Combinatorial Games

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Combinatorial Games

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Preface

The subject of combinatorics is only slowly acquiring respectability and combinatorial games will clearly take longer than the rest of combinatorics. Perhaps this partly stems from the puritanical view that anything amusing can't possibly involve any worthwhile mathematics.

In the past, "game theory" has meant the subject delineated by von Neumann and Morgenstern, which has found wide, though usually unsuccessful, application in economics, management, military strategy, and other useful forms of human activity. Combinatorial games, with complete information, no chance moves, and no place for bluffing or coalitions, are of little interest to the classical game theorist, who knows that there is always at least one pure optimal strategy.

Why, then, should we be interested in combinatorial games?

Aviezri Fraenkel gives some cogent reasons in the introduction to his Selected Bibliography at the end of this volume.

There are many connexions with other parts of mathematics, only a few of which have so far begun to be explored, and only one of which is seriously considered here. Vera Pless explains the several connexions with coding theory, through which we make contact with most of the branches of the main stream of combinatorics, including graph theory.

A whole new theory of number, including infinitesimals and transfinite numbers, has emerged as a special case of the theory of games. This is introduced by John Conway in the second chapter. Investigation of this remarkable area is only slowly gaining momentum, in spite of the early appearance of Donald Knuth's popularization, *Surreal Numbers*. Perhaps this only served to perpetuate the myth that we are dealing with a frivolous subject.

As Aviezri Fraenkel explains, complexity theory is very well illustrated by combinatorial games, which supply a plethora of examples of harder problems than most of those which have been considered in the past.

We have been able to do no more than touch on the theory of "hot" games, which are, of course, the interesting ones from a practical, as well as a theoretical point of view. Elwyn Berlekamp explains the significant progress that he has been making with the analysis of endgames in Go, a game long thought to be even more intractable than Chess.

We introduce "impartial" games in Chapter 3. These "tepid" games are of minimal interest to the classical game theorist, but there are plenty of unsolved problems in this area, as well as in the rest of the subject. A notable one is how to deal with the "misère play" of impartial games. A small but important break-through was recently made by William Sibert, and Thane Plambeck is now unveiling the misère analysis of several games which had earlier seemed intransigent.

A list of open problems is given towards the end of the book. While some of these are undoubtedly hard, inroads are being made into others even as I write, and a new generation of graduate students will find a rich vein of material waiting to be investigated.

As examples of what has been and what remains to be done, Richard Nowakowski presents in the last chapter some specific examples of games. Welter's Game is now understood, but see the quotation of Berlekamp by Fraenkel in section 6.2 of the "complexity" chapter. In Conway's game of Sylver Coinage, on the other hand, much remains to be discovered, though G.L. Sicherman and others are slowly revealing more and more of the truth. Finally, Berlekamp's masterly analysis of the well-known children's game of Dots-and-Boxes illustrates the many levels at which a seemingly simple game may be played, and the many quite sophisticated techniques which may be used in its analysis.

We are indebted to Academic Press for permission to reproduce some text and a number of figures from John Conway's *On Numbers and Games* and from *Winning Ways for your Mathematical Plays* by Berlekamp, Conway and Guy.

Thanks to the Short Course Committee for suggesting and to Jim Maxwell and Monica Foulkes for organizing the course that has supplied the *raison d'être* for the present volume, and to Alison Buckser and other members of the American Mathematical Society's staff for its careful and expert production.

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What is a Game?

Richard K. Guy

1 Introduction

We only skim the surface of the vast subject of combinatorial games. For more breadth, depth and detail, consult both of the books:

Elwyn R. Berlekamp, John H. Conway and Richard K. Guy, Winning Ways for your Mathematical Plays, Academic Press, London & New York, 1982;

John H. Conway, On Numbers and Games, *London Math. Soc. Monograph* **6**, Academic Press, London & New York, 1956;
which we will frequently refer to as WW and ONAG respectively.

Two other surveys are Fraenkel (1980), who considers the complexity of games, and Guy (1983), who explores the connexions between games and graphs.

Fraenkel contrasts Nim with Go, the former with a very simple winning strategy, the latter very complicated. In later lectures in this course, Fraenkel will discuss complexity, I will go into some detail about impartial games, whose prototype is the game of Nim, and Berlekamp will tell you about his discoveries concerning the game of Go. Nim has no cycles in its game graph, no interaction between tokens, and is impartial; Go has cycles and interaction and is partizan. The spectrum between the two games spans the complexity gap between polynomial, Pspace-complete, and Extime-complete games. In existential problems such as the travelling salesman problem, high complexity is a liability, but in games and cryptanalysis, it can be an asset.

The paper is in final form and no version of it will be submitted for publication elsewhere.
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Fraenkel also maintains a valuable bibliography of the subject, copies of which may be obtained from him at the Weizmann Institute, Rehovot, Israel. A recent edition appears at the end of the present volume.

Guy surveys the connexions between combinatorial game theory and graph theory: graphs of games; games on graphs (Hackenbush, von Neumann's game, Rims, Rails, Lucasta, Sprouts); the ways graphs can be used to elucidate puzzles (Tantalizer, Rubik's Cube, Fifteen Puzzle, magic squares); and the occurrence of Euler's formula in Berlekamp's analysis of Dots-and-Boxes [WW, 507–550].

2 What We Mean by a Combinatorial Game

[WW, 16–17]

Our games are unlike those of "classical" game theory, that find application in economics, management, and military strategy. Our games (almost always!) satisfy the following conditions:

1. There are just two players, often called Left and Right. There can be no question of **coalitions**.

2. There are several, usually finitely many, **positions**, and often a particular **starting position**.

3. There are clearly defined **rules** that specify the two sets of **moves** that Left and Right can make from a given position to its **options**.

4. Left and Right move alternately, in the game as a whole.

5. In the **normal play** convention a player unable to move **loses**.

6. The rules are such that play will always come to an end because some player will be unable to move. This is called the **ending condition**. There are no games which are drawn by repetition of moves.

7. Both players know what is going on; there is **complete information**. There is no occasion for **bluffing**.

8. There are no **chance moves**: no dealing of cards; no rolling of dice.

Think about games that you know. How far do they satisfy these eight conditions?

Ludo, **Snakes-and-Ladders** and **Backgammon** all have complete information, but contain chance moves, since they all use dice.

Battleships, **Kriegspiel**, **Three-Finger Morra** and **Scissors-Paper-Stone** have no chance moves but the players do not have complete information about the disposition of their opponents' pieces or fingers. Moreover, in the finger games, the players move simultaneously rather than alternately.

Tic-Tac-Toe (**Noughts-and-Crosses**) fails 5. because a player unable to move is not necessarily the loser, since ties are possible. **Chess** also fails 5. and contains positions that are *tied* by stalemate (in which the last player does *not* win) and positions that are *drawn* by infinite play (of which perpetual check is a special case). The words "tied" and "drawn" are often used interchangeably, though with slight transatlantic differences, for games which are neither won nor lost. We suggest that **drawn** be used for cases when this happens because play is drawn out indefinitely and **tied** for cases when play definitely ends but the rules do not award a win to either player.

Monopoly fails on several counts. As in Ludo, there are chance moves and there may be more than two players. The players don't have complete information about the arrangement of the cards and the game could, theoretically, go on for ever.

In **Poker** much of the interest derives from the incompleteness of the information, the chance moves and the possibility of coalitions.

Bridge is a two-person game, each "person" being a team of two, but the players do not even have complete information about their own cards.

Nim is played with heaps of beans. When it is your turn to move, choose a heap and remove as many beans from it as you wish; perhaps the whole heap, but at least one bean. **Grundy's Game** is also played with heaps of beans. A move now is to split a heap into two heaps of *unequal* size, so that heaps of one or two cannot be split. **Wythoff's Game** is played with just two heaps. A move is to remove any number of beans from one heap, or *equal* numbers of beans from both heaps. This last option is an example of a **nondisjunctive** move: it doesn't satisfy the condition for the **sum** of two games, which we will define later. Nim, Grundy's Game, and Wythoff's Game each satisfy all of our eight conditions, together with the additional one that the options from any given position are the same for each player, regardless of whose turn it is to move. Such games are called **impartial**; those games in which the options for the two players are not all alike are called **partizan**.

Dots-and-Boxes is won by the player scoring the larger number of boxes, so that it doesn't satisfy the normal play convention. However, as Richard Nowakowski explains in the last chapter, it can almost always be treated as an impartial game, satisfying the normal play convention. Part of its theory uses that of **Kayles** and of **Dawson's Kayles**. These two games are played with rows of skittles: in Kayles a move removes just one skittle, or two adjacent ones, so that the game repeatedly splits into a sum of smaller games. Similarly in Dawson's Kayles, in which the move is to remove any skittle, provided that its immediate neighbors, if any, are also removed.

Sylver Coinage is an impartial game which uses the **misère play convention** that the last person to play *loses.* Misère games are usually very difficult to analyze, though a recent breakthrough was made by William Sibert and John Conway who have found the complete analysis of **Misère Kayles.**

Go is a good example of a "hot" partizan game, i.e. one in which, in the great majority of positions, each player is eager to make the next move. As Elwyn Berlekamp explains in a later chapter, he has recently been making good progress with analyzing the concluding stages of the game, using his generalization of the idea of "overheating" [WW 170–174].

We will often refer to a move as being "good" if it wins, and "bad" if it won't. In theory it usually suffices to find any good move, or to show than no good move exists. But in real life games there are many other criteria for choosing between your various options. If you're *losing*, then all your options are bad in the above sense, but in practice they're not all equal, and you might prefer one that makes the situation too complicated for your opponent to analyze (the **Enough Rope Principle**).

It is hard to draw the line between mathematics and psychology. There are even cases where you should prefer a bad move to a good one! Your opponent might be learning how to play a game with which you're already familiar. In this case you'll probably be able to win a few times despite the bad moves you deliberately make so as not to give away your strategy. Or one move, theoretically the best, might gain you only a dollar, while another, which loses a dollar, might win you a hundred if your opponent fails to find the subtle winning reply. Or you may be a baby-sitter, whose job is much more peaceful if your opponent wins. Or a card-sharp who's losing while the stakes are low, in anticipation of winning later when the stakes are higher.

3 Game Graphs and Trees

A game may be visualized as a digraph: the nodes are the positions and the arcs are the options. The arcs may be thought of as colored, say

bLue, Red, or grEen

according as the option is available

to Left only, to Right only, or to Either player.

Alternatively, we may distinguish between different plays of the game, i.e. different dipaths in the digraph, by duplicating the nodes as necessary and representing the game by a rooted tree. The root is the starting position and the arcs are directed away from the root.

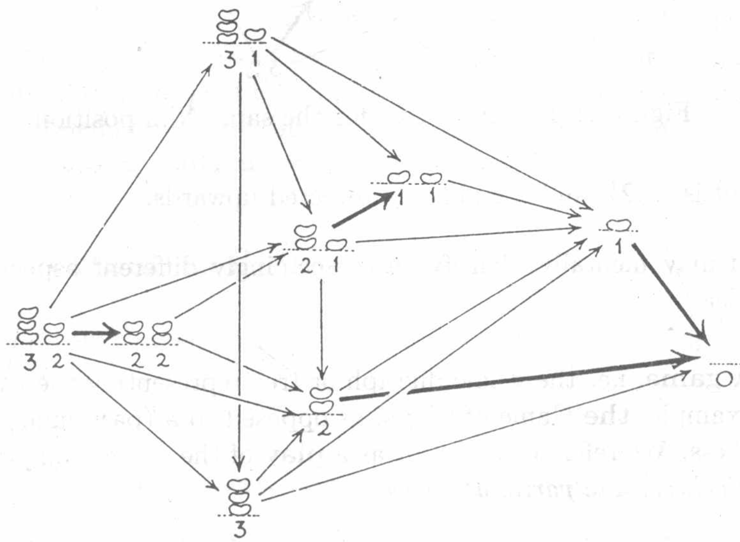

Figure 1: The game graph for the Nim position {3,2}.
From the leftmost position the next player can win
by adopting the strategy indicated by the heavy arrows.

Figures 1 and 2 show the game graph and the game tree for the position {3,2} in a game of Nim: two heaps, one with three beans, the other with two. Nim is an example of an impartial game, in every position of which the same set of options is available to either player: think of the arcs in Figures 1 and 2 as being colored green. Nim is played with a number of heaps of beans. The typical option, for either player, is to choose a heap and remove from it as many beans as you wish: the whole heap maybe, but at least one bean.

Notice the difference between the **complete analysis** of a game, and a **winning strategy**. Figure 2 is a complete analysis: for a winning strategy it suffices to describe the four black arrows.

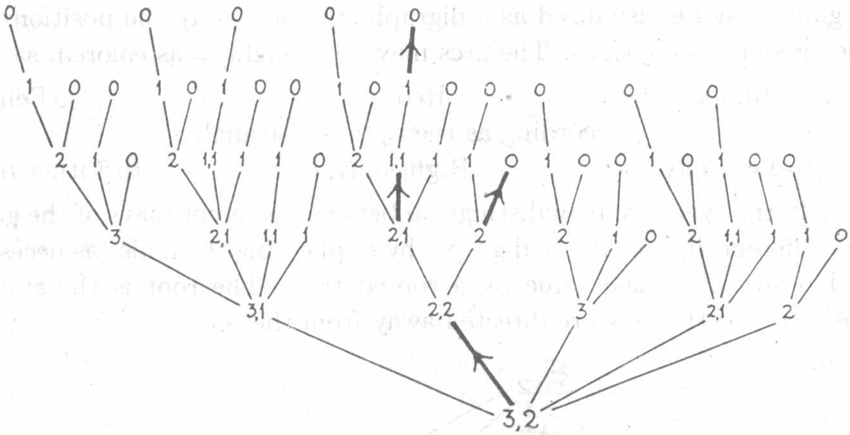

Figure 2: The game tree for the same Nim position.

The root is {3,2}, and the arcs are directed upwards.

You may mentally identify three seemingly different aspects of the same idea.

1. A **game**, i.e. the whole digraph or tree representing the game. For example, **the** Game of Chess, as opposed to **a** (particular) game of chess. We refer to the latter as a **play** of the game: compare *le jeu d'échecs, une partie d'échecs.*

2. A **position** in a game; a particular node of the digraph, perhaps the root of the tree. For example, the standard opening position in Chess, ready for a play of the game.

3. The **ordered pair** of sets of options available to the two players from a given position, e.g.

{Pa3,Pa4,...,Ph3,Ph4,Sa3,Sc3,Sf3,Sh3|Pa6,Pa5,...,Ph6,Ph5,Sa6,Sc6,Sf6,Sh6}

A position, such as the rightmost in Figure 1, or any zero in Figure 2, from which neither player has any option, is a **terminal position**, at which the game ends. The **outcome** is then specified by the rules. It may be a win for Left, or a win for Right, possibly accompanied by some score or payoff. The rules may not specify a winner, so that the game may end in a **tie**. For present purposes we will adopt the **normal play convention** that the winner is the player who has just made the last move: equivalently, **last player winning**; if you can't move, you lose. We won't have time to say very much about the **misère play convention**, which accords the win to a player unable to move: **last player losing**. Analysis is far more difficult in this case.

To ensure that we *have* a last player, our games must end. We assume that they satisfy the **ending condition**: that there is no infinite sequence of options. Notice that this condition prohibits *all* infinite sequences, not merely those in which Left and Right make alternate moves. In order to give **values** to our games, we need to consider the possibility of several consecutive moves by the same player. This can occur in the play of the **sum** of two or more games, as we shall see.

A game that does not satisfy the ending condition is called a **loopy game**. Its digraph will contain a directed circuit or an infinite directed path. The outcome may be a **draw**: note that we distinguish between a *tied* game and one *drawn out* by infinite play. Chess exhibits both kinds of outcome: stalemate is a tie, but perpetual check, repetition of moves, or insufficient mating material are equivalent to draws.

4 The Formal Definition of a Game

This is deceptively simple: each game is an ordered pair of sets of games:

$$G = \{\{G^{L_1}, G^{L_2}, \ldots\}|\{G^{R_1}, G^{R_2}, \ldots\}\}.$$

To avoid proliferation of braces, we write this more compactly as

$$G = \{G^L|G^R\}$$

where we must remember that G^L and G^R are *sets* of Left and Right options, which may, for example, be infinite, or empty. Indeed the definition is inductive, and the empty set is the basis for the induction, which starts with the **Endgame**

$$\{\oslash|\oslash\} = \{\ |\ \},$$

in which neither player has an option, and which we will denote by 0 (**zero**).

Here, and from now on, we use several symbols, which are familiar in elementary arithmetic, with the strong implication that we can manipulate games in the same way that we manipulate numbers in ordinary arithmetic. Some games behave like numbers and we call them numbers, **but to justify the manipulations takes more space than we have here,** so turn to pages 71–96 of ONAG if you would like more detail and further examples. See also the next chapter, Numbers and Games, by John Conway.

It's helpful to attach ordinal numbers, or **birthdays**, to games, and to introduce the idea of **simplicity** [WW, 23–27]. When a move is made in a game, it becomes *simpler* in the sense that we arrive at a position with an earlier birthday. All definitions and proofs are inductive in that they are assumed to have been made for all simpler games. The basis, as we have already stated, is the simplest game of all, the Endgame, born on day zero.

On day one we have two sets, the empty set and the set $\{0\}$ consisting of the Endgame; so that we can visualize 2^2 games. Their game trees (in which Left's moves slope up to the left and Right's moves slope up to the right), together with their names, are shown in Fig. 3.

$0 = \{\ |\ \}$ $1 = \{0|\ \}$ $-1 = \{\ |0\}$ $* = \{0|0\}$

Figure 3: The four simplest games, born on days zero and one.

We quote from p. 72 of ONAG:

> The simplest game of all is the *Endgame*, 0. I courteously offer you the first move in this game, and call upon you to make it. You lose, of course, because 0 is defined as the game in which it is never legal to make a move.
>
> In the game $1 = \{0|\}$, there is a legal move for Left, which ends the game, but at no time is there any legal move for Right. If I play Left, and you Right, and you have first move again (only fair, as you lost the previous game) you will lose again, being unable to move even from the initial position. To demonstrate my skill, I shall now start from the same position, make my legal move to 0, and call upon you to make yours.
>
> Of course you are now beginning to suspect that Left always wins, so for our next game, -1, you may play as Left and I as Right! For the last of our examples, the new game $* = \{0|0\}$, you may play whichever role you wish, provided that for this privilege you allow me to play first.

In summary:

The Endgame is the prototype of games in which the next player loses, since no option is available: a **second player win.**

The game 1 is a **Left win**, no matter who starts: if Louise starts, she goes to $\{\ |\ \} = 0$ and Richard has no option and loses; if Richard starts, he has no option and loses even more quickly.

The game -1 is a **Right win**, no matter who starts.

The game $\{0|0\} = *$ ("Star") is the simplest game which is not a number [WW, 40]. It is a **first player win.**

5 The Four Outcome Classes

If we adopt the normal play convention, every game belongs to just one of four outcome classes [ONAG, Theorem 50] which are exemplified by the four games we've just seen. The terminology and notation are displayed in Figure 4.

If, in a game G		Right starts	
		& L has a winning strategy	& R has a winning strategy
Left starts	& R has a winning strategy	ZERO $G = 0$ 2nd wins	NEGATIVE $G < 0$ R wins
	& L has a winning strategy	POSITIVE $G > 0$ L wins	FUZZY $G \| 0$ 1st wins

Figure 4: The four outcome classes.

It is convenient to combine these outcome classes and symbols in pairs.

If	Left	Right	Left	Right	has a winning strategy
provided	Right	Left	Left	Right	starts, then we
write	$G \geq 0$	$G \leq 0$	$G \vert\!\rhd 0$	$G \lhd\!\vert 0$	corresponding to
the	1st col	1st row	2nd row	2nd col	of Figure 4.

6 The Negative of a Game

A device to breathe new life into an otherwise one-sided contest is to allow a novice opponent, when he feels he is losing, to turn the board around, to reverse the roles of the two players, to handicap his more skilled adversary, by asking her to defend what appears to him to be an inferior position. This replaces the game by its negative. Formally, the **negative** of G,

$$-G = \{-G^R \mid -G^L\}$$

is defined inductively [WW, 35]. Remember that $-G^R$, for example, is short for the set $\{-G^{R_1}, -G^{R_2}, \ldots\}$, whose members are simpler games than $-G$, and hence have been defined earlier.

7 Sums of Games

There are many ways of playing two or more games simultaneously, but often the most natural is what we call the **sum**, or **disjunctive compound** [ONAG, 75; WW, 33]. Nim, for example, is the sum of a number of games of one-heap Nim. In the sum of two or more component games, the player whose turn it is to move selects one component and makes a legal move in it:

$$G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}.$$

Once again this is an inductive definition: $G^R + H$, for example, represents the set of options $\{G^{R_1} + H, G^{R_2} + H, \ldots\}$ each of which is a simpler game than $G + H$, so that addition there is already defined.

It's not hard to see that sums are commutative and associative, that $G + 0 = G$, and [ONAG, Theorem 51] that $G + (-G) = 0$. In that last sentence we've used zero in two quite different senses. In $G + 0 = G$ we intended 0 to mean the Endgame, $\{\ \mid\ \}$. In $G + (-G) = 0$ we intended

"= 0" to mean "is a zero game", that is "belongs to the (very large!) equivalence class of games for which the second player has a winning strategy". Check that $1 + (-1) = 0$ and that $* + * = 0$, so that we can speak of the games $1 + (-1)$ and $* + *$ as having the same **value**, 0, as the Endgame, even though their **forms** are different.

More generally, we will say that two games are **equivalent**, and have the same **value**, and write $G = H$, if the game $G + (-H)$ is a second player win. With the above definitions of sum, negative and zero, the set of all games forms a commutative group. Moreover, games form a partially ordered set, and we write $G > H$ just if $G - H > 0$, that is, if Left can win the sum $G + (-H)$, no matter who starts. Our notation is justified by theorems such as the following, proved in [ONAG, 76]. If $G \geq 0$ and $H \geq 0$, then $G + H \geq 0$. If H is a zero game (that is, a win for the second player), then $G + H$ has the same outcome as G. If $H - K$ is a zero game, then $G + H$ and $G + K$ have the same outcome.

8 The Games Born on Day Two

As day two dawns we have four games to play with, and so $2^4 = 16$ sets of games. There are 16 choices for Left's options and 16 for Right's, giving a potential of 256 games on day two. However, things are not *quite* that complicated, in that for each player, some options are clearly preferable to others. The four games born on day one can be arranged in the lattice (in the poset sense, rather than the geometrical sense) of Figure 5, in which Left's preferences are placed higher, and Right's are lower.

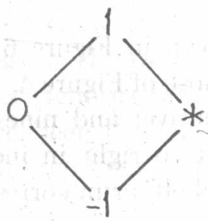

Figure 5: The lattice of games born on day one.

The only set of options for which there is any doubt in either player's mind about the best move, is the incomparable pair $\{0, *\}$. So, for a player's options we need consider only six possibilities: the empty set, the four singletons, and this incomparable pair. Among the resulting 6^2 possibilities for games born on day two, just 22 are inequivalent and