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An Introduction to Complex Analysis in Several Variables

LARS HÖRMANDER

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PREFACE

Two recent developments in the theory of partial differential equations have caused this book to be written. One is the theory of overdetermined systems of differential equations with constant coefficients, which depends very heavily on the theory of functions of several complex variables. The other is the solution of the so-called $\bar{\partial}$ Neumann problem, which has made possible a new approach to complex analysis through methods from the theory of partial differential equations. Solving the Cousin problems with such methods gives automatically certain bounds for the solution, which are not easily obtained with the classical methods, and results of this type are important for the applications to overdetermined systems of differential equations. It has therefore seemed natural to give a self-contained exposition of complex analysis from the point of view of the theory of partial differential equations. Since we have concentrated on topics which are suitable for such a treatment, analytic spaces will not be discussed. Instead we have included some theorems on Banach algebras as another example of the applications to analysis of the theory of functions of several complex variables.

This book is only a slight modification of lecture notes from a course given by the author at Stanford University during the Spring and Summer quarters of 1964. The aim has not been to achieve completeness in any direction but to provide an easy introduction to complex analysis for readers whose main interest is in analysis. For this reason it has been assumed only that the reader knows a certain amount of real function theory, more specifically the elements of integration theory, distribution theory, functional analysis, and the calculus of differential forms. Very little algebra is used. In Chapter I the elementary theory of functions of a single complex variable is recalled briefly. The main reason for this is to introduce the central problems in a familiar case as a guide for the general case. Chapter I also includes some classical facts, such as the Cauchy integral formula for solutions of the inhomogeneous Cauchy-Riemann equations, which unfortunately are missing in many elementary texts. The last section of Chapter I develops the facts concerning subharmonic functions which are needed. Since most readers should pass quickly to Chapter II, we wish to mention that the main point of the Hartogs theorem on separate analyticity has been inserted there.

Chapter II starts with classical facts concerning power series expansions, domains of holomorphy, and pseudoconvex domains. Following a classical paper of Oka, rewritten in the spirit of differential equations, existence theorems for the Cauchy-Riemann equations in Runge domains are then proved. This is done to illustrate the Oka-Cartan methods in a very simple case which is sufficient for the main applications to the theory of Banach algebras. These are given in Chapter III where a preliminary section recalls the basic facts concerning such algebras. Both Chapter III and section 2.7 can be bypassed without any loss of the continuity.

In Chapter IV the Cauchy-Riemann equations are solved in domains of holomorphy by means of a variant of the $\bar{\partial}$ Neumann problem. At the same time a solution of the Levi problem is obtained, that is, the identity of pseudoconvex domains and domains of holomorphy is shown. These results are extended to Stein manifolds in Chapter V. It is proved that Stein manifolds can be embedded in complex vector spaces of high dimension. Chapter V ends with a proof that complex structures can be defined on a manifold by giving a system of Cauchy-Riemann equations satisfying a certain integrability condition.

Chapter VII is devoted to the theory of coherent analytic sheaves on Stein manifolds. The proofs are based on the existence theory for the Cauchy-Riemann equations established in Chapter V and the local theory presented in Chapter VI. A final section is devoted to "cohomology with bounds" for sheaves over $\mathbf{C}^n$ with polynomial generators. Used there are the existence theorems for the Cauchy-Riemann equations proved in Chapter IV. The book ends with applications to overdetermined systems of differential equations.

Preface

I am greatly indebted to colleagues and students at Stanford University who helped improve the original notes, and also to the National Science Foundation for supporting the work through grant GP 2426 at Stanford University during the summer of 1964.

LARS HÖRMANDER

Princeton, New Jersey
January 1966

Preface to second edition

The main change in this edition is that section 4.4 has been improved. A number of references have also been added, particularly to work in the spirit of that section, and a few misprints have been corrected.

Lund in February 1973 LARS HÖRMANDER

LIST OF SYMBOLS

$\complement A$ is the complement of A (in some larger set understood from the context).

$\varnothing$ is the empty set.

$A \setminus B$ is a notation for $A \cap \complement B$.

$A \pm B = \{a \pm b; a \in A, b \in B\}$ if A and B are subsets of an abelian group.

$A \subset\subset B$ means that A is relatively compact in B, that is, A is contained in a compact subset of B.

∂A is the boundary of A.

$\partial_0 A$ denotes the distinguished boundary when A is a polydisc.

$C^k(\Omega)$, where Ω is an open set in $\mathbf{R}^N$ (or a C^∞ manifold) is the space of k times continuously differentiable complex valued functions in Ω, $0 \leq k \leq \infty$.

$C_0{}^k(A)$, where A is a subset of a C^∞ manifold Ω, denotes the set of functions in $C^k(\Omega)$ vanishing outside a compact subset of A.

supp f denotes the support of f, which is the closure of the smallest set outside which f vanishes (see p. 3).

D is sometimes used as a shorter notation for $C_0{}^\infty(\Omega)$ (see p. 78).

$A(\Omega)$ is the space of analytic functions in Ω (see pp. 1, 23).

$P(\Omega)$ is the space of plurisubharmonic functions in Ω (see p. 44).

$L^2(\Omega,\varphi)$ is the space of measurable functions in Ω such that (see pp. 78, 113)

$$\|u\|_\varphi^2 = \int |u|^2 e^{-\varphi}\, dx < \infty.$$

$\mathscr{D}'(\Omega)$ is the space of Schwartz distributions in Ω.

$\mathscr{E}'(\Omega)$ is the subspace of distributions with compact support.

W^s is the space of L^2 functions in $\mathbf{R}^N$ with all derivatives of order $\leq s$ in the sense of distribution theory belonging to L^2 (see p. 85).

$W^s(\Omega,\text{loc})$, where Ω is an open set in a C^∞ manifold, is the set of functions in Ω which agree on every compact subset of a coordinate patch with some function W^s in the coordinate space (see pp. 85, 119).

$L^2(\Omega,\text{loc})$ is the same as $W^0(\Omega,\text{loc})$.

$\mathscr{F}_{(p,q)}$, where $\mathscr{F}$ is any of the previous spaces, denotes the set of all forms of type (p,q) with coefficients in $\mathscr{F}$ (see p. 24).

List of Symbols

$\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ (see pp. 1 and 22).
$\partial^\alpha = (\partial/\partial z_1)^{\alpha_1} \cdots (\partial/\partial z_n)^{\alpha_n}$ (p. 26), where
α is a multiorder $= (\alpha_1, \cdots, \alpha_n)$ with α_j non-negative integers,
$|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$
$\wedge$ denotes exterior multiplication.
d is the exterior differentiation.
∂ and $\bar{\partial}$ are the components of d of type (1,0) and (0,1) (see pp. 22, 24).
u^*f, where f is a form and u a map, is defined on p. 23.
I (or J or K) often denotes a multi-index, that is, a sequence $(i_1, \cdots, i_p)$ of integers between 1 and n, the dimension of the space considered. We write $|I| = p$, and Σ'_I indicates that summation is restricted to multi-indices with $i_1 < i_2 < \cdots < i_p$.
$\hat{K}_\Omega$ is defined on pp. 8, 37, 109.
$\hat{K}_\Omega{}^P$ is defined on p. 46.
$\tilde{K}$ is defined on p. 53.
$\gamma_z f$ denotes the germ of f at z (see p. 152).
A_z denotes the set of germs at z of analytic functions.
D_T is the domain of the operator T.
R_T is the range of the operator T.
$d\lambda$ denotes the Lebesgue measure.
$\hat{f}$ denotes the Gelfand transform (or Fourier transform) of f.
$H^p(\mathscr{U}, \mathscr{F})$ is a cohomology group of the covering $\mathscr{U}$ with values in the sheaf $\mathscr{F}$ (see p. 177).
$H^p(X, \mathscr{F})$ is a cohomology group of the paracompact space X with values in the sheaf $\mathscr{F}$ (see p. 178).
$R[z]$ denotes the set of polynomials in one variable z with coefficients in the ring R.

CONTENTS

Chapter I

ANALYTIC FUNCTIONS OF ONE COMPLEX VARIABLE

Summary. In the first two sections we recall the simplest properties of analytic functions which follow from the Cauchy integral formula. Then follows a discussion of approximation theorems (the Runge theorem) and existence theorems for meromorphic functions (the Mittag-Leffler and Weierstrass theorems). These are the one-dimensional case of the Cousin problems around which the theory of analytic functions of several variables has developed. Finally we prove some basic theorems concerning subharmonic functions.

1.1. Preliminaries. Let u be a complex valued function in $C^1(\Omega)$,† where Ω is an open set in the complex plane $\mathbf{C}$, which we identify with $\mathbf{R}^2$. If the real coordinates are denoted by x, y, and $z = x + iy$, we have $2x = z + \bar{z}$, $2iy = z - \bar{z}$, so that the differential of u can be expressed as a linear combination of dz and $d\bar{z}$,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial \bar{z}}d\bar{z}, \tag{1.1.1}$$

where we have used the notations

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y}\right), \qquad \frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{1}{i}\frac{\partial u}{\partial y}\right). \tag{1.1.2}$$

Definition 1.1.1. *A function $u \in C^1(\Omega)$ is said to be analytic (or holomorphic) in Ω if $\partial u/\partial \bar{z} = 0$ in Ω (the Cauchy–Riemann equation), or equivalently if du is proportional to dz. For analytic functions one also writes u' instead of $\partial u/\partial z$; thus $du = u'\,dz$ if u is analytic. The set of all analytic functions in Ω is denoted by $A(\Omega)$.*

† For the notation used in this book not otherwise explained, see list of symbols on p. ix.

Examples. (1) For every integer n we have $d(z^n) = nz^{n-1}\,dz$ (for $z \neq 0$ if $n < 0$). Hence every polynomial $p(z) = \Sigma_0^n a_k z^k$ is an analytic function, and $p'(z) = \Sigma_1^n k a_k z^{k-1}$. (2) If we define $e^z = e^x(\cos y + i \sin y)$, we obtain $d\,e^z = e^z\,dz$ so e^z is analytic.

Since the differential operator $\partial/\partial\bar{z}$ is linear, it is obvious that linear combinations with complex coefficients of analytic functions are analytic. From the product rule $d(uv) = u\,dv + v\,du$ we obtain the product rule for the operators $\partial/\partial z$ and $\partial/\partial\bar{z}$. Hence the product of analytic functions is analytic.

Let u be analytic in Ω and let v be analytic in (an open set containing) the range of u. Then the function $z \to v(u(z))$ is analytic in Ω, for the chain rule gives

$$dv = v'(u)\,du = v'(u)u'(z)\,dz,$$

which also implies that $\partial v/\partial z = (\partial v/\partial u)(\partial u/\partial z)$.

We shall finally study the inverse of an analytic function. First note that since $du = u'\,dz$, the map $dz \to du$ is a rotation followed by a dilation in the ratio $|u'|$. Hence the Jacobian of the map $z \to u(z)$, considered as a map of $\mathbf{R}^2$ into $\mathbf{R}^2$, is equal to $|u'|^2$. If $u'(z_0) \neq 0$, it follows therefore from the implicit function theorem that u maps a neighborhood of z_0 homeomorphically on a neighborhood of $u_0 = u(z_0)$, and that the inverse map $u \to z(u)$ is also continuously differentiable in a neighborhood of u_0. Since $u(z(w)) = w$, the chain rule gives $u'(z(w))\,dz = dw$, so z is an analytic function of w and $\partial z(w)/\partial w = 1/u'(z(w))$.

1.2. Cauchy's integral formula and its applications. Let ω be a bounded open set in $\mathbf{C}$, such that the boundary $\partial\omega$ consists of a finite number of C^1 Jordan curves. Stokes' formula gives, if $u \in C^1(\bar{\omega})$,

$$\int_{\partial\omega} u\,dz = \iint_{\omega} du \wedge dz, \tag{1.2.1}$$

or if we note that $du \wedge dz = \partial u/\partial\bar{z}\,d\bar{z} \wedge dz = 2i\,\partial u/\partial\bar{z}\,dx \wedge dy$

$$\int_{\partial\omega} u\,dz = 2i\iint_{\omega} \partial u/\partial\bar{z}\,dx \wedge dy = \iint_{\omega} \partial u/\partial\bar{z}\,d\bar{z} \wedge dz. \tag{1.2.2}$$

(This can of course be proved directly by integrating the right-hand side.) Here $\partial\omega$ is oriented so that ω lies to the left of $\partial\omega$. An immediate consequence is that $\int_{\partial\omega} u\,dz = 0$ if $u \in C^1(\bar{\omega})$ and u is analytic in ω. Moreover, we obtain Cauchy's integral formula:

Theorem 1.2.1. *If $u \in C^1(\bar{\omega})$, we have*

$$u(\zeta) = (2\pi i)^{-1}\left\{\int_{\partial\omega} \frac{u(z)}{z-\zeta}\,dz + \iint_{\omega} \frac{\partial u/\partial \bar{z}}{z-\zeta}\,dz \wedge d\bar{z}\right\}, \qquad \zeta \in \omega. \tag{1.2.3}$$

Proof. Put $\omega_\varepsilon = \{z\,;z \in \omega,\ |z-\zeta| > \varepsilon\}$ where $0 < \varepsilon <$ the distance from ζ to $\complement\omega$. If we apply (1.2.2) to $u(z)/(z-\zeta)$ and note that $1/(z-\zeta)$ is analytic in ω_ε, we obtain

$$\iint_{\omega_\varepsilon} \partial u/\partial\bar{z}\,(z-\zeta)^{-1}\,d\bar{z} \wedge dz = \int_{\partial\omega} u(z)(z-\zeta)^{-1}\,dz - \int_0^{2\pi} u(\zeta + \varepsilon e^{i\theta})i\,d\theta.$$

Since $(z-\zeta)^{-1}$ is integrable over ω and u is continuous at ζ, we obtain (1.2.3) by letting $\varepsilon \to 0$.

Conversely, we shall prove

Theorem 1.2.2. *If μ is a measure with compact support† in* **C**, *the integral*

$$u(\zeta) = \int (z-\zeta)^{-1}\,d\mu(z)$$

defines an analytic C^∞ function outside the support of μ. In any open set ω where $d\mu = (2\pi i)^{-1}\varphi\,dz \wedge d\bar{z}$ for some $\varphi \in C^k(\omega)$, we have $u \in C^k(\omega)$ and $\partial u/\partial\bar{z} = \varphi$ if $k \geq 1$.

Proof. That $u \in C^\infty$ outside the support K of μ is obvious since $(z-\zeta)^{-1}$ is a C^∞ function of (z,ζ) when $z \in K$ and $\zeta \in \complement K$, and since $\partial(z-\zeta)^{-1}/\partial\bar{\zeta} = 0$ when $\zeta \neq z$, the analyticity follows by differentiation under the sign of integration. To prove the second statement we first assume that $\omega = \mathbf{R}^2$. After a change of variables we can write

$$u(\zeta) = -(2\pi i)^{-1}\iint \varphi(\zeta - z)z^{-1}\,dz \wedge d\bar{z}.$$

Since z^{-1} is integrable on every compact set, it is legitimate to differentiate under the sign of integration at most k times and the integrals obtained are continuous. Hence $u \in C^k$ and

$$\begin{aligned}\partial u/\partial\bar{\zeta} &= -(2\pi i)^{-1}\iint \partial\varphi(\zeta - z)/\partial\bar{\zeta}\,z^{-1}\,dz \wedge d\bar{z}\\ &= (2\pi i)^{-1}\iint (z-\zeta)^{-1}\,\partial\varphi(z)/\partial\bar{z}\,dz \wedge d\bar{z}.\end{aligned}$$

Application of Theorem 1.2.1 with u replaced by φ and ω equal to a disc containing the support of φ now gives $\partial u/\partial\bar{\zeta} = \varphi$. Finally, if ω is arbitrary, we can, for every $z_0 \in \omega$, choose a function $\psi \in C_0^{\,k}(\omega)$ which

† The support of a measure or function is the smallest closed set outside which it is equal to 0.

is equal to 1 in a neighborhood V of z_0. If $\mu_1 = \psi\mu$ and $\mu_2 = (1 - \psi)\mu$, we have $u = u_1 + u_2$ where

$$u_j(\zeta) = \int (z - \zeta)^{-1} \, d\mu_j(\zeta).$$

Since μ_1 is equal to $(2\pi i)^{-1}\psi\varphi \, dz \wedge d\bar{z}$ and $\psi\varphi \in C_0^k(\mathbf{R}^2)$, we have $u_1 \in C^k$ and $\partial u_1/\partial\bar{\zeta} = \psi\varphi$. Since μ_2 vanishes in V, it follows that $u \in C^k(V)$ and that $\partial u/\partial\bar{\zeta} = \varphi$ in V. The proof is complete.

Corollary 1.2.3. *Every $u \in A(\Omega)$ is in $C^\infty(\Omega)$. Hence $u' \in A(\Omega)$ if $u \in A(\Omega)$.*

Proof. This follows from Theorems 1.2.1 and 1.2.2 applied to discs ω with $\bar{\omega} \subset \Omega$.

More precise information is given in the next theorem.

Theorem 1.2.4. *For every compact set $K \subset \Omega$ and every open neighborhood $\omega \subset \Omega$ of K there are constants C_j, $j = 0, 1, \cdots$, such that*

$$\sup_{z \in K} |u^{(j)}(z)| \leq C_j \|u\|_{L^1(\omega)}, \qquad u \in A(\Omega), \tag{1.2.4}$$

where $u^{(j)} = \partial^j u/\partial z^j$.

Proof. Choose $\psi \in C_0^\infty(\omega)$ so that $\psi = 1$ in a neighborhood of K. If $u \in A(\Omega)$, we have $\partial(\psi u)/\partial\bar{z} = u\partial\psi/\partial\bar{z}$ and consequently Theorem 1.2.1 applied to ψu gives

$$\psi(\zeta)u(\zeta) = (2\pi i)^{-1} \int u(z) \, \partial\psi/\partial\bar{z} \, (z - \zeta)^{-1} \, dz \wedge d\bar{z}. \tag{1.2.5}$$

Since $\psi = 1$ in a neighborhood of K and $|z - \zeta|$ is bounded from below when $\zeta \in K$ and z is in the support of $\partial\psi/\partial\bar{z}$, differentiation of (1.2.5) leads immediately to (1.2.4).

Corollary 1.2.5. *If $u_n \in A(\Omega)$ and $u_n \to u$ when $n \to \infty$, uniformly on compact subsets of Ω, it follows that $u \in A(\Omega)$.*

Proof. Application of (1.2.4) to $u_n - u_m$ shows that $\partial u_n/\partial z$ converges uniformly. Since $\partial u_n/\partial\bar{z} = 0$, it follows that $\partial u_n/\partial x$ and $\partial u_n/\partial y$ converge uniformly on compact sets. Hence $u \in C^1$ and $\partial u/\partial\bar{z} = \lim \partial u_n/\partial\bar{z} = 0$.

Corollary 1.2.6. (*Stieltjes-Vitali*) *If $u_n \in A(\Omega)$ and the sequence $|u_n|$ is uniformly bounded on every compact subset of Ω, there is a subsequence u_{n_j} converging uniformly on every compact subset of Ω to a limit $u \in A(\Omega)$.*

Proof. As in Corollary 1.2.5, we obtain from Theorem 1.2.4 that there are uniform bounds for the first-order derivatives of u_n on any compact set. Hence this sequence is equicontinuous and the corollary follows from Ascoli's theorem and Corollary 1.2.5.

Corollary 1.2.7. *The sum of a power series*

$$u(z) = \sum_0^\infty a_n z^n$$

is analytic in the interior of the circle of convergence.

Proof. The series converges uniformly in every smaller disc.

Theorem 1.2.8. *If u is analytic in $\Omega = \{z;\ |z| < r\}$, we have*

$$u(z) = \sum_0^\infty u^{(n)}(0)\, z^n/n!$$

with uniform convergence on every compact subset of Ω.

Proof. Let $r_1 < r_2 < r$. We have by (1.2.3)

$$u(z) = (2\pi i)^{-1} \int_{|\zeta| = r_2} u(\zeta)/(\zeta - z)\, d\zeta, \qquad |z| \le r_1. \tag{1.2.6}$$

Since

$$(\zeta - z)^{-1} = \sum_0^\infty z^n \zeta^{-n-1}, \qquad |z| \le r_1, \qquad |\zeta| = r_2,$$

and the series is uniformly and absolutely convergent, the theorem follows if we integrate term by term, noting that (1.2.6) gives

$$u^{(n)}(0) = n!(2\pi i)^{-1} \int_{|\zeta| = r_2} u(\zeta)\zeta^{-n-1}\, d\zeta.$$

Corollary 1.2.9. (*The uniqueness of analytic continuation.*) *If $u \in A(\Omega)$ and there is some point z in Ω where*

$$u^{(k)}(z) = 0, \quad \textit{for all } k \ge 0, \tag{1.2.7}$$

it follows that $u = 0$ in Ω if Ω is connected.

Proof. The set of all $z \in \Omega$ satisfying (1.2.7) is obviously closed in Ω, and by Theorem 1.2.8 it is also open. Since it is non-empty by assumption, it must be equal to Ω.

Corollary 1.2.10. *If u is analytic in the disc $\Omega = \{z;\ |z| < r\}$ and if u is not identically* 0, *one can write u in one and only one way in the form*

$$u(z) = z^n v(z)$$

where n is an integer ≥ 0 *and* $v \in A(\Omega)$, $v(0) \neq 0$ (*which means that* $1/v$ *is also analytic in a neighborhood of* 0).

Proof. The proof is obvious.

Theorem 1.2.11. *If u is analytic in* $\{z;\ |z - z_0| < r\} = \Omega$ *and if* $|u(z)| \leq |u(z_0)|$ *when* $z \in \Omega$, *then u is constant in* Ω.

Proof. We may assume that $u(z_0) \neq 0$. Since

$$u(z_0) = (2\pi)^{-1} \int_0^{2\pi} u(z_0 + \rho e^{i\theta})\, d\theta$$

when $0 < \rho < r$, we obtain

$$\int_0^{2\pi} (1 - u(z_0 + \rho e^{i\theta})/u(z_0))\, d\theta = 0.$$

The real part of the integrand is ≥ 0 and $= 0$ only when $u(z_0) = u(z_0 + \rho e^{i\theta})$. This proves the theorem.

Corollary 1.2.12. (*Maximum principle.*) *Let* Ω *be bounded and let* $u \in C(\bar{\Omega})$ *be analytic in* Ω. *Then the maximum of* $|u|$ *in* $\bar{\Omega}$ *is attained on the boundary.*

Proof. If the maximum is attained in an interior point, Theorem 1.2.11 and Corollary 1.2.9 prove that u is constant in the component of Ω containing that point and therefore $|u|$ assumes the same value at some boundary point.

1.3. The Runge approximation theorem. From Theorem 1.2.8 it follows in particular that a function which is analytic in a disc can be approximated uniformly by polynomials in z on any smaller disc. In particular, every entire function can be approximated by polynomials uniformly on every compact set. We shall now give a general approximation theorem.

Theorem 1.3.1. (*Runge.*) *Let* Ω *be an open set in* $\mathbf{C}$ *and K a compact subset of* Ω. *The following conditions on* Ω *and on K are equivalent:*

(a) *Every function which is analytic in a neighborhood of K can be approximated uniformly on K by functions in* $A(\Omega)$.

(b) *The open set* $\Omega \setminus K = \Omega \cap \complement K$ *has no component which is relatively compact in* Ω.

(c) *For every* $z \in \Omega \setminus K$ *there is a function* $f \in A(\Omega)$ *such that*

$$|f(z)| > \sup_K |f|. \tag{1.3.1}$$

By the remarks preceding the theorem we obtain the following special case by taking $\Omega = \mathbf{C}$.

Corollary 1.3.2. *Every function which is analytic in a neighborhood of the compact set K can be approximated by polynomials uniformly on K if and only if $\complement K$ is connected, or equivalently, for every $z \in \complement K$ there is a polynomial f such that* (1.3.1) *is valid.*

Proof of Theorem 1.3.1. We first prove that (c) $\Rightarrow$ (b) and that (a) $\Rightarrow$ (b). Thus assume that (b) is not valid, that is, that $\Omega \setminus K$ has a component O such that $\bar{O}$ is compact and $\subset \Omega$. Then the boundary of O is a subset of K and the maximum principle gives

$$\sup_{\bar{O}} |f| \le \sup_K |f|, \qquad f \in A(\Omega), \tag{1.3.2}$$

which contradicts (c). If (a) were valid we could for every f which is analytic in a neighborhood of K choose $f_n \in A(\Omega)$ so that $f_n \to f$ uniformly on K. Application of (1.3.2) to $f_n - f_m$ proves that f_n converges uniformly in $\bar{O}$ to a limit F. We have $F = f$ on the boundary of O, and F is analytic in O and continuous in $\bar{O}$. In particular, we can choose $f(z) = 1/(z - \zeta)$ if $\zeta \in O$, and then we have $(z - \zeta)F(z) = 1$ on the boundary of O, hence $(z - \zeta)F(z) = 1$ in O. This gives a contradiction when $z = \zeta$.

To prove that (b) $\Rightarrow$ (a) it suffices to show that every measure μ on K which is orthogonal to $A(\Omega)$ is also orthogonal to every function f which is analytic in a neighborhood of K, for the theorem is then a consequence of the Hahn–Banach theorem. Set

$$\varphi(\zeta) = \int (z - \zeta)^{-1} \, d\mu(z), \qquad \zeta \in \complement K.$$

By Theorem 1.2.2, φ is analytic in $\complement K$, and when $\zeta \in \complement \Omega$ we have

$$\varphi^{(k)}(\zeta) = k! \int (z - \zeta)^{-k-1} \, d\mu(z) = 0 \quad \text{for every } k,$$

for the function $z \to (z - \zeta)^{-k-1}$ is analytic in Ω if $\zeta \in \complement \Omega$. Hence $\varphi = 0$ in every component of $\complement K$ which intersects $\complement \Omega$. Since $\int z^n \, d\mu(z) = 0$ for every n and $(z - \zeta)^{-1}$ can be expanded in a power series in z which converges uniformly on K if $|\zeta| > \sup_K |z|$, we also have $\varphi = 0$ in the unbounded component of $\complement K$. Now (b) guarantees that $\Omega \setminus K$ has no component which is relatively compact in Ω, and we conclude that $\varphi = 0$ in $\complement K$.

Choose a function $\psi \in C_0^{\infty}(\omega)$, where ω is a neighborhood of K in which f is analytic, and choose ψ so that $\psi = 1$ on K. Then we have

$$f(z) = \psi(z) f(z) = (2\pi i)^{-1} \iint f(\zeta) \, \partial\psi(\zeta)/\partial\bar{\zeta} \, (\zeta - z)^{-1} \, d\zeta \wedge d\bar{\zeta}, \qquad z \in K.$$