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M. Green J. Murre C. Voisin

Algebraic Cycles and Hodge Theory

Torino, 1993

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Algebraic Cycles and Hodge Theory

Lectures given at the 2nd Session of the
Centro Internazionale Matematico Estivo
(C.I.M.E.) held in Torino, Italy,
June 21-29, 1993

Editors: A. Albano, F. Bardelli

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- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

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Lecture Notes in Mathematics 1594

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FOREWORD

The Second 1993 C.I.M.E. Session "Algebraic Cycles and Hodge Theory" was held at Villa Gualino, Torino, from June 21 to June 29, 1993.

There were three series of main lectures and some seminars: this volume contains the texts of the three series of main lectures and of those seminars most closely related to them, providing results or examples that are directly relevant to some part of these main lectures.

The theory of algebraic cycles is today still one of the most difficult and most beautiful areas of algebraic geometry (and of all mathematics): notable open problems include the Hodge conjecture, the relations among the several equivalence relations between algebraic cycles, the connections with the properties of some cohomology theories.

Our main goal in organizing this C.I.M.E. Session was to gather together some of the leading mathematicians active in this area, to assess the present state of the art and to describe the possible future developments.

Thus the three series of main lectures dealt with:
i) Infinitesimal methods in Hodge Theory, delivered by Mark L. Green (U.C.L.A., USA)
ii) Algebraic cycles and algebraic aspects of cohomology and K-theory, delivered by J.P. Murre (Rijksuniversiteit, Leiden, The Netherlands)
iii) Transcendental methods in the study of algebraic cycles, delivered by Claire Voisin (Université Paris-Sud, Orsay, France)

To complete this rough outline of the volume, it suffices to say a few words about the seminars that have been selected for inclusion in the text.

The first one, by G.P. Pirola, reports on joint work with A. Collino: they compute the infinitesimal invariant of the normal function associated to the cycle C^+-C^- in its Jacobian and derive from this computation a nice refinement of Ceresa's theorem and a Torelli theorem in the spirit of Griffiths in genus three. These results are obtained by applying M. Green's technique of computing the infinitesimal invariant of a normal function and can be regarded therefore as an exemplification and as a striking application of this technique. They are closely related to the first series of main lectures, and in the summer of 1993, new, surprising and important results were obtained from what was proved in the seminar.

The second seminar by Bert Van Geemen ties in closely with the lectures of J.P. Murre and his treatment of the Hodge conjecture: the author restricts his attention to abelian varieties, in particular to those of Weil type, and studies them by means of the Mumford-Tate group, a topic that cannot be left aside in a course like this one. Finally the author points out some relations between theta functions and cycles on some particular abelian fourfolds.

The last seminar, by S. Müller-Stach, deals with height pairings and reveals a connection between mixed Hodge structures (already treated in M. Green's lectures) and Deligne cohomology (see J.P. Murre's lectures) by using the theory of logarithmic currents (see M. Green's lectures). Thus this topic finds here an ideal context in which to be outlined and discussed.

We are very happy to note that the lectures did an outstanding job and that all the participants contributed with interest and enthusiasm to creating a very stimulating atmosphere throughout the session. It is fair to say that its spirit has been captured well by the texts of this volume. We wish to thank the C.I.M.E. Foundation which made all of this possible.

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INFINITESIMAL METHODS IN HODGE THEORY

June 1993

Mark L. Green

LECTURE 1

During most of my years as an undergraduate student, I thought that algebra was my favorite subject. However, in my senior year of college, I took a course from Victor Guillemin. This was my first course in geometry, and the main theorem was De Rham's Theorem. This had a lasting effect on my mathematical interests, as the reader can observe. Much as Aristophanes thought that men and women were originally one creature trying (often in vain) to become reunited, so mathematicians often search through life (once again, often in vain) for problems that will bring together the various parts of mathematics that they love. For me, my contact with the area of infinitesimal methods in Hodge theory was one moment when, briefly, this actually happened.

My goal in these lectures is to cover the material necessary to the understanding of the Nori Connectedness Theorem, with stops for other interesting results along the way. Hodge theory, like algebraic geometry as a whole, is rich in having many levels of abstraction at which to approach any given idea. Unfortunately, as one rises to higher levels of abstraction and mathematical power, one tends to get further and further away from the underlying geometry. What I have attempted to do here is to try to make accessible some of these various levels by starting with the most geometric formulation and gradually introducing more abstract formulations. Thus some proofs are given more than once, in hopes that this will clarify how the machinery works.

I would like to thank my fellow lecturers, Jacob Murre and Claire Voisin, for their camaraderie and mathematical inspiration; I feel privileged to have shared a podium with them. I want to express my deep gratitude to Fabio Bardelli, who had the insight to realize that this subject needed a series of expository lectures and who mapped out for the three of us his vision of what should be covered; I could not have wished for a better or wiser organizer of the scientific program. I would also like to thank Alberto Albano for courageously and warm-heartedly stepping in when Fabio became ill, and carrying off the conference in a successful and enjoyable way.

De Rham's Theorem states that every real cohomology class on a smooth manifold M can be represented by a closed $\mathcal{C}^\infty$ differential form ω, and that two closed forms represent the same cohomology class if and only if they differ by an exact form $d\tau$, where τ is a $\mathcal{C}^\infty$ differential form. If we denote by $A^k(M)$ the $\mathcal{C}^\infty$ k-forms on M and $A^\bullet(M), d$ the complex of $\mathcal{C}^\infty$ differential forms with exterior derivative, we denote

$$H^k_{\mathrm{DR}}(M) = H^k(A^\bullet(M)).$$

THEOREM (De Rham's Theorem). *For a smooth manifold M, for all k,*

$$H^k_{\mathrm{DR}}(M) \cong H^k(M, \mathbf{R}).$$

The map is given by

$$\omega \mapsto [\sigma \mapsto \int_\sigma \omega].$$

Once Einstein had discovered general relativity, it was realized that the electromagnetic field ($\vec{E} = (E_1, E_2, E_3)$ the electric field and $\vec{B} = (B_1, B_2, B_3)$ the magnetic field) could be represented on four-dimensional space-time by a 2-form

$$\Omega = \sum_{i=1}^{3} E_i dx_i \wedge dt + B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_1 + B_3 dx_1 \wedge dx_2,$$

and that two out of four of Maxwell's equations in free space could be written as

$$d\Omega = 0.$$

The other two equations do not come out as naturally, but if one considers the 2-form

$$*\Omega = E_1 dx_2 \wedge dx_3 + E_2 dx_3 \wedge dx_1 + E_3 dx_1 \wedge dx_2 - \sum_{i=1}^{3} B_i dx_i \wedge dt,$$

then the other two equations are

$$d * \Omega = 0.$$

The relationship between Ω and $*\Omega$ is not invariant under smooth change of coordinates, but it is invariant under changes of coordinates which preserve the Lorentz metric on space-time. It was physical considerations of this kind which led Hodge to discover the Hodge $*$-operator and to formulate the Hodge Theorem.

The most natural mathematical motivation for the Hodge Theorem is to ask whether one can find one "natural" differential form ω representing each cohomology class. It is appealing to find some measure of the "size" of a differential form and then look for the "smallest" element of the set $\{\omega + d\tau\}$ for fixed closed form ω as τ varies over all smooth forms of a given degree. One might define the size first pointwise and then integrate over M. This is done as follows:

We need to remember some standard constructions. If V, W are vector spaces with a positive-definite inner product, then $V \otimes W$ may be given a natural positive-definite inner product so that if e_i and f_j are orthonormal bases for V, W respectively, then $e_i \otimes f_j$ is an orthonormal basis for $V \otimes W$. Secondly, if $W \subseteq V$ and V has a positive-definite inner product, then W inherits one by restriction and V/W by orthogonal projection. If V has a positive definite inner product, then V^* inherits one naturally in such a way that the dual basis of an orthonormal basis is orthonormal.

Let V be an oriented n-dimensional vector space over $\mathbf{R}$ equipped with a positive-definite inner product. Then for any k, $\wedge^k V$ may be given a natural positive-definite inner product by combining these two standard constructions, since $\wedge^k V \subseteq \otimes^k V$. Thus if M is an oriented Riemannian manifold of dimension n and ω is a smooth k-form on M, then for any $p \in M$, we can apply the construction above to $T^*_{p,M}$ with the induced inner product to obtain a length $||\omega||^2_p$. If M is an oriented compact Riemannian manifold, then we define

$$||\omega||^2_M = \int_M ||\omega||^2_p dV,$$

where dV is the element of volume. This is a positive-definite inner product on the space of smooth k-forms on M.

DEFINITION. *A smooth k-form ω on a compact Riemannian manifold M is* **harmonic** *if $d\omega = 0$ and*

$$||\omega||_M \leq ||\omega + d\tau||_M$$

for all smooth $(k-1)$-forms τ. We denote the set of harmonic k-forms on M by $\mathcal{H}^k(M)$.

There is slightly different way to describe the inner product on forms. The inner product on a vector space V gives a natural map

$$V \otimes V \rightarrow \mathbf{R}.$$

This in turn gives a natural isomorphism $V \rightarrow V^*$ of V with its dual. If we take $\wedge^k$ of this isomorphism, we obtain an isomorphism $\wedge^k V \rightarrow \wedge^k V^*$. Taking volume gives a natural isomorphism $\wedge^n V \cong \mathbf{R}$. Wedge product gives a map

$$\wedge^k V \times \wedge^{n-k} V \rightarrow \wedge^n V \cong \mathbf{R},$$

and since this is a non-degenerate pairing, it gives a natural isomorphism $\wedge^k V \cong \wedge^{n-k} V^*$, and now using $\wedge^k$ of the isomorphism induced by the inner product, we can identify the factor on the right with $\wedge^{n-k} V$. Putting all this together, we obtain a natural map

$$*: \wedge^k V \rightarrow \wedge^{n-k} V,$$

and this is the Hodge $*$-operator. This is defined pointwise and thus extends to a map $*: A^k(M) \rightarrow A^{n-k}(M)$. The basic facts are:

LEMMA. *(1) For $\alpha, \beta \in \wedge^k V$,*

$$\alpha \wedge *\beta = \beta \wedge *\alpha = (\alpha, \beta)\mathrm{Vol},$$

where $\mathrm{Vol} \in \wedge^n V$ is the element of volume;
(2) If $e_1, \ldots, e_n$ is an oriented orthonormal basis for V, then $*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \pm e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$, where $\{j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\} - \{i_1, \ldots, i_k\}$ and the sign

is chosen so that $e_{i_1} \wedge \cdots e_{i_k} \wedge \pm e_{j_1} \wedge \cdots \wedge e_{j_{n-k}} = e_1 \wedge \cdots \wedge e_n$*;*
(3) For $\alpha \in \wedge^k V$, $*^2\alpha = (-1)^{k(n-k)}\alpha$.

COROLLARY. *For* $\alpha, \beta \in A^k(M)$,

$$(\alpha, \beta)_M = \int_M \alpha \wedge *\beta.$$

We would like to construct an adjoint for $d\colon A^k(M) \to A^{k+1}(M)$.

PROPOSITION. *Let* $d^*\omega = (-1)^{(k+1)(n-k)+1} * d * \omega$ *for all* $\omega \in A^k(M)$. *Then*

$$(d^*\omega, \phi)_M = (\omega, d\phi)_M$$

for all $\omega \in A^k(M), \phi \in A^{k-1}(M)$.

PROOF: By Stokes Theorem. □

DEFINITION. *The* **Laplace operator** $\Delta\colon A^k(M) \to A^k(M)$ *is defined by*

$$\Delta = dd^* + d^*d.$$

PROPOSITION. *For* $\omega \in A^k(M)$, *the following are equivalent:*
(1) ω *is harmonic;*
(2) $d\omega = 0$ *and* $d^*\omega = 0$;
(3) $\Delta\omega = 0$.

PROOF: (1)←(2): If ω is harmonic, then $d\omega = 0$. Now for any constant ϵ,

$$\begin{aligned}||\omega + \epsilon d\tau||_M^2 &= (\omega + \epsilon d\tau, \omega + \epsilon d\tau)_M \\ &= ||\omega||_M^2 + 2\epsilon(\omega, d\tau)_M + \epsilon^2 ||d\tau||_M^2 \\ &= ||\omega||_M^2 + 2\epsilon(d^*\omega, \tau)_M + \epsilon^2 ||d\tau||_M^2.\end{aligned}$$

For ϵ small, we see that ω harmonic implies that $(d^*\omega, \tau)_M = 0$ for all $\tau \in A^{k-1}(M)$, and this forces $d^*\omega = 0$.
(2)← (1): By the formula above, if $d\omega = 0$ and $d^*\omega = 0$, then

$$||\omega + d\tau||_M^2 = ||\omega||_M^2 + ||d\tau||_M^2,$$

and thus

$$||\omega||_M < ||\omega + d\tau||_M$$

if $d\tau \neq 0$.
(2)→ (3) $\Delta\omega = dd^*\omega + d^*d\omega = d(0) + d^*(0) = 0$.
(3) →(2): We have for any ω that

$$\begin{aligned}(\Delta\omega, \omega)_M &= (dd^*\omega, \omega)_M + (d^*d\omega, \omega)_M \\ &= (d^*\omega, d^*\omega)_M + (d\omega, d\omega)_M \\ &= ||d^*\omega||_M^2 + ||d\omega||_M^2.\end{aligned}$$

If $\Delta\omega = 0$, then the left hand side is zero, and hence the right hand side is, which implies $d^*\omega = 0, d\omega = 0$. □

PROPOSTION. *There is a natural injection*

$$\mathcal{H}^k(M) \to H^k_{\mathrm{DR}}(M)$$

sending a harmonic k-form ω to its De Rham cohomology class.

PROOF: The only thing to be proved is that if a harmonic form is exact, then it is 0. If $\omega \in \mathcal{H}^k(M)$ and 0 belongs to the De Rham class of ω, then by minimality $|\omega|_M \leq |0|_M = 0$, so $\omega = 0$. □

Of course, it is not clear that harmonic forms exist, i.e. that there is a form of minimal size in each De Rham class. To see that a sequence of smooth forms in a De Rham class with sizes converging to the infimum of the sizes of forms in that class must converge to a smooth form requires some basic results from the theory of elliptic operators. The final result, which we quote here, is:

THEOREM (The Hodge Theorem). *For M a compact oriented Riemannian manifold, the natural map*

$$\mathcal{H}^k(M) \to H^k_{\mathrm{DR}}(M)$$

is an isomorphism, i.e. every De Rham class is represented by a unique harmonic form.

Beautiful as it is, the Hodge Theorem by itself is not quite enough for the purposes of Hodge theory. One sometimes needs the full package of consequences of elliptic operator theory. The eigenspaces of Δ are finite-dimensional and are spanned by smooth functions, and the eigenvalues are ≥ 0 and march off to infinity. Every L^2 k-form can be expressed as the L^2 limit of sums of eigenforms of Δ. If one takes a k-form, projects it on the space orthogonal to $\mathcal{H}^k(M)$, and then multiplies its projection on the λ-eigenspace of Δ by $\frac{1}{\lambda}$, one obtains the Green's operator. The fact we will need to quote is that:

THEOREM (Existence of Green's Function). *For M a compact oriented Riemannian manifold, there exists a unique operator $G: A^k(M) \to A^k(M)$ such that G commutes with d and d^*, $G(\mathcal{H}^k(M)) = 0$, and*

$$\mathrm{Id} = \pi_{\mathcal{H}} + \Delta G,$$

where $\pi_{\mathcal{H}}$ is the orthogonal projection $A^k(M) \to \mathcal{H}^k(M)$.

If a compact orientable manifold M has a metric with nice differential-geometric properties, it is possible to draw interesting conclusions about the cohomology of M. However, it is really in the case of complex manifolds, especially Kähler manifolds, that the Hodge Theorem pays truly powerful geometric dividends.

As with the Hodge theorem, there is a certain amount of preliminary multilinear algebra that goes into the story. The main difference is how much interesting multilinear algebra goes on, and how subtle some of the results are.

DEFINITION. *Let V be a vector space over* $\mathbf{R}$. *An* **almost-complex structure** *on V is an endomorphism $J \in \mathrm{End}(V)$ such that $J^2 = -\mathrm{Id}$.*

DEFINITION. *Let V, J be a real vector space with almost-complex structure, and $V_\mathbf{C} = V \otimes_\mathbf{R} \mathbf{C}$, with J extended to $V_\mathbf{C}$ in the canonical way. Define $V^{1,0}, V^{0,1}$ respectively as the $+i$ and $-i$ eigenspaces of J on $V_\mathbf{C}$.*

PROPOSITION. *Let V, J be a real vector space with almost-complex structure. Then $V_\mathbf{C} = V^{1,0} \oplus V^{0,1}$. Further, $\dim_\mathbf{R} V = \dim_\mathbf{C} V^{1,0} = \dim_\mathbf{C} V^{0,1}$.*

PROOF: The eigenvalues of J occur in conjugate pairs, and clearly the direct sum of $V^{1,0}$ and $V^{0,1}$ injects into $V_\mathbf{C}$. It thus suffices to prove that $\dim_\mathbf{C} V^{1,0} = \dim_\mathbf{R} V$. The map $v \mapsto (iv + Jv) \oplus (-iv + Jv)$ takes $V_\mathbf{C} \to V^{1,0} \oplus V^{0,1}$ and is injective, which shows that $\dim_\mathbf{C} V_\mathbf{C} \leq 2\dim_\mathbf{C} V^{1,0}$, which is enough. □

DEFINITION. *Let V, J be a real vector space with an almost-complex structure. A positive definite inner product $(,)$ is* **hermitian** *if J is an isometry, i.e. $(Jv, Jw) = (v, w)$ for all $v, w \in V$.*

PROPOSITION. *Let V, J be a real vector space with almost-complex structure and hermitian metric $(,)$.*
(1) The map $\omega: V \otimes V \to \mathbf{R}$ defined by

$$\omega(v, w) = (Jv, w)$$

is a real alternating form;
(2) If we extend ω to an element of $\wedge^2 V_\mathbf{C}^*$, then ω is zero when restricted to $V^{1,0} \otimes V^{1,0}$ and $V^{0,1} \otimes V^{0,1}$;
(3) ω gives a non-degenerate pairing when restricted to $V^{1,0} \otimes V^{0,1}$;
(4) If we extend $(,)$ to be complex linear in the first variable and conjugate linear in the second variable, then it is a positive definite Hermitian inner product on $V^{1,0}$.

PROOF: (1) $(Jv, w) = (J^2 v, Jw) = -(v, Jw) = -(Jw, v)$.
(2) For the purposes of proving (2) and (3), extend $(,)$ to be complex linear in both entries. If $v, w \in V^{1,0}$, then $(Jv, w) = i(v, w) = (v, Jw) = -(Jv, w)$, so $(v, w) = 0$ and thus $\omega(v, w) = 0$. Similarly for $V^{0,1}$.
(3) If $v \in V^{1,0}$ then for some $w \in V_\mathbf{C}$, $(Jv, w) \neq 0$, as otherwise $Jv = 0$ and hence $v = 0$. If $w = w^{1,0} + w^{0,1}$ is the decomposition of w under the direct sum decomposition $V_\mathbf{C} = V^{1,0} \oplus V^{0,1}$, then by (2), $(Jv, w) = (Jv, w^{0,1})$, and this proves the pairing is non-degenerate in the first factor. A similar argument works for the second factor.
(4) If $v = a + ib \in V^{1,0}$, where $a, b \in V$, then $(v, \bar{v}) = (a+ib, a-ib) = (a, a) + (b, b)$, from which positive-definiteness is clear. □

DEFINITION. *Let V, J be a real vector space with almost-complex structure and hermitian metric $(,)$. Let ω be as in the preceding Proposition. Then ω is called the* **alternating form associated to** $(,)$.

DEFINITION. *For a complex manifold M, a $\mathcal{C}^\infty$* **form of type** (p,q) *is a $\mathcal{C}^\infty$ section of the bundle $\wedge^p T^{1,0*} \otimes \wedge^q T^{0,1*}$; we will denote the set of these by $A^{p,q}(M)$.*

DEFINITION. *Let M be a complex manifold with almost-complex structure $J: T_M \to T_M$. A Riemannian metric on the underlying real manifold of M is* **hermitian** *if it is hermitian with respect to J on $T_{M,p}$ for every point $p \in M$. The* **associated (1,1) form** *ω of the hermitian metric is defined by taking ω_p to be the alternating form associated to the metric on $T_{M,p}$ for every $p \in M$.*

COROLLARY. *The associated (1,1)-form of a hermitian metric is a real 2-form on the underlying real manifold of M and has type (1,1).*

DEFINITION. *A hermitian metric on a complex manifold M is said to be a* **Kähler metric** *if the associated $(1,1)$ form ω is closed. In this case, ω is called the* **Kähler form**. *The element of $H^2_{\mathrm{DR}}(M)$ determined by ω is called the* **Kähler class**. *If the Kähler class belongs to the image of $H^2(M, \mathbf{Z})$, the metric is said to be a* **Hodge metric**.

EXAMPLE. *Fubini-Study metric*

On $\mathbf{P}^n$, if we let

$$0 \to S \to V \otimes \mathcal{O}_{\mathbf{P}^n} \to Q \to 0$$

be the tautological sub-bundle sequence, then it is well-known that

$$T_{\mathbf{P}^n} \cong \mathrm{Hom}(S, Q).$$

If we put a Hermitian metric on the complex vector space V, then it induces natural Hermitian metrics on S and Q by restriction and orthogonal projection. This in turn induces a natural metric on $S^* \otimes Q \cong \mathrm{Hom}(S, Q)$. This metric is invariant under the action of the unitary group on V, and this forces the associated (1,1)-form ω to be closed. Since $H^2(\mathbf{P}^n, \mathbf{R})$ is 1-dimensional, adjusting the metric by a constant makes ω integral, and thus $\mathbf{P}^n$ has a Hodge metric, the **Fubini-Study metric**.

PROPOSITION. *A smooth projective variety $M \subseteq \mathbf{P}^N$ has a Hodge metric obtained by restricting the Fubini-Study metric.*

PROOF: It is elementary to see that the restriction of a Hodge metric is Hodge, since the operations of restricting and taking associated (1,1)-form commute. □

We quote the following famous consequence of the Kodaira Embedding Theorem:

THEOREM (Kodaira). *If a compact hermitian complex manifold admits a Hodge metric, then there exists an embedding of M in some $\mathbf{P}^N$ such that the Hodge metric is $\frac{1}{k}$ times the restriction of the Fubini-Study metric, for some positive integer k.*

Using geodesic coordinates, it is easy to see that at every point p of a Riemannian manifold M, there are local coordinates $x_1, \ldots, x_n$ for M centered at p such that

$$(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij} + O(\|x\|^2).$$

However, it is not true that at every point p of a hermitian complex manifold, there are local holomorphic coordinates $z_1, \ldots, z_n$ centered at p such that

$$(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) = \delta_{ij} + O(\|z\|^2).$$

The following proposition makes the Kähler condition quite natural (or at least as natural as it is going to get.)

PROPOSITION. *Let M be a complex hermitian manifold. The following are equivalent:*
(1) The metric is Kähler;
(2) At every point p of M, there are local holomorphic coordinates $z_1, \ldots, z_n$ centered at p such that

$$(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) = \delta_{ij} + O(\|z\|^2).$$

PROOF: If $z_1, \ldots, z_n$ are local holomorphic coordinates on a complex manifold M, let

$$h_{ij} = (\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}).$$

Then

$$\omega = i \sum_{ij} h_{ij} dz_i \wedge d\bar{z}_j.$$

If (2) holds, then all first partials of the h_{ij} vanish at the origin, and hence $d\omega = 0$ there; since the point was arbitrary, $d\omega = 0$ and the metric is Kähler. Conversely, if $d\omega = 0$, and we choose holomorphic coordinates $z_1, \ldots, z_n$ so that

$$h_{ij} = \delta_{ij} + \sum_k a_{ij}^k z_k + \sum_k \bar{a}_{ji}^k \bar{z}_k + O(\|z\|^2),$$

then the change of variables $z_i = w_i + q_i(w, w)$, where the q_i are homogeneous and quadratic in the w's, changes the linear term of h_{ij} by $\frac{\partial q_i}{\partial z_j} + \overline{\frac{\partial q_j}{\partial z_i}}$. Thus a_{ij}^k is changed by $\frac{\partial^2 q_i}{\partial z_j \partial z_k}$. The condition $d\omega = 0$ at p is equivalent to $a_{ij}^k = a_{kj}^i$ for all i, j, k, and thus if we take $q_j = -\sum_{ik} a_{ij}^k$, the coordinates $w_1, \ldots, w_n$ satisfy (2). □