Lecture Notes in Mathematics 1578

Joseph Bernstein Valery Lunts

Equivariant Sheaves and Functors

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- a table of contents;
- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

Further remarks and relevant addresses at the back of this book.

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Contents

Introduction.

Let $f : X \to Y$ be a continuous map of locally compact spaces. Let $Sh(X)$, $Sh(Y)$ denote the abelian categories of sheaves on X and Y, and $D(X)$, $D(Y)$ denote the corresponding derived categories (maybe bounded $D = D^b$ or bounded below $D = D^+$ if necessary). It is well known that there exist functors

$$f^*,\ f_*,\ f^!,\ f_!,\ D,\ Hom,\ \otimes$$

between the categories $D(X)$ and $D(Y)$, which satisfy certain identities.

Now assume that X, Y are in addition G-spaces for a topological group G, and that f is a G-map. Instead of sheaves let us consider the equivariant sheaves $Sh_G(X)$, $Sh_G(Y)$. One wants to have triangulated categories $D_G(X)$, $D_G(Y)$ - "derived categories of equivariant sheaves" - together with all the above functors. More precisely, there should exist the forgetful functor

$$For : D_G \to D,$$

so that the functors in categories D_G are compatible with the usual ones in categories D under this forgetful functor. Simple examples show that the derived category $D(Sh_G)$ of the abelian category Sh_G cannot be taken for D_G (unless the group G is discrete). The main purpose of this work is to introduce the suitable category D_G and to define the corresponding functors.

Actually, we get more structure. Namely, let $\phi : H \to G$ be a homomorphism of groups, X be an H-space, Y be a G-space, and $f : X \to Y$ be a map compatible with the homomorphism ϕ. In this situation we have functors of inverse and direct image

$$Q_f^* : D_G(Y) \to D_H(X),$$

$$Q_{f*} : D_H(X) \to D_G(Y).$$

The direct image functor Q_{f*} is probably the most interesting one. It does not in general commute with the forgetful functor.

For a connected Lie group G we give an algebraic description of the triangulated category $D_G(pt)$ in terms of DG-modules over a natural DG-algebra $\mathcal{A}_G$. This description is our main tool in applications of the theory. As an example of an application we "compute" the equivariant intersection cohomology (with compact supports) of toric varieties.

Let us explain briefly the structure of the text. Part I is devoted mainly to the definition of the category $D_G(X)$ and of various functors. In part II we use DG-modules to study the category $D_G(pt)$ and discuss equivariant cohomology. Finally, in the last part III the general theory is applied to toric varieties, which yields some applications to combinatorics.

This text summarizes the work which started some five years ago. During this period the authors were partially supported by the NSF.

Part I. Derived category $D_G(X)$ and functors.

0. Some preliminaries.

0.1. Let G be a topological group and X be a topological space. We say that X is a G-**space** if G acts continously on X. This means that the multiplication map

$$m: \ G \times X \to X, \quad (g,x) \mapsto gx$$

is continuous.

Let X, Y be G-spaces. A continuous map $f : X \to Y$ is called a G-**map** if it commutes with the action of G on X and Y.

More generally, let $\phi : H \to G$ be a homomorphism of topological groups. Let X be an H-space and Y be a G-space and $f : X \to Y$ be a continuous map. We call f a ϕ-**map** if

$$f(hx) = \phi(h)f(x)$$

for all $x \in X, \ h \in H$.

Let X be a G-space. We denote by $\overline{X} := G\backslash X$ the quotient space (the space of G-orbits) of X and by $q : X \to \overline{X}$ the natural projection. By definition q is a continuous and open map.

0.2. Let X be a G-space. Consider the diagram of spaces

$$G \times G \times X \ \underset{d_2}{\overset{d_0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{d_1}}}}} \ G \times X \ \underset{d_1}{\overset{d_0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{s_0}}}}} \ X$$

where

$$d_0(g_1, \ldots, g_n, x) = (g_2, \ldots, g_n, g_1^{-1}x)$$

$$d_i(g_1, \ldots, g_n, x) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_n, x), \quad 1 \leq i \leq n-1$$

$$d_n(g_1, \ldots, g_n, x) = (g_1, \ldots, g_{n-1}, x)$$

$$s_0(x) = (e, x)$$

A G-**equivariant sheaf** on X is a pair (F, θ), where $F \in Sh(X)$ and θ is an isomorphism

$$\theta : d_1^* F \simeq d_0^* F,$$

satisfying the cocycle condition

$$d_0^*\theta \circ d_2^*\theta = d_1^*\theta, \quad s_0^*\theta = id_F.$$

We will always assume that F is an abelian sheaf or, more generally, a sheaf of R - modules for some fixed ring R.

A **morphism** of equivariant sheaves is a morphism of sheaves $F \to F'$ which commutes with θ.

Equivariant sheaves form an abelian category which we denote by $Sh_G(X)$.

Examples.

1. $Sh_G(G) \simeq R - mod.$
2. If G is a connected group, then $Sh_G(pt) \simeq R - mod.$

Remark. In case G is a discrete group, a G-equivariant sheaf is simply a sheaf F together with an action of G which is compatible with its action on X (cf. [Groth]).

0.3. Consider the quotient map $q : X \to \overline{X}$. Let $H \in Sh(\overline{X})$. Then $q^*(H) \in Sh(X)$ is naturally a G-equivariant sheaf. This defines a functor

$$q^* : Sh(\overline{X}) \to Sh_G(X).$$

Let $F \in Sh_G(X)$. Then the direct image $q_*F \in Sh(\overline{X})$ has a natural action of G. Denote by $q_*^G F = (q_*F)^G$ the subsheaf of G-invariants of q_*F. This defines a functor

$$q_*^G : Sh_G(X) \to Sh(\overline{X}).$$

Definition. A G-space X is **free** if

a) the stabilizer $G_x = \{g \in G | gx = x\}$ of every point $x \in X$ is trivial, and

b) the quotient map $q : X \to \overline{X}$ is a locally trivial fibration with fibre G.

A free G-space X is sometimes called a principal G-homogeneous space over $\overline{X}$.

The following lemma is well known.

Lemma. Let X be a free G-space. Then the functor $q^* : Sh(\overline{X}) \to Sh_G(X)$ is an equivalence of categories. The inverse functor is $q_*^G : Sh_G(X) \to Sh(\overline{X})$.

0.4. The last lemma shows that in case of a free G-space we may identify the equivariant category $Sh_G(X)$ with the sheaves on the quotient $Sh(\overline{X})$. Hence in this case one may define the **derived category** $D_G(X)$ of equivariant sheaves on X to be the derived category of the abelian category $Sh_G(X)$, i. e.

$$D_G(X) := D(Sh_G(X)) = D(Sh(\overline{X})).$$

If X is not a free G-space, the category $D(Sh_G(X))$ does not make much sense in general. (However, it is still the right object in case G is a discrete group (see section 8 below)).

It turns out that in order to give a good definition of $D_G(X)$ one has first of all to resolve the G-space X, i.e. replace X by a free G-space, and then to use the above naive construction of D_G for a free space. This allows us to define all usual functors in D_G with all usual properties.

It is possible to give a more abstract definition of D_G using simplicial topological spaces (see Appendix B). However, we do not know how to define functors using this definition and hence never use it.

1. Review of sheaves and functors.

This section is a review of the usual sheaf theory on locally compact spaces and on pseudomanifolds. The subsections on the smooth base change (1.8) and on acyclic maps (1.9) will be especially important to us. We will mostly follow [Bo1].

1.1. Let X be a topological space. We fix a commutative ring R with 1 and denote by C_X the constant sheaf of rings on X with stalk R. We denote by $Sh(X)$ the abelian category of C_X-modules (i.e., sheaves of R-modules) on X.

Let $f : X \to Y$ be a continuous map of topological spaces. We denote by $f^* : Sh(Y) \to Sh(X)$ the inverse image functor and by $f_* : Sh(X) \to Sh(Y)$ the direct image functor. The functor f^* is exact and $f^*(C_Y) = C_X$. The functor f_* is left exact and we denote by $R^i f_*$ its right derived functors.

Our main object of study is the category $D^b(X)$ - the bounded derived category of $Sh(X)$. We also consider the bounded below derived category $D^+(X)$.

A continuous map $f : X \to Y$ defines functors

$$f^* : D^b(Y) \to D^b(X) \quad \text{and} \quad Rf_* : D^+(X) \to D^+(Y).$$

Remark. Since we mostly work with derived categories, we usually omit the sign of the derived functor and write f_* instead of Rf_*, $\otimes$ instead of $\overset{L}{\otimes}$ and so on.

1.2. Truncated derived categories (see [BBD])

For any integer a we denote by $D^{\leq a}(X)$ the full subcategory of objects $A \in D^+(X)$ which satisfy $H^i(A) = 0$ for $i > a$. The natural imbedding $D^{\leq a}(X) \to D^+(X)$ has a right adjoint functor $\tau_{\leq a} : D(X)^+ \to D^{\leq a}(X)$, which is called the truncation functor.

Similarly we define the subcategory $D^{\geq a}(X) \subset D^+(X)$ and the truncation functor $\tau_{\geq a} : D^+(X) \to D^{\geq a}(X)$.

Given a segment $I = [a, b] \subseteq \mathbf{Z}$ we denote by $D^I(X)$ the full subcategory $D^{\geq a}(X) \cap D^{\leq b}(X) \subset D^b(X)$.

Subcategories $D^{\geq a}(X)$, $D^{\leq b}(X)$ and $D^I(X)$ are closed under extensions (i.e. if in an exact triangle $A \to B \to C$ objects A and C lie in a subcategory, then B also lies in the subcategory). All these subcategories are preserved by inverse image functors.

If $J \subset I$, we have a natural fully faithful functor $D^J(X) \to D^I(X)$. The category $D^b(X)$ can be reconstructed from the system of finite categories $D^I(X)$, namely

$$D^b(X) = \lim_{I} D^I(X).$$

Since all functors $D^J(X) \to D^I(X)$ are fully faithful, there are no difficulties in defining this limit.

In the case when $I = [0,0]$ the subcategory $D^I(X)$ is naturally equivalent to $Sh(X)$. This is the heart of the category $D^b(X)$ with respect to t-structure defined by truncation functors τ (see [BBD]).

1.3. We assume that the coefficient ring R is noetherian of finite homological dimension (in fact we are mostly interested in the case when R is a field, usually of characteristic 0). Then we can define functors of tensor product $\otimes : D^b(X) \times D^b(X) \to D^b(X)$ and $Hom : D^b(X)^0 \times D^+(X) \to D^+(X)$ (see [Bol], V.6.2 and V.7.9).

1.4. For locally compact spaces one has additional functors $f_!, f^!$ and the Verdier duality functor D. In order to define these functors we will work only with a special class of topological spaces. Namely, we say that a topological space X is **nice** if it is Hausdorff and locally homeomorphic to a pseudomanifold of dimension bounded by $d = d(X)$ (see [Bol]).

Every nice topological space is locally compact, locally completely paracompact and has finite cohomological dimension (see [Bol]). In particular every object in $D^b(X)$ can be realized by a bounded complex of injective sheaves. In fact we could consider instead of nice spaces the category of topological spaces satisfying these properties.

Let $f : X \to Y$ be a continuous map of nice topological spaces. Then following [Bol] we define functors $f_*, f_! : D^b(X) \to D^b(Y)$, and $f^*, f^! : D^b(Y) \to D^b(X)$.

Functors described above are connected by some natural morphisms. We will describe some of them; one can find a pretty complete list in [GoMa]. These properties are important for us since we would like them to hold in the equivariant situation as well.

We denote by $\mathcal{T}$ the category of topological spaces.

In the rest of this section 1 (except for 1.9) we assume that all spaces are nice.

1.4.1. We have the following natural functorial isomorphisms.

$$Hom(A \otimes B, C) \simeq Hom(A, Hom(B, C)).$$

$$f^*(A \otimes B) \simeq f^*(A) \otimes f^*(B).$$

1.4.2. Composition. Given continuous maps $f : X \to Y$ and $g : Y \to Z$ there are natural isomorphisms of functors $(fg)^* = g^* \cdot f^*, (fg)^! = g^! \cdot f^!, (fg)_* = f_* \cdot g_*, (fg)_! = f_! \cdot g_!$.

1.4.3. Adjoint functors. The functor f^* is naturally left adjoint to f_* and the functor $f_!$ is naturally left adjoint to $f^!$.

1.4.4. There is a canonical morphism of functors $f_! \to f_*$ which is an isomorphism when f is proper.

1.4.5. Exact triangle of an open subset. Let $U \subset X$ be an open subset, $Y = X \setminus U, i : Y \to X$ and $j : U \to X$ natural inclusions. Then for every $F \in D^b(X)$ adjunction morphisms give exact triangles

$$i_! i^! F \to F \to j_* j^* F$$

and

$$j_! j^! F \to F \to i_* i^* F.$$

In this case $i_! = i_*$ and $j_!$ are extensions by zero, $j^* = j^!$ is the restriction to an open subset, $i_! i^!$ is the derived functor of sections with support in Y.

1.4.6. Base change. In applications we usually fix a topological space S (a base) and consider the category $\mathcal{T}/S$ of topological spaces over the base S. An object of this category is a pair $X \in \mathcal{T}$ and a map $X \to S$.

Every continuous map $\nu : T \to S$ defines a base change $\sim: \mathcal{T}/S \to \mathcal{T}/T$ by $X \mapsto \widetilde{X} = X \times_S T$.

Given a space X/S we will use the projection $\nu : \widetilde{X} \to X$ to define a base change functor $\nu^* : D^b(X) \to D^b(\widetilde{X})$. This functor commutes with functors f^* and $f_!$, i.e. there are natural functorial isomorphisms

$$\nu^* f^* = f^* \nu^* \quad \text{and} \quad \nu^* f_! = f_! \nu^*.$$

Similarly, there are natural isomorphisms

$$\nu^! f^! = f^! \nu^! \quad \text{and} \quad \nu^! f_* = f_* \nu^!.$$

1.4.7. Properties of the functor $f^!$.

The object $D_f := f^!(C_Y) \in D^b(X)$ is called the dualizing object of f.

1. We say that the map f is locally fibered if for every point $x \in X$ there exist neighbourhoods U of x in X and V of $y = f(x)$ in Y such that the map $f : X \to Y$ is homeomorphic to a projection $F \times V \to V$.

Assume that f is locally fibered. Then for every $A \in D^b(Y)$ there is a natural isomorphism

$$f^!(A) \simeq f^*(A) \otimes f^!(C_Y)$$

(see [Ve2]).

2. Let $f : X \hookrightarrow Y$ be a closed embedding. We say that f is relatively smooth if there exists an open neighbourhood U of X in Y, such that $U = X \times \mathbf{R}^d$ and f is the embedding of the zero section $f(x) = (x, 0)$. Let $p : U \to X$ be the projection. An object $F \in D^b(Y)$ is called smooth relative to X if $F_U = p^* F'$ for some $F' \in D^b(X)$.

Assume $f : X \to Y$ is a relatively smooth embedding. Then $D_f \in D^b(X)$ is invertible (see 1.5 below). Let $F \in D^b(Y)$ be smooth relative to X. Then we have a natural isomorphism in $D^b(X)$

$$f^! F = f^* F \otimes D_f.$$

In particular the dualizing object D_Y (see 1.6.1 below) of Y is smooth relative to X and we have

$$D_X = f^! D_Y = f^* D_Y \otimes D_f.$$

3. Let

$$\begin{array}{ccc} Z_p & \xrightarrow{j} & Z \\ \downarrow f & & \downarrow f \\ \{p\} & \xhookrightarrow{i} & W \end{array}$$

be a pullback square, where $f : Z \to W$ is a locally trivial fibration, and $j : Z_p \to Z$ is the inclusion of the fiber. Then we have a canonical isomorphism of functors

$$j^! \cdot f^* = f^* \cdot i^!.$$

1.5. Twist. An object $L \in D^b(X)$ is called **invertible** if it is locally isomorphic to $C_X[n]$ - the constant sheaf C_X placed in degree $-n$. Then for $L^{-1} := Hom(L, C_X)$ the natural morphism $L \otimes L^{-1} \to C_X$ is an isomorphism. Every invertible object L defines a twist functor $L : D^b(X) \to D^b(X)$ by $A \longmapsto L \otimes A$. If L, M are invertible objects, then $N = L \otimes M$ is also invertible and the twist by N is isomorphic to the product of twists by L and M. In particular, the twist functor by L has an inverse given by the twist by L^{-1}.

The twist is compatible with all basic functors. For example $L \otimes (A \otimes B) \simeq (L \otimes A) \otimes B$ and $L \otimes Hom(A, B) = Hom(A, L \otimes B) = Hom(L^{-1} \otimes A, B)$.

Fix a base S and an invertible object L in $D^b(S)$. It defines a family of twist functors L in categories $D^b(X)$ for all spaces X/S; namely if $p : X \to S$ and $A \in D^b(X)$, then $L(A) = p^*(L) \otimes A$. This twist is compatible with all our functors, i.e., for every continuous map $f : X \to Y$ over the base S there are canonical isomorphisms of functors

$$f^* L = L f^*, \ f^! L = L f^!, \ f_* L = L f_*, \ f_! L = L f_!.$$

These isomorphisms are compatible with isomorphisms in 1.4.

1.6. Verdier duality

1.6.1. Let us fix an invertible object D_{pt} in $D^b(pt)$ and call it a dualizing object over the point. For any nice topological space X we define its **dualizing object** $D_X \in D^b(X)$ to be $p^!(D_{pt})$, where $p : X \to pt$. If X is a smooth manifold of dimension d the dualizing object D_X is invertible (1.5) and locally isomorphic to $C_X[d]$. Using this dualizing object we define the Verdier duality functor $D : D^b(X) \to D^b(X)$ by $D(A) = Hom(A, D_X)$.

For any object $A \in D^b(X)$ we have a canonical functorial biduality morphism

$$A \to D(D(A)).$$

1.6.2. Theorem (Verdier duality). *For any continuous map f there are canonical functorial isomorphisms*

$$Df_! = f_*D \quad \textit{and} \quad f^!D = Df^*.$$

1.6.3. Different choices of the object D_{pt} give rise to different duality functors, which differ by a twist. We will choose the standard normalization $D_{pt} = C_{pt}$ (see [Bo1]).

Remark. This standard normalization is not always natural. For example, if $R = k(M)$ is an algebra of functions on a nonsingular algebraic variety M, the natural choice for D_{pt} is a dualizing module for M, equal to Ω_M $[dimM]$.

1.7. Smooth maps. Let $f : X \to Y$ be a continuous map of topological spaces. We say that f is smooth of relative dimension d if for every point $x \in X$ there exist neighborhoods U of x in X and V of $f(x)$ in Y such that the restricted map $f : U \to V$ is homeomorphic to the projection $V \times \mathbf{R}^d \to V$.

For a smooth map f the dualizing object $D_f \in D^b(X)$ is invertible and is locally isomorphic to $C_X[d]$.

1.8. Smooth base change. Consider a smooth base change $\nu : T \to S$. If X is a nice topological space (see 1.4.), then the space $\widetilde{X} = X \times_S T$ is also nice. The crucial observation, which makes our approach possible, is that in this situation the base change functor $\nu^* : D^b(X) \to D^b(\widetilde{X})$ essentially commutes with all other functors.

Theorem (Smooth base change).
(i) We have canonical functorial isomorphisms

$$\nu^*(A \otimes B) \;=\; \nu^*(A) \otimes \nu^*(B),$$

$$\nu^*(Hom(A,B)) = Hom(\nu^*(A),\nu^*(B)).$$

(ii) Let $f : X \to Y$ be any map of spaces over S. Let us denote by the same symbol the corresponding map $\widetilde{X} \to \widetilde{Y}$. Then for $A \in D^b(X)$, $B \in D^b(Y)$ we have canonical isomorphisms

$$\nu^* f_*(A) \simeq f_*\nu^*(A), \quad \nu^* f_!(A) \simeq f_!\nu^*(A)$$

$$\nu^* f^*(B) \simeq f^*\nu^*(B), \quad \nu^* f^!(B) \simeq f^!\nu^*(B).$$

These isomorphisms are compatible with isomorphisms in 1.4.

(iii) The Verdier duality commutes with ν^* up to a twist by the (invertible) dualizing object D_ν of $\nu : T \to S$. Namely

$$D(\nu^*(A)) = D_\nu \otimes \nu^*(D(A)).$$

This isomorphism is compatible with the identities in 1.6. For example, if we identify $\nu^*(DD(A)) \simeq DD(\nu^* A)$ using the last isomorphism then ν^* preserves the biduality morphism (1.6.1).

We will discuss this theorem in Appendix **A**.

1.9. Acyclic maps. Fix $n \geq 0$. In this section we consider general topological spaces. The proofs are given in Appendix A below.

1.9.1. Definition. We say that a continuous map $f : X \to Y$ is **n-acyclic** if it satisfies the following conditions:

a) For any sheaf $B \in Sh(Y)$ the adjunction morphism $B \to R^0 f_* f^*(B)$ is an isomorphism and $R^i f_* f^*(B) = 0$ for $i = 1, 2, \ldots, n$.

b) For any base change $\widetilde{Y} \to Y$ the induced map $f : \widetilde{X} = X \times_Y \widetilde{Y} \to \widetilde{Y}$ satisfies the property a).

We say that f is ∞-acyclic if it is n-acyclic for all n.

It is convenient to rewrite the condition a) in terms of derived categories. Namely, consider the functor $\sigma = \tau_{\leq n} \cdot f_* : D^b(X) \to D^b(Y)$. Then the adjunction morphisms $B \to f_* f^*(B)$ and $f^* f_*(A) \to A$ define functorial morphisms $\tau_{\leq n}(B) \to \sigma f^*(B)$ and $f^*\sigma(A) \to \tau_{\leq n}(A)$.

The condition a) can be now written as

a′) For any sheaf $B \in Sh(Y) \subset D^b(Y)$ the natural morphism $B \to \sigma f^*(B)$ is an isomorphism.

1.9.2. It turns out that for an n-acyclic map $f : X \to Y$ large pieces of the category $D^b(Y)$ can be realized as full subcategories in $D^b(X)$. Namely, let us say that an object $A \in D^+(X)$ **comes from** Y if it is isomorphic to an object of the form $f^*(B)$

for some $B \in D^+(Y)$. We denote by $D^+(X|Y) \subset D^+(X)$ the full subcategory of objects which come from Y.

Let us fix a segment $I = [a,b] \subset \mathbf{Z}$ and consider the truncated subcategory $D^I(X|Y) = D^I(X) \cap D^+(X|Y)$.

Proposition (see Appendix **A**). *Let $f : X \to Y$ be an n-acyclic map, where $n \geq |I| = b - a$ (resp ∞-acyclic). Then*
(i) The functor $f^* : D^I(Y) \to D^I(X|Y)$ (resp. $f^* : D^+(Y) \to D^+(X|Y)$) is an equivalence of categories. The inverse functor is given by $\sigma = \tau_{\leq b} \circ f_* : D^b(X) \to D^b(Y)$ (resp. $f_* : D^+(X) \to D^+(Y)$).
(ii) The functor f^* gives a bijection of the sets of equivalence classes of exact triangles in $D^I(Y)$ and $D^I(X|Y)$ (resp. in $D^+(Y)$ and $D^+(X|Y)$). In other words a diagram (T) in $D^I(Y)$ is an exact triangle iff the diagram $f^*(T)$ in $D^I(X)$ is an exact triangle.
(iii) The subcategory $D^I(X|Y) \subset D^b(X)$ (resp. $D^+(X|Y) \subset D^+(X)$) is closed under extensions and taking direct summands.

1.9.3. The following lemma gives a criterion, when an object $A \in D^I(X)$ comes from Y.

Lemma. *Suppose we have a base change $q : \widetilde{Y} \to Y$ in which q is epimorphic and admits local sections. Set $\widetilde{X} = X \times_Y \widetilde{Y}$ and consider the induced map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$. Then*
(i) The induced map $\widetilde{f}$ is n-acyclic if and only if f is n-acyclic.
(ii) Suppose $f, \widetilde{f}$ are n-acyclic. Let $A \in D^I(X)$, where $|I| \leq n$. Then A comes from Y if and only if its base change $\widetilde{A} = q^*(A) \in D^I(\widetilde{X})$ comes from $\widetilde{Y}$.
(iii) The above assertions hold if we replace "n-acyclic" by "∞-acyclic" and D^I by D^+.

1.9.4. The following criterion, which is a version of the Vietoris-Begle theorem, gives us a tool for constructing n-acyclic maps.

We say that a topological space M is n-**acyclic**, if it is non-empty, connected, locally connected (i.e. every point has a fundamental system of connected neighborhoods) and for any R-module A we have $H^0(M, A) \simeq A$ and $H^i(M, A) = 0$ for $i = 1, 2, \ldots, n$.

Criterion. *Let $f : X \to Y$ be a locally fibered map (1.4.7). Suppose that all fibers of f are n-acyclic. Then f is n-acyclic.*

1.10. Constructible complexes.

Suppose that X is a pseudomanifold with a given stratification $\mathcal{S}$ (see [Bo1] I.1). We denote by $D^b_c(X; \mathcal{S})$ the full subcategory of $\mathcal{S}$-constructible complexes in $D^b(X)$,