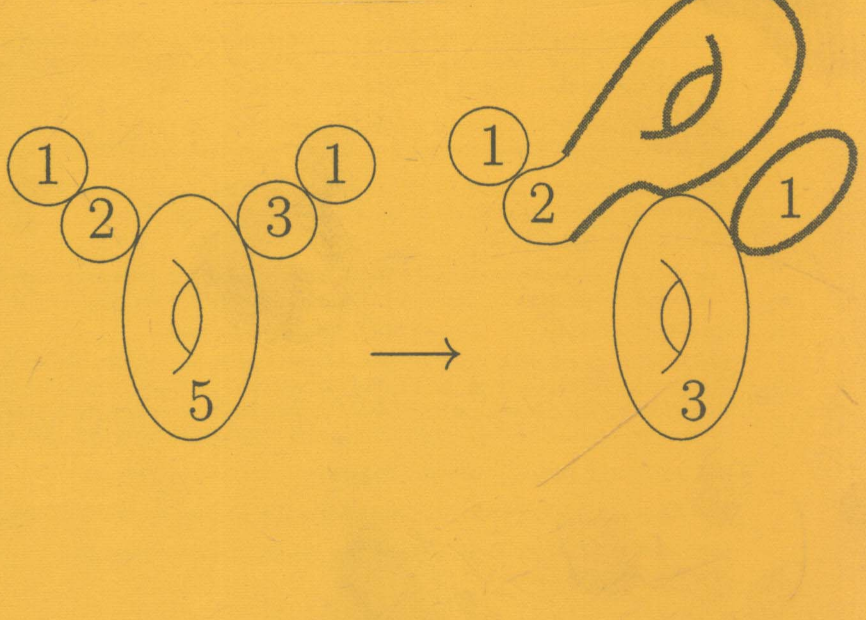
Lecture Notes in Mathematics

1886

Shigeru Takamura

Splitting Deformations of Degenerations of Complex Curves

Towards the Classification of Atoms of Degenerations, III



Springer

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Library of Congress Control Number: 2006923235

Mathematics Subject Classification (2000): 14D05, 14J15, 14H15, 32S30

ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692
ISBN-10 3-540-33363-0 Springer Berlin Heidelberg New York
ISBN-13 978-3-33363-0 Springer Berlin Heidelberg New York

DOI 10.1007/b138136

Typesetting: by the authors and SPi using a Springer LaTeX package
Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11735212 VA41/3100/SPi 5 4 3 2 1 0

Abstract

This is the third in our series of works which make a systematic study of degenerations of complex curves, and their splitting deformations. The principal aim of the present volume is to develop a new deformation theory of degenerations of complex curves. The construction of these deformations uses special subdivisors of singular fibers, which are characterized by some analytic and combinatorial properties. Intuitively speaking, given a special subdivisor, we will construct a deformation of the degeneration in such a way that the subdivisor is 'barked' (peeled) off from the singular fiber. The construction of these "barking deformations" are very geometric and related to deformations of surface singularities (in particular, cyclic quotient singularities) as well as the mapping class groups of Riemann surfaces (complex curves) via monodromies; moreover the positions of the singularities of a singular fiber appearing in a barking deformation is described in terms of the zeros of a certain polynomial which is expressed in terms of the Riemann theta function and its derivative. In addition to the solid foundation of the theory, we provide several applications, such as (1) a construction of interesting examples of splitting deformations which leads to the class number problem of splitting deformations and (2) the complete classification of absolute atoms of genus from 1 to 5. For genus 1 and 2 cases, this result recovers those of B. Moishezon and E. Horikawa respectively.

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Introduction

Wading through,
And wading through,
Yet green mountains still.
(Santoka "Somokuto[1]")

This is the third in our series of works on degenerations of complex curves. (We here use "complex curve" instead of "Riemann surface".) The aim of the present volume is to develop a new deformation theory of degenerations of complex curves. This theory is very geometric and a particular class of subdivisors contained in singular fibers plays a prominent role in the construction of deformations. It also reveals the close relationship between the monodromy of a degeneration and existence of deformations of the degeneration. Moreover, via some diagrams, we may visually understand how a singular fiber is deformed. These deformations are called *barking deformations*, because in the process of deformation, some special subdivisor of the singular fiber looks like "barked" (peeled) off. We point out that barking deformations have a remarkable cross-disciplinary nature; they are related to algebraic geometry, low dimensional topology, and singularity theory.

We will further develop our theory: In [Ta,IV], we describe the vanishing cycles of the nodes of the singular fibers appearing in barking families; we then apply this result to give the Dehn twist decompositions of some automorphisms of Riemann surfaces. In [Ta,V], we develop the moduli theory of splitting deformations, which as a special case, includes the theory of barking deformations over several parameters (in the present volume, we mainly discuss the one-parameter deformation theory).

Background

We will give a brief survey on history and recent development of degenerations of complex curves. Our review is not exhaustive but only covers related topics to our book.

[1] Translated by Hisashi Miura and James Green.

Degenerations of complex curves

A degeneration of complex curves is a one-parameter family of smooth complex curves, which degenerates to a singular complex curve. More precisely, let $\pi : M \to \Delta$ be a proper surjective holomorphic map from a smooth complex surface M to a small disk $\Delta := \{s \in \mathbb{C} : |s| < \delta\}$ such that $\pi^{-1}(0)$ is singular and $\pi^{-1}(s)$ for $s \neq 0$ is a smooth complex curve of genus g ($g \geq 1$); so the origin $0 \in \Delta$ is the critical value of π. (In what follows, unless otherwise mentioned, complex surfaces (curves) are always supposed to be smooth.) We say that $\pi : M \to \Delta$ is a *degeneration* of complex curves of genus g with the *singular fiber* $X := \pi^{-1}(0)$. For simplicity, we sometimes say "a degeneration of genus g".

Let $f : S \to C$ be a proper surjective holomorphic map from a compact complex surface S to a compact complex curve C, and then S is called a *fibered surface* (e.g. elliptic surface). We note that a degeneration appears as a local model of a fibered surface around a singular fiber: Let X be a singular fiber of $f : S \to C$, and then the restriction of f to a sufficiently small neighborhood (germ) of X in S is a degeneration. To classify fibered surfaces, it is important to understand their local structure — degeneration — around each singular fiber. It is also important to know when the signature $\sigma(S)$ (or some other invariant) of the fibered surface concentrates on singular fibers. Namely, when does the equality $\sigma(S) = \sum_i \sigma_{\mathrm{loc}}(M_i)$ holds?, where M_i is a germ of a singular fiber X_i in S, and $\sigma_{\mathrm{loc}}(M_i)$ denotes the local signature of M_i, and the summation runs over all singular fibers (see a survey [AK]). These questions motivate us to study degenerations and their invariants.

Apart from the (local) signature, we have another basic invariant "monodromy" of a degeneration, which also plays an important role in studying degenerations. Given a degeneration $\pi : M \to \Delta$ of complex curves of genus g, we may associate an element h of the symplectic group $Sp(2g : \mathbb{Z})$ acting on the homology group $H_1(\Sigma_g : \mathbb{Z})$, where Σ_g is a smooth fiber of $\pi : M \to \Delta$. The element h is defined as follows. We take a circle $S^1 := \{\,|s| = r\,\}$ contained in the disk Δ, and then $R := \pi^{-1}(S^1)$ is a real 3-manifold. The map $\pi : R \to S^1$ is a fibration (all fibers are diffeomorphic); that is, R is a Σ_g-bundle over S^1, where Σ_g is a smooth fiber of $\pi : M \to \Delta$. Topologically, R is obtained from a product space $\Sigma_g \times [0,1]$ by the identification of the boundary $\Sigma_g \times \{0\}$ and $\Sigma_g \times \{1\}$ via a homeomorphism γ of Σ_g. We say that γ is the *topological monodromy* of the degeneration $\pi : M \to \Delta$. (It measures how the complex surface M is twisted around the singular fiber X.) Then γ induces an automorphism $h := \gamma_*$ on $H_1(\Sigma_g : \mathbb{Z})$, which is called the *monodromy* of the degeneration. Note that h preserves the intersection form on $H_1(\Sigma_g : \mathbb{Z})$, and so $h \in Sp(2g : \mathbb{Z})$.

Monodromy already appeared in the early study of degenerations, notably the work of Kodaira [Ko1] on the classification of degenerations of elliptic curves (complex curves of genus 1). He showed that there are eight degenerations and determined their monodromies: The singular fibers of eight degenerations are respectively denoted by $I_n, I_n^*, II, III, IV, II^*, III^*, IV^*$. (Apart from the three types II, III, IV, each corresponds to an extended Dynkin diagram.) Kodaira also gave explicit construction of these eight degenerations.

Subsequently, Namikawa and Ueno [NU] carried out the classification of degenerations of complex curves of genus 2: there are about 120 degenerations. Namikawa and Ueno encountered with new phenomena, which did not occur in the genus 1 case: (1) The topological type of a degeneration is not necessarily determined by its singular fiber: There are topologically different degenerations of complex curves of genus 2 with the same singular fiber. (2) The monodromy does not determine the topological type of a degeneration. In fact, if $g \geq 2$, there are a lot of topologically different degenerations with the trivial topological monodromy. The reason is as follows: The mapping class group MCG_g of a complex curve of genus g has a natural homomorphism $MCG_g \to Sp(2g : \mathbb{Z})$ (homological representation), as $\gamma \in MCG_g$ induces an automorphism γ_* of $H_1(\Sigma_g : \mathbb{Z})$. The kernel of this homomorphism is the *Torelli group* T_g. (Note: If $g = 1$, then T_g is trivial (i.e. the above homomorphism is injective), whereas if $g \geq 2$, then T_g is nontrivial.) In particular, if $g \geq 2$, and the topological monodromy γ of a degeneration belongs to T_g, then $h := \gamma_*$ (monodromy) is the identity.

This fact indicates that monodromy is not powerful enough to classify degenerations. Moreover, as is suggested by Namikawa and Ueno's classification of 120 degenerations of genus 2, there seem a tremendous amount of degenerations of genus g, as g grows higher, and further classifications for genus 3, 4, ... got stuck. New development came from topology. Observe that in the converting process from a topological monodromy to a monodromy, some information may be lost, and hence it is natural to guess that a topological monodromy carries more information than a monodromy, and this is the starting point of the work of Matsumoto and Montesinos, which we shall explain. First of all, we note that the topological monodromy of a degeneration is a very special homeomorphism; it is either periodic or pseudo-periodic (see [Im], [ES], [ST]). Here, a homeomorphism γ of a complex curve C is *periodic* if for some positive integer m, γ^m is isotopic to the identity, and *pseudo-periodic* if for some loops (simple closed curves) $l_1, l_2, \ldots, l_n$ on C, the restriction γ on $C \setminus \{l_1, l_2, \ldots, l_n\}$ is periodic. A Dehn twist γ along a loop l on C is an example of a pseudo-periodic homeomorphism, as the restriction of γ to $C \setminus l$ is isotopic to the identity.

Remark 1 There is a classical study of pseudo-periodic homeomorphisms due to Nielsen [Ni1] and [Ni2]; he referred to a pseudo-periodic homeomorphism as algebraically finite type.

For a pseudo-periodic homeomorphism γ, let m be the integer as above, i.e. γ^m on $C \setminus \{l_1, l_2, \ldots, l_n\}$ is isotopic to the identity. Then γ^m is generated by Dehn twists along $l_1, l_2, \ldots, l_n$. According to the direction of the twist, a Dehn twist is called *right* or *left*. A pseudo-periodic homeomorphism γ is *right* or *left* provided that γ^m is generated only by right or left Dehn twists. The complex

structure on a degeneration poses a strong constraint on the property of its **topological monodromy**. Using the theory of Teichmüller spaces, Earle–Sipe [ES] and Shiga–Tanigawa [ST] demonstrated that any topological monodromy is a right pseudo-periodic homeomorphism — in [MM2], it is called a pseudo-periodic homeomorphism of *negative type*. For example, if the singular fiber is a Lefschetz fiber (a reduced curve with one node), then the topological monodromy is a right Dehn twist along a loop l on a smooth fiber C. Note that the singular fiber is obtained from C by pinching l; in other words, l is the vanishing cycle.

Matsumoto–Montesinos theory

Matsumoto and Montesinos established the converse of the result of Earle–Sipe and Shiga–Tanigawa. Namely, given a periodic or right pseudo-periodic homeomorphism γ, they constructed a degeneration with the topological monodromy γ. Their argument is quite topological, using "open book construction". In [Ta,II], we gave algebro-geometric construction, clarifying the relationship between topological monodromies and quotient singularities.

We denote by $\mathcal{P}_g$ the set of periodic and right pseudo-periodic homeomorphisms of a complex curve of genus g, and denote by $\widehat{\mathcal{P}}_g$ the conjugacy classes of $\mathcal{P}_g$. Next, we denote by $\mathcal{D}_g$ the set of degenerations of complex curves of genus g, and denote by $\widehat{\mathcal{D}}_g$ its topologically equivalent classes. The main result of Matsumoto and Montesinos [MM2] is as follows:

Theorem 2 (Matsumoto and Montesinos [MM2]) *The elements of $\widehat{\mathcal{P}}_g$ are in one to one correspondence with the elements of $\widehat{\mathcal{D}}_g$.*

One important consequence of this theorem is that the topological classification of degenerations completely reduces to the classification of periodic and right pseudo-periodic homeomorphisms.

Matsumoto and Montesinos [MM2] also determined the shape (configuration) of the singular fiber of a degeneration in terms of the data of its topological monodromy — screw numbers and ramification data. Here, we must take care when using the word "shape", because a shape depends on the choice of model of a degeneration, and it changes under blow up or down. Algebraic geometers usually work with the relatively minimal model of a degeneration — a degeneration is *relatively minimal* if any irreducible component of its singular fiber is not an exceptional curve (a projective line with the self-intersection number -1). However, from the viewpoint of topological monodromies, the relatively minimal model is not so natural. The most natural one is the normally minimal model, because it reflects the topological monodromy very well [MM2]. We now review the definition. Express a singular fiber X as a divisor: $X = \sum_i m_i \Theta_i$ where Θ_i is an irreducible component and a positive integer m_i is its multiplicity. Then $\pi : M \to \Delta$ is called *normally minimal* if X satisfies the following conditions:

(1) the reduced curve $X_{\text{red}} := \sum_i \Theta_i$ is normal crossing (i.e. any singularity of X_{red} is a node), and
(2) if Θ_i is an exceptional curve, then Θ_i intersects other irreducible components at at least three points.

We point out that a relatively minimal degeneration, after successive blow up, becomes a normally minimal one, which is uniquely determined from the relatively minimal degeneration.

In what follows, unless otherwise mentioned, we assume that a degeneration is normally minimal. According to whether the topological monodromy is periodic or pseudo-periodic, the singular fiber is *stellar* (star-shaped) or *constellar* (constellation-shaped). Here, a singular fiber X is called stellar[2] if its dual graph is stellar (star-shaped): X has a central irreducible component (*core*), and several chains of projective lines emanating from the core (see Figure 4.2.1, p61). Such a chain of projective lines is called a *branch* of X. A constellar singular fiber is obtained by bonding branches of stellar fibers, and a resulting chain of projective lines after bonding is called a *trunk*; it is a bridge joining two stellar singular fibers.

The number of the singular fibers of genus g increases rapidly, as g grows higher; this is because a constellar singular fiber is constructed from stellar singular fibers in an inductive way with respect to the genus. For instance, a constellar singular fiber of genus 2 is bonding of two stellar singular fibers of genus 1. (Precisely speaking, there is also a constellar singular fiber of genus 2 obtained from one stellar singular fiber of genus 1 by bonding its two branches.) A constellar singular fiber of genus 3 is either bonding of three stellar singular fibers of genus 1, or bonding of two stellar singular fibers of genus 1 and 2. And as g grows, the partition of the integer g increases rapidly, and accordingly the number of constellar singular fibers increases rapidly.

Based on the work of Matsumoto and Montesinos, Ashikaga and Ishizaka [AI] proposed an algorithm to carry out the topological classification of degenerations of given genus. Although the practical computation becomes difficult as genus grows higher, their algorithm settled down the topological classification problem of degenerations at least theoretically. They applied their algorithm to achieve the topological classification for the genus 3 case (see [AI]): The number of degenerations is about 1600, and among them there are about 50 degenerations with stellar singular fibers. (For any genus, the number of stellar singular fibers is much less than that of constellar singular fibers.)

Morsification

There are about 8, 120, and 1600 degenerations of genus 1, 2, and 3 respectively, and as the genus grows higher, the number of degenerations increases

[2] We have a similar notion in singularity theory, that is, a star-shaped singularity: A singularity V is *star-shaped* if the dual graph of the exceptional set in the resolution space of V is star-shaped, e.g. a singularity with $\mathbb{C}^{\times}$-action. See [OW], [Pn].

rapidly. This fact motivates us to consider another kind of classification — "classification of degenerations modulo deformations". Before we explain it, we review related materials from Morse theory, which elucidates the relationship between the shapes of smooth manifolds and smooth functions on them. One of the key ingredients of Morse theory is the Morse Lemma, asserting that we may perturb a smooth function $f : M \to \mathbb{R}$ in such a way that $f_t : M \to \mathbb{R}$ has only non-degenerate critical points. A non-degenerate critical points is stable under arbitrary perturbation, and so the Morse lemma ensures that we may split critical points of f into stable ones under perturbation. Of course, the Morse lemma is a result in the smooth category, but its spirit is carried over to the complex category, for instance, *Morsification of singularities*: When does an isolated singularity V admits a deformation $\{V_t\}$ such that V_t for $t \neq 0$ possesses only A_1-singularities? (It is known that any hypersurface isolated singularity admits a Morsification, e.g. see Dimca [Di] p82)

We next explain *Morsification of singular fibers*, which was advocated by M. Reid [Re]. First of all, we review splitting deformations.

Splitting deformations of degenerations

Let $\Delta^\dagger := \{t \in \mathbb{C} : |t| < \varepsilon\}$ be a sufficiently small disk. Suppose that $\mathcal{M}$ is a complex 3-manifold, and $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a proper flat surjective holomorphic map. We set $M_t := \Psi^{-1}(\Delta \times \{t\})$ and $\pi_t := \Psi|_{M_t} : M_t \to \Delta \times \{t\}$. (Hereafter, we denote $\Delta \times \{t\}$ simply by Δ, so that $\pi_t : M_t \to \Delta$.) We say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *deformation family* of $\pi : M \to \Delta$ if $\pi_0 : M_0 \to \Delta$ coincides with $\pi : M \to \Delta$. In this case, $\pi_t : M_t \to \Delta$ is referred to as a *deformation* of $\pi : M \to \Delta$.

Suppose that $\pi_t : M_t \to \Delta$ for $t \neq 0$ has at least two singular fibers, say, $X_1, X_2, \ldots, X_n$ $(n \geq 2)$. Then we say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *splitting family* of the degeneration $\pi : M \to \Delta$, and that $\pi_t : M_t \to \Delta$ is a *splitting deformation* of $\pi : M \to \Delta$. In this case, we say that the singular fiber $X = \pi^{-1}(0)$ *splits into* $X_1, X_2, \ldots, X_n$.

To the contrary, if a singular fiber X admits no splitting deformations at all, the degeneration $\pi : M \to \Delta$ is called *atomic*. The singular fiber of the atomic degeneration is called an *atomic fiber*. (Caution: This terminology is not completely rigorous, because a singular fiber does not determine the topological type of a degeneration, so we must use it with care.) A Lefschetz fiber (i.e. a reduced curve with one node) and a multiple $m\Theta$ of a smooth curve Θ, where $m \geq 2$ is an integer, are examples of atomic fibers (see [Ta,I]).

A *Morsification* of a degeneration $\pi : M \to \Delta$ is a splitting family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ such that for $t \neq 0$, all singular fibers of $\pi_t : M_t \to \Delta$ are atomic fibers. Unfortunately this notion is too restrictive, as many degenerations of high genus seem to admit no Morsifications. Instead, we work with a weaker notion "a finite-stage Morsification", defined as follows. If $\pi : M \to \Delta$ is not atomic, take a splitting family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$, say, X splits into $X_1, X_2, \ldots, X_n$ (the first-stage splitting). If all singular fibers $X_1, X_2, \ldots, X_n$

are atomic, the first-stage splitting is a Morsification. If some X_i is not atomic, then take a sufficiently small neighborhood M_i of X_i in M_t, and then consider the restriction of π_t to M_i, which is a degeneration $\pi_i : M_i \to \Delta$ (called the *fiber germ* of X_i in $\pi_t : M_t \to \Delta$). Next, take a splitting family $\Psi_i : \mathcal{M}_i \to \Delta \times \Delta^\dagger$ of $\pi_i : M_i \to \Delta$, say, X_i splits into $X_{i,1}, X_{i,2}, \ldots, X_{i,m}$ (the second-stage splitting). Repeating this process, we finally reach to a set of atomic fibers, say, $X'_1, X'_2, \ldots, X'_l$: Under the finite-stage Morsification, X splits into atomic fibers $X'_1, X'_2, \ldots, X'_l$. In this case, we obtain a smooth 4-manifold M' together with a locally holomorphic map $\pi' : M' \to \Delta$ such that (1) M' is diffeomorphic to M and (2) all singular fibers $X'_1, X'_2, \ldots, X'_l$ of π' are atomic. Here, "locally holomorphic map" means that M' has a complex structure around X'_i, and π' is holomorphic with respect to this complex structure. A finite-stage Morsification of a degeneration is useful for studying the topological types of fibered algebraic surfaces.

There is another motivation from algebraic geometry to study Morsification, inspired by the following question: How does an invariant of a degeneration (e.g. local signature, Horikawa index [AA1]) behave under splitting. Specifically, let $\mathrm{inv}(\pi)$ be some invariant of a degeneration $\pi : M \to \Delta$. Suppose that $\pi_t : M_t \to \Delta$ is a splitting deformation, which splits the singular fiber X into singular fibers $X_1, X_2, \ldots, X_n$. Then find a formula of the form

$$\mathrm{inv}(\pi) = \sum_{i=1}^{n} \mathrm{inv}(\pi_i) + c,$$

where $\pi_i : M_i \to \Delta$ is a fiber germ of X_i in M_t, and c is a "correction term". For these problems, we refer the reader to excellent surveys [AE], [AK], and also [Re].

A primary concern of the Morsification problem of degenerations is to classify all atomic degenerations. The number of atomic degenerations of genus g must be much less than that of all degenerations of genus g, and so this problem leads us to a reasonable classification — *classification of degenerations modulo deformations*.

When is a degneration atomic? Before we discuss this problem, we explain several methods to construct splitting families.

Double covering method for hyperelliptic degenerations

A hyperelliptic curve C is a complex curve which admits a double covering $C \to \mathbb{P}^1$ branched over $2g+2$ points on $\mathbb{P}^1$, where $g = \mathrm{genus}(C)$. (All complex curves of genus 1 and 2 are hyperelliptic.) A degeneration $\pi : M \to \Delta$ is called *hyperelliptic* provided that any smooth fiber $\pi^{-1}(s)$ is a hyperelliptic curve. In this case, the total space M is expressed as a double covering $M \to \mathbb{P}^1 \times \Delta$ branched over a complex curve (*branch curve*) B in $\mathbb{P}^1 \times \Delta$, and conversely from this double covering, we may recover the hyperelliptic degeneration $\pi : M \to \Delta$. (Precisely speaking, instead of M, we need to take