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Edited by A. Dold and B. Eckmann

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INTRODUCTION

La Douzième Ecole d'Eté de Calcul des Probabilités de Saint-Flour s'est tenue du 22 Août au 8 Septembre 1982 et a rassemblé, outre les conférenciers, une cinquantaine de participants dans les locaux accueillants du Foyer des Planchettes.

Les trois conférenciers, Messieurs Dudley, Kunita et Ledrappier ont entièrement repris la rédaction de leurs cours qui constitue maintenant un texte de référence et ceci justifie le nombre d'années mis pour les publier.

En outre les exposés suivants ont été faits par les participants et ils ont été publiés dans le numéro 76 des Annales Scientifiques de l'Université de Clermont-Ferrand II :

M. HAREL	Convergence pour les processus empiriques éclatés
L.M. LE NY	Forme produit pour des réseaux multiclasses à routages dynamiques
G. LETAC	Mesures sur le cercle et convexes du plan
H.I. PEREIRA	Rate of convergence towards a Frechet type limit distribution
C. SUNYACH	Condition pour qu'une transformation d'un espace uniforme soit une contraction stricte et applications

La frappe du manuscrit a été assurée par les Départements de Kyushu University et de l'Université de Clermont II et nous remercions pour leur soin et leur efficacité les secrétaires qui se sont chargées de ce travail délicat.

Nous exprimons enfin notre gratitude à la Société Springer Verlag qui permet d'accroître l'audience internationale de notre Ecole en accueillant une nouvelle fois ces textes dans la collection Lecture Notes in Mathematics.

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A COURSE ON EMPIRICAL PROCESSES

PAR R.M. DUDLEY

Preface

In this course I have tried to organize and present currently known limit theorems for empirical processes based on variables which are independent and identically distributed with some law P, in rather general (at least, multidimensional) sample spaces. For each limit theorem (Glivenko-Cantelli law of large numbers, Donsker central limit theorem, or law of the iterated logarithm) the object is to find the weakest possible conditions on a class of sets in the sample space, or functions on the space, such that convergence holds simultaneously and uniformly over the class. In this scheme, cumulative empirical distribution functions appear in the quite special case where the class of sets is the class of intervals $(-\infty,x]$ in the line or a Euclidean space. The limit theorems have been extended to much more general classes of sets and functions, especially in the past five years.

Bringing together the material brought up some questions, some of which could be answered, so that there are a few new results in the course. Recent improvements in technique (due to V. I. Kolčinskii and David Pollard) also allow shorter proofs of some older results. I will next briefly survey the contents of the course and mention what is new. (Readers new to the subject may prefer to begin with Chapter 1.)

The main emphasis will be on central limit theorems (Donsker classes). Laws of large numbers are treated in Sections 6.1, 11.1 and 11.2. Some laws of the iterated logarithm and corresponding strong invariance principles will be only briefly mentioned (Sections 1.1, 4.2); such results are proved in a forthcoming paper with Walter Philipp, "Invariance principles for sums of Banach space valued random elements and empirical processes".

Since the supremum norm for the space of all bounded functions on an infinite set is non-separable, there are difficulties of measurability (too many Borel sets) and even of choosing the space in which the empirical process and limiting Gaussian process take their values. The above-mentioned work with W. Philipp develops a new approach to the foundations of empirical processes via P. Major's formulation of Donsker's invariance principle (Section 1.1 below). For a possibly non-measurable function f on a probability space let f^* be the essential infimum of the set of measurable functions g with $g > f$. Let $(S, ||.||)$ be a Banach space, generally non-separable. It turns out that a useful set of inequalities for random variables in separable Banach spaces carry over to $||\cdot||^*$, with only minor modifications in their proofs (Chap. 3). Thus the foundations given in Chapters 1-4 can avoid most of the measurability problems in previous versions of the theory. The new approach avoids the need to define convergence of laws, laws themselves, or sigma-algebras in non-separable metric or Banach spaces.

In connection with the Vapnik-Červonenkis property, which restricts only the intersections of sets in a class with finite sets, some further measurability conditions are really needed, as shown by an example (10.3.3) of Mark Durst and the author.

Limit theorems for empirical measures uniformly over classes $\mathcal{F}$ of functions are not only special cases of limit theorems in a Banach space S, but are actually equivalent to such theorems (Sec. 4.1), where $\mathcal{F}$ is (any norming subset of) the unit ball in the dual of S. For example, the classical strong law of large numbers of E. Mourier in separable Banach spaces, and the Blum-DeHardt Glivenko-Cantelli theorem, both follow from an easy extension of DeHardt's theorem on empirical measures to possibly unbounded functions (Sec. 6.1). Among central limit theorems in separable Banach spaces which do not restrict the geometry of the space, perhaps the best up to now is that of Naresh Jain and Michael Marcus. The Jain-Marcus theorem follows directly from a new central limit theorem of Pollard for empirical processes (Sec. 11.3).

There are at least three kinds of hypotheses sufficient for a central limit theorem uniformly over a class $\mathcal{F}$. If $\mathcal{F}$ is countable, special conditions are available (Chapter 5). In Chap. 6, for each $\varepsilon > 0$, $\mathcal{F}$ is covered by finitely many "brackets" $[f_i, f_j] = \{f: f_i \leq f \leq f_j\}$ where $||f_j - f_i||_p < \varepsilon$ in some $L^p(P)$ norm. If the number $N_p(\varepsilon)$ of brackets needed is always finite, we have a law of large numbers (Sec. 6.1). If $\mathcal{F}$ is uniformly bounded and $\int_0^1 (\log N_1(x^2))^{1/2}\, dx < \infty$ we have a central limit theorem (Sec. 6.2). The hypothesis on $N_1(x^2)$ is sharp (I. S. Borisov's theorem, Sec. 6.3).

In Chap. 7, $N_p(\varepsilon)$ is estimated for concrete classes of functions (satisfying uniform differentiability and Hölder conditions) and sets (regions with differentiable boundaries, convex sets and lower layers).

Chapter 8 is based on my paper "Empirical and Poisson processes on classes of sets or functions too large for central limit theorems". It is shown that the law of the iterated logarithm fails (by a factor of $(\log n)^{1/2}$) for the class of lower layers in the plane. The same holds for the convex sets in 3-space and the class of sets in $\mathbb{R}^d$ with uniformly C^{d-1} smooth boundaries.

Chapter 9 brings in the Vapnik-Červonenkis combinatorial condition on a class $\mathcal{C}$ of sets: for some n, no set with n elements has all its 2^n subsets cut out by sets in $\mathcal{C}$. In the purely combinatorial part of the theory, there are notable results of Vapnik and Červonenkis themselves, N. Sauer, P. Assouad, R. Wenocur and others, yet questions remain open in sets with as few as 6 elements (but many pairs of collections of subsets!).

Section 10.1 notes (a folk theorem?) that the empirical measure is a sufficient statistic for an unknown P. The rest of Chap. 10 develops measurability structures (admissibility, Suslin property) to be used in Chap. 11.

Section 11.1 introduces a kind of entropy, due independently to Kolčinskii and Pollard, which provides a common generalization for a) Vapnik-Červonenkis conditions and b) metric entropy of functions in the supremum norm. Section 11.2 proves the Vapnik-Červonenkis-Steele necessary and sufficient condition on a class $\mathcal{C}$ of sets (with suitable measurability structure) for the weak <u>or</u> strong law of large numbers (Glivenko-Cantelli theorem) uniformly over $\mathcal{C}$, for a given law P. (The recent extension by Vapnik and Červonenkis to uniformly bounded classes of functions is also mentioned.) The proof that the weak law implies the strong law in this case, due to Mike Steele, uses Kingman's subadditive ergodic theorem. Also, a new simpler proof by Kolčinskii and Pollard's methods is given for the Vapnik-Červonenkis weak law (which Steele just referred to in his proof).

Section 11.3 proves Pollard's central limit theorem, but with his separability replaced by the "admissible Suslin" condition from Chap. 10. Here $\mathcal{F}$ is of the form $\{Fg: g \in \mathcal{G}\}$ where $\int F^2 dP < \infty$ and $\mathcal{G}$ is a uniformly bounded class of functions satisfying an integral condition for Pollard's entropy, which does not depend on P. Thus if $F = 1$ one finds classes of functions satisfying the uniform central limit theorem for all P, hence useful in nonparametric statistics where P is unknown (Sec. 12.1).

If $\mathcal{G}$ is a class of indicators of sets, finiteness of Pollard's entropy is equivalent to the Vapnik-Červonenkis property (Sec. 11.1). Also, if the central limit theorem holds uniformly over a class $\mathcal{C}$ of sets for all laws P defined on $\mathcal{C}$, then $\mathcal{C}$ must have the Vapnik-Červonenkis property, a result of Mark Durst and the author (Sec. 11.4). In this sense, Vapnik-Červonenkis classes (with mild measurability conditions, which are satisfied for classes in reasonable applications) are exactly the right classes of sets for non-parametric statistics.

I am very grateful to all those who sent me preprints or reprints of their work on empirical processes, and to those whose comments led to corrections and improvements in the course: to Patrice Assouad, Erich Berger, Yves Derrienic, Joseph Fu, Lucien LeCam, Pascal Massart, Jim Munkres, Walter Philipp, He Sheng Wu, and especially to Lucien Birgé and Tom Salisbury, my heartiest thanks. I would much appreciate hearing from readers of any further errors or improvements.

Many thanks, as well, to Irene Fontaine-Gilmour for excellent typing.

Chapter 1 Introduction

1.1 Invariance principles

The classical results of Donsker (1951, 1952) have recently been put into simpler yet stronger forms as follows. Let λ be Lebesgue measure on $[0,1]$ and let $N(m,\sigma^2)$ denote the normal law with mean m and variance σ^2. Call a probability space nonatomic if there is a random variable on it with law λ. "Independent and identically distributed with law P" will be abbreviated "i.i.d. P".

Theorem 1.1.1. Let P be any probability law on $\mathbb{R}^1$ with mean 0 and variance 1. Then on any nonatomic probability space there exist random variables $X_1, X_2, \ldots$ i.i.d. P, and $Y_1, Y_2, \ldots$ i.i.d. $N(0,1)$, such that

(1.1.2) $\lim_{n\to\infty} n^{-1/2} \max_{k\leq n} |\sum_{j=1}^{k} X_j - Y_j| = 0$ in probability.

Remark. The sequence $\{X_j\}_{j\geq 1}$ is not, of course, independent of the sequence $\{Y_j\}_{j\geq 1}$.

Let $S_n := X_1 + \ldots + X_n$ (throughout, ":=" means "equals by definition"). Let $S_0 := 0$, $f_n(k/n) := S_k/n^{1/2}$, $k = 0, 1, \ldots, n$, and let f_n be linear on each interval $[k/n, (k+1)/n]$. Donsker (1951) proved that the law of f_n, in the separable Banach space $C[0,1]$ of continuous real functions on $[0,1]$ with supremum norm, converges to that of standard Brownian motion. Theorem 1.1.1 is easily seen to imply Donsker's theorem, yet Theorem 1.1.1 itself avoids mentioning any infinite-dimensional sample space such as $C[0,1]$. Further, Theorem 1.1.1 replaces convergence in law by a stronger form of convergence, in probability.

The formulation of Theorem 1.1.1 emerged from proofs of Donsker's "invariance principle" in the books of Breiman (1968, Theorem 13.8, p. 279) and Freedman (1971, p. 83, (130)) and was brought out explicitly by Major (1976, p. 222).

The term "invariance principle", by the way, refers to the fact that the limit in distribution of any functional such as $\max_{k\leq n} S_k/n^{1/2}$ is the same, or invariant, for all X_j i.i.d. with mean 0 and variance 1.

Theorem 1.1.1 extends in a natural way to finite-dimensional Euclidean spaces, and an alternate result implies laws of the iterated logarithm, as follows. On $\mathbb{R}^d$