerned with operations in generalized cohomology theories and with formal groups. His was not an easy task. Andrzej worked essentially alone. Polish topologists were at that time continuing the tradition of their pre-war school. Andrzej's friend and Ph.D. student wrote*): "He wanted to understand the deepest and most difficult theorems found by his contemporaries. At that beautiful time of great discoveries Andrzej faced the difficult obstacle of being alone. He put a lot of effort into overcoming this difficulty, and also conveying his knowledge to others." Andrzej began to lecture on algebraic topology and to organize seminars as soon as he joined the University of Warsaw in 1962. For nine years, from 1967, he was the *spiritus movens* of the Summer School on Algebraic Topology held annually in Gdańsk. He moved to Gdańsk in 1971. From 1969 until his death he led a seminar on transformation groups. Wojtek Pulikowski was one of the participants.

Wojtek graduated in 1969 from Poznań and moved to Warszawa and then to Gdańsk. In 1973 Wojtek obtained his Ph.D. for the work on equivariant bordism theories indexed by representations and returned to Poznań. He invested great effort into organizing seminars, summer schools and meetings on various topics in algebraic and differential topology. At the same time he continued teaching his students and before long directed their research towards transformation groups. Wojtek was a born teacher, able to convey not only his knowledge but also his passion, enthusiasm, and interest in the subject. He wrote a number of papers on equivariant homology theories, but he spent most of his time in teaching - which he did with joy and love. His friends and students all owe him a great deal.

Besides mathematics, both Andrzej and Wojtek had another passion - mountains. And in the mountains both of them met their death in August 1975, Andrzej climbing the Tirach Mir peak in the Hindu Kush mountains and Wojtek in an accident in the Beskidy mountains in Poland.

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*) R.Rubinsztein: "Andrzej Jankowski (1938-1975)", Wiadomości Matematyczne, vol. XXIII (1980), pp. 85-91.

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Bounds on the torus rank

C. Allday and V. Puppe

For a topological space X let $rk_o(X) := \max\{\dim T$, where T is a torus which can act on X almost freely (i.e. with only finite isotropy subgroups)$\}$ be the torus rank of X . Stephen Halperin has raised the following question (s.[11]):

(H$\mathbb{Q}$) Is it true that $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) \geq 2^{rk_o(X)}$ for any simply connected reasonable space X ?

In this context "reasonable" (s. [11]) is a technical condition which assures that one can apply the A. Borel version of P.A. Smith theory (s.[4],[5],[12]) and Sullivan's theory of minimal models (s.[13],[10],[14]). In particular any connected finite CW-complex is certainly "reasonable", but X being connected, paracompact, finitistic (s.[5] p. 133) and of the rational homotopy type of a CW-complex would also suffice.

In the first section we give some lower bounds for $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q})$ if X allows an almost free action of an n-dimensional torus $G = T^n$. These results are obtained using only the additive structure in $H^*(X;Q)$ (and a version of the localization theorem (s. [2])) and hold for rather general spaces X , e.g. simply connectedness is not needed; but the bounds we get are far below the desired $2^{rk_o(X)}$.

The second section gives bounds on the torus rank in terms of the cohomology of X , where a very special structure of the cohomology ring $H^*(X;\mathbb{Q})$, i.e. X being a rational cohomology Kähler space, is used.

The third section is concerned with relations between properties of the minimal model M(X) of X (in particular the rational homotopy Lie algebra $L_*(X)$), $rk_o(X)$ and $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q})$. Halperin observed (s.[11], 1.5) that the results of [1], in particular the inequality $rk_o(X) \leq -\chi_\pi(X)$, where $\chi_\pi(X)$ is the rational homotopy Euler characteristic (s.[1], Theorem 1), implies an affirmative answer to his question if X is a homogenous space G/K, $K \subset G$ compact, connected Lie groups. Among other things we describe another class of spaces for which $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) \geq 2^{rk_o(X)}$ holds, but the bound on the torus rank given by the rational homotopy Euler characteristic is not sharp in many cases (compare also [11], 4.4) and does not suffice to answer (H$\mathbb{Q}$). Indeed, for this class the knowledge of the additive structure of $\pi_*(X) \otimes \mathbb{Q}$ is not enough; it is essential to use the Lie algebra structure of $L_*(X) \cong \pi_*(\Omega X) \otimes \mathbb{Q}$.

1. Let X be a connected, paracompact, finitistic space which has the rational homotopy type of a CW-complex and on which a torus $G = T^n$ of dimension n acts almost freely. If M(X) is the minimal model of X over the field $\mathbb{C}$ of complex numbers and $R := H^*(BG;\mathbb{C}) \cong \mathbb{C}[t_1,\dots,t_n]$, then the R-cochain algebra $C_G(X) := R \tilde{\otimes}_{\mathbb{C}} M(X)$ (where the twisting of the boundary, indicated by "~", reflects the G-action) is a model for the Borel construction X_G.

For any $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{C}^n$ we denote by $\mathbb{C}^\alpha$ the field $\mathbb{C}$ together with the R-algebra structure given by the evaluation map $\varepsilon^\alpha\colon R = \mathbb{C}[t_1,\dots,t_n] \to \mathbb{C}$, $t_i \longrightarrow \alpha_i$ for $i = 1,\dots,n$. The cochain algebra $C_G(X)^\alpha$ (over $\mathbb{C}$) is defined to be the tensor product $C_G(X)^\alpha := \mathbb{C}^\alpha \otimes_R C_G(X)$. Theorem (4.1) of [2] implies that $H^*(C_G(X)^\alpha) = 0$ for all $\alpha \neq 0$ (since the G-action is assumed to be almost free). It follows from a theorem of E.H. Brown (s. [7], (9.1), compare [2], (2.3)) that there exists a twisted boundary on $D_G(X) := R \tilde{\otimes} H^*(X;\mathbb{C})$ which makes $R \tilde{\otimes} H^*(X;\mathbb{C})$ homotopy equivalent to $R \tilde{\otimes} M(X)$ as R-cochain complexes. We therefore get (for an almost free action) that $H(D_G(X)^\alpha) = 0$ for all $\alpha \neq 0$ and we shall use this information to obtain the following proposition:

<u>(1.1) Proposition</u>: Under the above hypothesis one has

a) $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) \geq 2n$ for all $n = 1,2,\dots$

b) $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) \geq 2(n+1)$ for all $n \geq 3$

<u>Proof</u>: We can of course assume that $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q})$ is finite, and the fact that the action has no fixed point implies that the Euler characteristic $\chi(X)$ is zero. Let $x_1,\dots,x_k$ be a homogenous $\mathbb{Q}$-basis of $H^{ev}(X;\mathbb{Q})$ with $|x_1| \geq \dots > |x_k| = 0$ and $y_1,\dots,y_k$ a homogenous $\mathbb{Q}$-basis of $H^{odd}(X;\mathbb{Q})$ with $|y_1| \geq \dots \geq |y_k| > 0$ ($|\ |$ denotes degree). Since $|t_i| = 2$ for $i = 1,\dots,n$ the twisted boundary $\tilde{d}$ on $R \tilde{\otimes} H^*(X;\mathbb{C})$ is given by two $k \times k$-matrices $P = (p_{ij})$ and $Q = (q_{ij})$, where the entries p_{ij}, q_{ij} are homogenous polynomials in the variables $t_1,\dots,t_n$ of degree $\neq 0$, i.e.

$$\begin{aligned} \tilde{d}y_1 &= p_{11}x_1 + \dots + p_{1k}x_k & \qquad \tilde{d}x_1 &= q_{11}y_1 + \dots + q_{1k}y_k \\ &\ \vdots & &\ \vdots \\ \tilde{d}y_k &= p_{k1}x_1 + \dots + p_{kk}x_k & \qquad \tilde{d}x_k &= q_{k1}y_1 + \dots + q_{kk}y_k . \end{aligned}$$

If $p_{ij} \neq 0$ (resp. $q_{ij} \neq 0$) then $|y_i| > |x_j|$ and $|p_{ij}| = |y_i|-|x_j|+1$ (resp. $|x_i| > |y_j|$ and $|q_{ij}| = |x_i|-|y_j|+1$), in particular $\tilde{d}x_k \equiv 0$ (i.e. $q_{kj} \equiv 0$ for all $j = 1,\ldots,k$).

The equation $\tilde{d}\circ\tilde{d} = 0$ is equivalent to $PQ = QP = 0$ and the vanishing of $H(\mathcal{D}_G(X)^\alpha)$ for any $\alpha \in \mathbb{C}^n \setminus \{0\}$ then means that $\operatorname{rk} P(\alpha) + \operatorname{rk} Q(\alpha) = k$ for all $\alpha \in \mathbb{C}^n \setminus \{0\}$, where $\operatorname{rk} P(\alpha)$ denotes the rank of the $k\times k$-matrix over $\mathbb{C}$ obtained from P by evaluating the polynomials p_{ij} at the point $\alpha \in \mathbb{C}^n$ (similar for $\operatorname{rk} Q(x)$). The semi-continuity of $\operatorname{rk} P(\alpha)$ and $\operatorname{rk} Q(\alpha)$ (as a function of α)(together with $\operatorname{rk} P(\alpha) + \operatorname{rk} Q(\alpha) = k$ for $\alpha \neq 0$) then implies that $\operatorname{rk} P(\alpha)$ and $\operatorname{rk} Q(\alpha)$ have to be constant on $\mathbb{C}^n \setminus \{0\}$.

To prove part a) on only needs to observe that the variety $V(p_{1k},\ldots,p_{kk})$ can only consist of the point $0 \in \mathbb{C}^n$. If the polynomials p_{ik}, $i = 1,\ldots,k$ would have a common zero $\alpha \in \mathbb{C}^n \setminus \{0\}$ then "at the point α" the cycle x_k could not be a boundary and hence $H(\mathcal{D}_G(X)^\alpha)$ would not vanish. Since the p_{ik}, $i = 1,\ldots,k$ are k polynomials in n variable one gets $k \geq n$ (otherwise $V(p_{1k},\ldots,p_{kk}) \cap \mathbb{C}^n \setminus \{0\} \neq \emptyset$).

To get the slight improvement b) a considerably more involved argument is necessary:
We assume $k = n$ and will show that this implies $n \leq 2$.

<u>Case 1</u>: Let $|y_1| > |x_i|$ for all i, i.e. the top dimensional classes have odd degree. Again $V(p_{1n},\ldots,p_{nn}) = 0$ and it now follows that $p_{1n},\ldots,p_{nn}$ is a regular sequence in $R = \mathbb{C}[t_1,\ldots,t_n]$. Therefore the condition $QP = 0$ implies that all the q_{ij}'s are contained in the ideal $\langle p_{1n},\ldots,p_{nn}\rangle \subset R$ generated by p_{in}, $i = 1,\ldots,n$. (From $\sum_{j=1}^{n} q_{ij}p_{jn}=0$ it follows that the equivalence class of $q_{ij}\, p_{jn}$ in $R/(p_{1n},\ldots,\hat{p}_{jn},\ldots,p_{nn})$ is zero. The regularity of the sequence $p_{1n},\ldots,p_{nn}$ then implies that the class of q_{ij} is already zero in $R/(p_{1n},\ldots,\hat{p}_{jn},\ldots,p_{nn})$. Hence $q_{ij} \in (p_{1n},\ldots,\hat{p}_{jn},\ldots,p_{nn}) \subset (p_{1n},\ldots,p_{nn})$ for all $i,j = 1,\ldots,n$.) Since $|q_{ij}| = |x_i|-|y_j|+1 < |y_1|+1 = |p_{1n}|$ one actually has $q_{ij} \in (p_{2n},\ldots,p_{nn})$ for all $i,j = 1,\ldots,n$. (This is where we use the assumption that the top classes have odd degree and - as one sees from the above inequality - the weaker assumption "$|y_1|+|y_k|>|x_1|$" would suffice.) Choose $\alpha \in V(p_{2n},\ldots,p_{nn}) \cap (\mathbb{C}^n \setminus \{0\})$. Then $q_{ij}(\alpha) = 0$ for $i,j = 1,\ldots,n$ and hence $\operatorname{rk} Q(\alpha) = 0$. Since $\operatorname{rk} Q$ is constant on $\mathbb{C}^n \setminus \{0\}$ we get $Q \equiv 0$ and $\operatorname{rk} P$ must therefore be maximal ($= n$) on $\mathbb{C}^n \setminus \{0\}$. Since $\det P$ is a polynomial in the variables $t_1,\ldots,t_n$ this can only happen if $n = 1$.

<u>Case 2</u>: Let $|x_1| > |y_i|$ for all i, i.e. the top classes have even de-

gree. We have $q_{nj} \equiv 0$ for $j = 1,\dots,n$; $V(p_{1n},\dots,p_{nn}) = 0$, i.e. $p_{1n},\dots,p_{nn}$ is a regular sequence (as before), and in addition $p_{i1} \equiv 0$ for $i = 1,\dots,n$ (for degree reasons); $V(q_{11},\dots,q_{1n}) = 0$, i.e. $q_{11},\dots,q_{1n}$ is a regular sequence, since otherwise x_1 would give a non-zero element in $H(D_G(X)^{\alpha})$ for any $\alpha \in V(q_{11},\dots,q_{1n}) \cap (\mathbb{C}^n \setminus \{0\})$. Analogous to case 1 we get from $QP = PQ = 0$ that $q_{ij} \in (p_{1n},\dots,p_{nn})$ and $p_{ij} \in (q_{11},\dots,q_{1n})$ for all $i,j = 1,\dots,n$. In particular $(p_{1n},\dots,p_{nn}) = (q_{11},\dots,q_{1n})$. Since $|p_{1n}| > |p_{ij}|$ for all (i,j) with $j < n$ it follows that $p_{ij} \in (p_{2n},\dots,p_{nn})$ if $(i,j) \neq (1,n)$. For $n > 1$ choose $\alpha \in V(p_{2n},\dots,p_{nn}) \cap (\mathbb{C}^n \setminus \{0\})$, then $\operatorname{rk} P(\alpha) = 1$ and therefore $\operatorname{rk} Q(\alpha) = n-1$. This implies $\operatorname{rk} Q = n-1$ on $\mathbb{C}^n \setminus \{0\}$. Since $q_{nj} \equiv 0$ for all $j = 1,\dots,n$ the $(n-1) \times (n-1)$ minors $Q_1,\dots,Q_n$ of the matrix

$$\begin{matrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{n-11} & \cdots & q_{n-1n} \end{matrix}$$

have to form a regular sequence (Q_j is obtained by skipping the j-th column)

The expension formula for the determinant (with respect to the first row) of the matrix

$$\begin{matrix} q_{11} & \cdots & q_{1n} \\ q_{11} & \cdots & q_{1n} \\ q_{21} & \cdots & q_{2n} \\ \vdots & & \vdots \\ q_{n-11} & \cdots & q_{n-1n} \end{matrix}$$

gives $q_{11}\, Q_1 - q_{12}\, Q_2 + \dots + (-1)^{n+1}\, q_{1n}\, Q_n = 0$.

As above one gets $q_{ij} \in (Q_1,\dots,Q_n)$ for all $i,j = 1,\dots,n$. This is only possible if $(n-1) = 1$ (otherwise $|q_{11}| < |Q_j|$ for all j, which gives a contradiction). This finishes the proof of (1.1).

<u>(1.2) Corollary</u>: If X is a paracompact, finitistic space which has the rational homotopy type of a CW-complex, then

a) $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) \geq 2\, \operatorname{rk}_o(X)$

b) $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) \geq 2(\operatorname{rk}_o(X)+1)$, if $\operatorname{rk}_o(X) \geq 3$.

<u>(1.3) Remark</u>: G. Carlsson has asked the analogous question to (H$\mathbb{Q}$) concerning spaces X on which an elementary abelian 2-group $G = (\mathbb{Z}/_2)^n$ acts freely (such that X becomes a finite G-CW complex). Results of Carlsson [8] and - using different methods - of W. Browder [6] in this direction imply in particular, that if G acts trivially in cohomology with $\mathbb{Z}/2$ coefficients s. [8] (resp. $\mathbb{Z}_{(2)} = \mathbb{Z}$ localized at 2 s. [6]) then the cohomology of X (with the corresponding coefficients) is non-zero in at least n+1 different dimensions.

The methods used to prove proposition (1.1) above can be applied in a similar fashion to free $(\mathbb{Z}/_2)^n$-actions (compare [2], 2.). One then obtains for a finite, free $(\mathbb{Z}/_2)^n$-CW complex X with trivial action on $H^*(X;\mathbb{Z}/_2)$:

a) $\dim_{\mathbb{Z}/_2} H^*(X;\mathbb{Z}/_2) \geq n+1$ for all n

b) $\dim_{\mathbb{Z}/_2} H^*(X;\mathbb{Z}/_2) \geq n+2$ for $n \geq 2$

Combining our approach with Carlsson's result leads to

c) $\dim_{\mathbb{Z}/_2} H^*(X;\mathbb{Z}/_2) \geq 2n$ for all n.

Browder's methods also work for $G = (\mathbb{Z}/_p)^n$, p an odd prime and he proves results analogous to the case p = 2 also for odd primes. The above approach would also work for p an odd prime (compare [2], 3.) but there are some technical complication arising from the more complicated structure of $H^*(B(\mathbb{Z}/_p)^n; \mathbb{Z}/_p)$ in case p is odd.

2. Let X be a connected paracompact finitistic space which is a rational Poincaré duality space of formal dimension 2m. In this section H^* denotes sheaf (or Alexander-Spanier or Cech) cohomology and H^*_Δ denotes singular cohomology. X will be called an agreement space if the natural transformation $H^*(X;\mathbb{Q}) \to H^*_\Delta(X;\mathbb{Q})$ is an isomorphism (e.g. X reasonable, as above).

(2.1) Definition: (compare [4]) The space X is said to be a rational cohomology Kähler space (CKS) if

(i) there exists $\omega \in H^2(X;\mathbb{Q})$ such that ω^m is non-zero in $H^{2m}(X;\mathbb{Q})\cong\mathbb{Q}$; and

(ii) the cup-product with ω^j: $H^{m-j}(X;\mathbb{Q}) \to H^{m+j}(X;\mathbb{Q})$ is an isomorphism for $0 \leq j \leq m$.

(2.2) Theorem: If X is a CKS, then $rk_o(X) \leq \alpha_1(X)$:= the maximal number of algebraically independent elements in $H^1(X;\mathbb{Q})$. In particular, $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) \geq 2^{rk_o(X)}$. Furthermore if X is an agreement space and if a torus G acts almost-freely on X, then $H^*(X;\mathbb{Q})$ and $H^*(X/G;\mathbb{Q}) \otimes H^*(G;\mathbb{Q})$ are isomorphic as graded $\mathbb{Q}$-algebras.

Proof: Suppose $G = T^n$ acts almost-freely on X. Let $EG \to BG$ be a universal principal G-bundle, and let $X_G = (X\times EG)/G$ be the Borel construction. Let $E_r^{p,q}$ be the rational cohomology Leray-Serre spectral sequence of $X_G \to BG$; and let s be the rank of the linear map d_2: $E_2^{o,1} = H^1(X;\mathbb{Q}) \to E_2^{2,o} \cong H^2(BG;\mathbb{Q})$. Choose $y_1,\dots,y_s \in H^1(X;\mathbb{Q})$ such that $d_2(y_i) = a_i$, $1 \leq i \leq s$, are linearly independent. Then, for $1 \leq i_1 < \ldots < i_{j+1} \leq s$, $d_2(y_{i_1} \ldots y_{i_{j+1}}) = \sum_{k=1}^{j+1} \pm a_{i_k} \otimes y_{i_1} \ldots \hat{y}_{i_k} \ldots y_{i_{j+1}}$.

Hence it follows by induction that $y_1 \ldots y_s \neq 0$. I.e. $s \leq \alpha_1(X)$. Now let K be the subtorus such that the ideal $(a_1,\ldots,a_s) = \ker[H^*(BG;\mathbb{Q}) \to H^*(BK;\mathbb{Q})]$. In particular, dim K = n-s. In the Leray-Serre spectral sequence of $X_K \to BK$, then $d_2\colon E_2^{0,1} \to E_2^{2,0}$ is zero. Thus, by Blanchard ([3]), the spectral sequence collapses; and so $X^K \neq \emptyset$. So K is trivial, and $n = s \leq \alpha_1(X)$.

If X is an agreement space, then it follows from the fibre bundle $X \to X_G \to BG$ that X_G is an agreement space also. Now, above, $d_2\colon H^1(X;\mathbb{Q}) \to H^2(BG;\mathbb{Q})$ is onto, since G is acting almost-freely: hence $H^2(BG;\mathbb{Q}) \to H^2(X_G;\mathbb{Q})$ is zero. On the other hand $G \to X \times EG \to X_G$ is the pull-back of $G \to EG \to BG$ via $X_G \to BG$; in particular it is orientable with respect to H^*_Δ. Thus X (homotopy equivalent to $X \times EG$) has a K.S.-model of the form $M(X_G) \otimes \Lambda(s_1,\ldots,s_n)$, where $\deg(s_i) = 1$, $1 \leq i \leq n$, and $d(s_i) = 0$, $1 \leq i \leq n$ (since $H^2(BG;\mathbb{Q}) \to H^2(X_G;\mathbb{Q})$ is zero). Hence $H^*_\Delta(X;\mathbb{Q}) \cong H^*_\Delta(X_G;\mathbb{Q}) \otimes H^*_\Delta(G;\mathbb{Q})$ as algebras. So $H^*(X;\mathbb{Q}) \cong H^*(X/G;\mathbb{Q}) \otimes H^*(G;\mathbb{Q})$, as algebras, by the Vietoris-Begle mapping theorem.

(2.3) Remarks: (i) An argument similar to the above, applied to a simple closed connected subgroup, shows that no non-abelian compact connected Lie group can act almost-freely on a CKS: and only condition (i) of definition (2.1) is needed for this.

(ii) Again if we assume only condition (i) of definition (2.1), then we get $rk_0(X) \leq \beta_1(X) := \dim_{\mathbb{Q}} H^1(X;\mathbb{Q})$.

(iii) Theorem (2.2) is "best possible", since T^{2m} is a CKS, and it can act freely on itself.

3. Let X be a simply connected reasonable space and let $L_*(X) = \pi_*(\Omega X) \otimes \mathbb{Q}$ denote its rational homotopy Lie algebra. The following result is proved in [2], Theorem (4.6)':

If $L_i(X) = 0$ for all odd i, then $rk_0(X) \leq \dim_{\mathbb{Q}} Z\,L_*(X)$, where $Z\,L_*(X)$ denotes the centre of the Lie algebra $L_*(X)$.

This improves the bound given by $-\chi_\pi(X) = \dim L_*(X)$ for this type of spaces and it is clear that one does need some improvement in this direction to get an affirmative answer to (H$\mathbb{Q}$) since $\dim_{\mathbb{Q}} H^*(X;\mathbb{Q}) \leq 2^{\dim L_*(X)}$ in the case at hand and equality holds only if the minimal model of X has trivial boundary (compare [2],(4.5)). In fact, the minimal model for a space X with $L_{odd}(X) = 0$ is the exterior algebra $\Lambda^*(V)$ over the vector space $V = \mathrm{Hom}(L_*(X),0)$ dual to $L_*(X)$ with a de-