INTRODUCTION TO

The Foundations of Mathematics

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Preface

This book grew out of a course in Foundations of Mathematics which I have given at the University of Michigan for over twenty years. The reason for instituting the course was simply the conviction that it was not good to have teachers, actuaries, statisticians, and others who had specialized in undergraduate mathematics, and who were to base their life's work on mathematics, leave the university without some knowledge of modern mathematics and its foundations. The training of these people consisted chiefly of "classical" mathematics and its applications—that part of mathematics which is based on pre-twentieth-century and, in large part, on pre-Cantorian ideas and methods.

It seemed, too, that a course in Foundations at about the senior level might serve to unify and extend the material covered in the traditional mathematics curriculum. The "compartmentalization" of the preparatory school—arithmetic, algebra, and geometry—is usually continued in college with a further dose of algebra, followed by courses in analytic geometry and calculus in which a little unification of preceding subjects takes place, but no time is spent on the *nature* of the material or its foundation.

Also, the growing realization that mathematical logic is a new and legitimate part of mathematics made it seem advisable to institute a course which would make manifest the importance of studies in the Foundations, and the reasons for inquiring into the nature of mathematics by either the tools of logic or other methods.

To my first class in the course I owe much for inspiration and encouragement. It consisted, with one or two exceptions, of approximately thirty actuarial students, most of them first-year graduate students. Their response was surprising; for I was aware of the antagonism of many professional mathematicians to any inquiry into the nature of mathematics, especially if it leads to any questioning of the validity of time-honored principles and methods (the analogue in mathematics, perhaps, of the historical lag in the cultivation of those sciences that study man's own behavior).

Perhaps their reaction was due to the change, possibly refreshing to them, from the type of teaching which treats mathematics as a "discipline," dogmatic in character. Whatever the reason, the course seemed to "take" with them and left no question about the desirability of repeating the experiment. (Even today I occasionally meet members of that original class, many of them now insurance company executives, and find them still able to recall topics that were discussed.)

In a few years the course ceased to be an experiment and became as well established as any other course in the curriculum. This development was aided, no doubt, by the realization that the course was also fulfilling other purposes. For those students who were going on to the doctorate in mathematics, it did spadework in the ideas and methods that were going to form the principal tools in their graduate courses and research. And for mature students in other fields, such as philosophy and the social sciences, it provided an insight into basic mathematical concepts without the necessity of first wading through the traditional courses in algebra, geometry, calculus, etc.

In the belief that such a course ought, perhaps, to be offered in most universities and colleges that train mathematicians either for teaching or for any of the professional fields, I decided to incorporate the material covered in a book which would serve as a basis for such a course. Unfortunately, a book in my own special field of research took precedence and delayed my starting on the present work for at least ten years. Moreover, it seemed desirable to include in the book material that it was not possible to crowd into a one-semester course, and which has usually been suggested for collateral reading; especially material that is in languages other than English. For it continues to be true that the students in American universities are generally not prepared to read in French and German; and most of the older and basic work in Foundations was originally published in German. The material in Chapter X (Intuitionism), for instance, has heretofore been available almost entirely only in German.

In a general way, the idea of the book is similiar to that which motivated J. W. Young's *Fundamental Concepts of Algebra and Geometry*, first published in 1911. In 1932 I discussed with Professor Young the desirability of a book such as this one; he agreed thoroughly that it was desirable to write it, if only to have available a book on fundamental concepts that would take into account the great strides that have been made in Foundations since the publication of his book.

As already indicated, I have given the material in the form of a course of one semester, with a calculus prerequisite. The students who have taken the course have, however, been at all levels, from undergraduate juniors to students already writing doctoral dissertations. Ideally, the course should be given at senior level or first-year graduate level. And, I have found, it is not necessary to insist on the calculus prerequisite for mature students in other fields, such as philosophy. (One of the most enthusiastic students in my experience was a medical student who was taking the course as a "cultural" subject.) No *mathematics* student, however, should take the course without having had calculus, and for the average student it is better that he have taken courses such as advanced calculus and projective geometry in order to develop the maturity requisite for abstract thinking.

Whether I have succeeded in getting down on paper a reasonable facsimile of what I have done in class only the reception of this book can tell. No false modesty prevents me from admitting the success of the course itself, as numerous past students will testify. But the enthusiasm and inspiration which come from facing a group of interested students are hard to duplicate in the seclusion of one's study, and it is difficult to recapture the many spontaneous ideas and illustrations that have revealed themselves in the classroom from time to time over the years. No doubt, now that the book is written, I shall occasionally recall some of them and regret that they are doomed to oblivion.

I have made it a general rule, incidentally, not to reveal to students my own opinion regarding a controversial point, despite frequent requests to do so. It has always seemed better to present, as emphatically as possible from the point of view of their proponents, such topics as are controversial; and frequently, in order to aggravate class discussion, it has been my custom to oppose the point of view of a student while secretly agreeing with him. There is nothing new about such methods of instruction, of course, and I bring them up here largely to afford occasion to remark that I am including my own opinions regarding the nature of mathematics in the material of Chapter XII. In the earlier chapters I have tried to follow my usual rule of acting as advocate for the view presented; and, in thus breaking my rule, I do so not only with a view to getting in my own "innings," but also to furnish additional fuel for the stirring up of controversy which, it seems to me, is the most effective stimulant to original and creative thinking. And if, after reading this book, the student is not aroused to the extent of "thinking about" mathematics, I shall have failed in one of my chief purposes in writing it.

For rapidity and exactness of reference, the decimal system of numbering sections has been employed. Cross references to items in the text are made by citing chapter and section; thus "IV 2.4" refers to

Chapter IV, Section 2.4. However, for reference to a section in the chapter under consideration, only the section number is used; thus "1.2" refers to Section 1.2 of the chapter in which the citation occurs.

The Bibliography is divided into two parts, the first listing books and longer memoirs, the second listing papers and shorter articles. References to the Bibliography are enclosed in brackets, those involving capital letters such as [B], [Ha], [H_1] referring to the first part of the Bibliography, and those involving single lower-case letters such as [a], [b] referring to the second part. Page or chapter numbers will frequently be included; thus "Hilbert [H_3; 6]" will be found under Hilbert's name in the first part of the Bibliography, item [H_3] (the reference, then, is to page 6 of Hilbert's *Foundations of Geometry*). Always, "f" indicates "footnote," and "ff" indicates "and the following page or pages."

To those colleagues and students who have given me encouragement and stimulation, I wish to express sincere thanks. I am especially grateful to Professors E. T. Bell, Leon Henkin, Paul Henle, and Leo Zippin, and to Dr. C. V. Newsom for suggestions and criticisms; but the errors and shortcomings to be found herein are not their fault and are present only in spite of their wise counsel.

For aid in a material way, thanks are due to the Office of Naval Research, under whose Contract N9onr-89300 the first draft of this work was written, as well as to the California Institute of Technology, which generously afforded an office and library facilities during the academic year 1949–1950 for my writing and research.

R. L. WILDER

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Suggestions for Use as a Textbook

The plan of presentation is based on the idea that an appreciation of the need for a study of the Foundations of Mathematics can come only through a prior acquaintance with Fundamental Concepts of Mathematics. It is the nature of these concepts and the contradictions to which their unrestricted use may lead that make one realize the necessity for going deeper into their background—Foundations.

Accordingly the book is divided into two parts: Part I, Fundamental Concepts and Methods of Mathematics, and Part II, Development of Various Viewpoints on Foundations. In Part I are presented those topics which are indispensable for work in modern mathematics. Here the student will become aware of the conventional character of much that he has heretofore taken for granted, as well as of the true intrinsic nature of such concepts as the infinite, the real number continuum, arithmetic, and geometry which he has continually encountered in his elementary and intermediate mathematics courses. When this material has been digested, the student's curiosity about the actual nature of the Foundations has usually been aroused, and he is ready for Part II.

By judicious selection of material, a satisfactory one-semester course can be based on the book. This has been my custom; for instance, such proofs as those of the equivalence of Choice Axiom, Comparability, and Well-ordering in Chapter V can be omitted, and some discussion of the principles involved, the reasons why the equivalence may induce some to reject the Choice Axiom, etc., can be substituted therefor. As a matter of fact, much of the material on the infinite in Chapters IV and V can be summarized with details omitted; and a like comment holds regarding Chapter VII.

However, it is probably best to leave the choice to the judgment of the individual instructor. One may wish to place the main emphasis on Foundations proper, spending most of the time on Part II; the bibliographical references should enable the instructor to supplement the material given, if he so desires. Another may wish to give a course emphasizing Fundamental Concepts, in which case Part I may be used exclusively, or may be supplemented by some discussion of the material in Part II without going into much detail, however.

If the book is used for a two-semester course, probably all details can be covered, although one can introduce variation, if desired, by substituting or augmenting here and there from the Bibliography. Problems are provided for the better understanding of the concepts discussed in Part I; these may be used for class discussion, or as a basis for written work.

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PART I · FUNDAMENTAL CONCEPTS AND METHODS OF MATHEMATICS

PART II · DEVELOPMENT OF VARIOUS VIEWPOINTS ON FOUNDATIONS

P A R T I

Fundamental Concepts and Methods of Mathematics

I

The Axiomatic Method

Since the axiomatic method as it is now understood and practiced by mathematicians is the result of a long evolution in human thought, we shall precede our discussion of it by a brief description of some older uses of the term *axiom*. The modern usage of the term represents a high degree of maturity, and a better understanding of it may be achieved by some acquaintance with the course of its evolution.

1 Evolution of the method

If the reader has at hand a copy of an elementary plane geometry, of a type frequently used in high schools, he may find two groupings of fundamental assumptions, one entitled "Axioms," the other entitled "Postulates." The intent of this grouping may be explained by such accompanying remarks as: "An *axiom* is a *self-evident truth*." "A *postulate* is a *geometrical fact* so simple and obvious that its validity may be assumed." The "axioms" themselves may contain such statements as: "The whole is greater than any of its parts." "The whole is the sum of its parts." "Things equal to the same thing are equal to one another." "Equals added to equals yield equals." It will be noted that such geometric terms as "point" or "line" do not occur in these statements; in some sense the axioms are intended to transcend geometry—to be "universal truths." In contrast, the "postulates" probably contain such statements as: "Through two distinct points one and only one straight line can be drawn." "A line can be extended indefinitely." "If L is a line and P is a point not on L, then through P there can be drawn one and only one line parallel to L." (Some so-called "definitions" of terms usually precede these statements.)

This grouping into "axioms" and "postulates" has its roots in an-