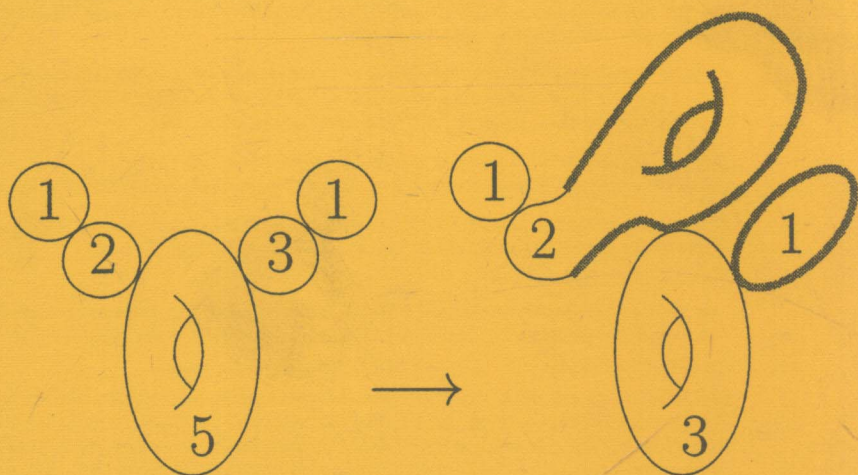


Shigeru Takamura

# Splitting Deformations of Degenerations of Complex Curves

Towards the Classification  
of Atoms of Degenerations, III

1886



Shigeru Takamura

# Splitting Deformations of Degenerations of Complex Curves.

Towards the Classification  
of Atoms of Degenerations, III

 Springer

Author

Shigeru Takamura

Department of Mathematics

Graduate School of Science

Kyoto University

Oiwakecho, Kitashirakawa

Sakyo-Ku, Kyoto 606-8502

Japan

*e-mail: takamura@math.kyoto-u.ac.jp*

Library of Congress Control Number: 2006923235

Mathematics Subject Classification (2000): 14D05, 14J15, 14H15, 32S30

ISSN print edition: 0075-8434

ISSN electronic edition: 1617-9692

ISBN-10 3-540-33363-0 Springer Berlin Heidelberg New York

ISBN-13 978-3-33363-0 Springer Berlin Heidelberg New York

DOI 10.1007/b138136

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable for prosecution under the German Copyright Law.

Springer is a part of Springer Science+Business Media

springer.com

© Springer-Verlag Berlin Heidelberg 2006

Printed in The Netherlands

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting: by the authors and SPi using a Springer L<sup>A</sup>T<sub>E</sub>X package

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper      SPIN: 11735212      VA41/3100/SPi      5 4 3 2 1 0

---

## Abstract

This is the third in our series of works which make a systematic study of degenerations of complex curves, and their splitting deformations. The principal aim of the present volume is to develop a new deformation theory of degenerations of complex curves. The construction of these deformations uses special subdivisors of singular fibers, which are characterized by some analytic and combinatorial properties. Intuitively speaking, given a special subdivisor, we will construct a deformation of the degeneration in such a way that the subdivisor is ‘barked’ (peeled) off from the singular fiber. The construction of these “barking deformations” are very geometric and related to deformations of surface singularities (in particular, cyclic quotient singularities) as well as the mapping class groups of Riemann surfaces (complex curves) via monodromies; moreover the positions of the singularities of a singular fiber appearing in a barking deformation is described in terms of the zeros of a certain polynomial which is expressed in terms of the Riemann theta function and its derivative. In addition to the solid foundation of the theory, we provide several applications, such as (1) a construction of interesting examples of splitting deformations which leads to the class number problem of splitting deformations and (2) the complete classification of absolute atoms of genus from 1 to 5. For genus 1 and 2 cases, this result recovers those of B. Moishezon and E. Horikawa respectively.

---

# Contents

<b>Introduction</b> .....	1
<b>Notation</b> .....	17
<hr/>	
<b>Part I Basic Notions and Ideas</b>	
<hr/>	
<b>1 Splitting Deformations of Degenerations</b> .....	23
1.1 Definitions .....	23
1.2 Splitting criteria via configuration of singular fibers .....	30
<b>2 What is a barking?</b> .....	33
2.1 Barking, I .....	33
2.2 Barking, II .....	37
<b>3 Semi-Local Barking Deformations: Ideas and Examples</b> ....	41
3.1 Semi-local example, I (Reduced barking) .....	41
3.2 Semi-local example, II (Multiple barking) .....	46
3.3 Semi-local example, III .....	48
3.4 Supplement: Numerical condition .....	51
3.5 Supplement: Example of computation of discriminant loci ....	53
<b>4 Global Barking Deformations: Ideas and Examples</b> .....	57
4.1 Preparation: Simplification lemmas .....	57
4.2 Typical examples of barking deformations .....	60
4.3 Supplement: Collision and Symmetry .....	74
4.3.1 Collision, I .....	74
4.3.2 Collision, II .....	76
4.3.3 Construction based on symmetry .....	78

---

**Part II Deformations of Tubular Neighborhoods of Branches**


---

<b>5</b>	<b>Deformations of Tubular Neighborhoods of Branches (Preparation)</b> .....	85
5.1	Branches .....	85
5.2	Deformation atlas .....	87
5.3	Subbranches .....	89
5.4	Dominant subbranches .....	91
5.5	Tame and wild subbranches .....	93
5.5.1	Supplement: Riemenschneider's work .....	98
<b>6</b>	<b>Construction of Deformations by Tame Subbranches</b> .....	99
6.1	Construction of deformations by tame subbranches .....	99
6.2	Supplement for the proof of Theorem 6.1.1 .....	104
6.2.1	Alternative construction .....	104
6.2.2	Generalization .....	104
6.3	Proportional subbranches .....	107
6.4	Singular fibers .....	109
<b>7</b>	<b>Construction of Deformations of type <math>A_l</math></b> .....	119
7.1	Deformations of type $A_l$ .....	119
7.2	Singular fibers .....	124
7.3	Supplement: Singularities of certain curves .....	129
7.4	Newton polygons and singularities .....	137
<b>8</b>	<b>Construction of Deformations by Wild Subbranches</b> .....	143
8.1	Deformations of ripple type .....	144
8.2	Singular fibers .....	150
<b>9</b>	<b>Subbranches of Types <math>A_l</math>, <math>B_l</math>, <math>C_l</math></b> .....	153
9.1	Subbranches of types $A_l$ , $B_l$ , $C_l$ .....	153
9.2	Demonstration of properties of type $A_l$ .....	160
9.3	Demonstration of properties of type $B_l$ .....	164
9.4	Demonstration of properties of type $C_l$ .....	166
<b>10</b>	<b>Construction of Deformations of Type <math>B_l</math></b> .....	177
10.1	Deformations of type $B_l$ .....	178
10.2	Singular fibers .....	180
<b>11</b>	<b>Construction of Deformations of Type <math>C_l</math></b> .....	183
11.1	Waving polynomials .....	183
11.2	Waving sequences .....	187
11.3	Deformations of type $C_l$ .....	191
11.4	Singular fibers .....	198
11.5	Supplement: The condition that $u$ divides $l$ .....	200
11.5.1	Proof of Lemma 11.5.1 .....	203

<b>12</b>	<b>Recursive Construction of Deformations of Type <math>C_l</math></b>	209
12.1	Ascending, descending, and stable polynomials	209
12.2	Technical preparation I	213
12.3	Recursive construction I	218
12.4	Technical preparation II	225
12.5	Recursive construction II	228
12.6	Examples of non-recursive deformations of type $C_l$	232
<b>13</b>	<b>Types <math>A_l</math>, <math>B_l</math>, and <math>C_l</math> Exhaust all Cases</b>	235
13.1	Results	235
13.2	Preparation	236
13.3	Case 1: $b = 0$	238
13.4	Case 2: $b \geq 1$	243
13.5	Conclusion	249
13.6	Supplement: Proof of Lemma 13.4.4	249
<b>14</b>	<b>Construction of Deformations by Bunches of Subbranches</b>	253
14.1	Propagation sequences	253
14.2	Bunches of subbranches	255
14.3	Example of a deformation by a wild bunch	260
<hr/>		
<b>Part III Barking Deformations of Degenerations</b>		
<hr/>		
<b>15</b>	<b>Construction of Barking Deformations (Stellar Case)</b>	265
15.1	Linear degenerations	265
15.2	Deformation atlas	267
15.3	Crusts	271
15.4	Deformation atlas associated with one crust	273
15.5	Reduced barking	275
<b>16</b>	<b>Simple Crusts (Stellar Case)</b>	279
16.1	Deformation atlases associated with multiple crusts	279
16.2	Multiple barking	281
16.3	Criteria for splittability	284
16.4	Singularities of fibers	288
16.5	Application to a constellar case	292
16.6	Barking genus	295
16.7	Constraints on simple crusts	299
<b>17</b>	<b>Compound barking (Stellar Case)</b>	303
17.1	Crustal sets	303
17.2	Deformation atlas associated with a crustal set	304
17.3	Example of a crustal set	306

<b>18</b>	<b>Deformations of Tubular Neighborhoods of Trunks</b>	309
18.1	Trunks	309
18.2	Subtrunks, I	311
18.3	Subtrunks, II	316
18.4	Other constructions of deformations	320
<b>19</b>	<b>Construction of Barking Deformations (Constellar Case)</b>	327
19.1	Notation	327
19.2	Tensor condition	329
19.3	Multiple barking (constellar case)	332
19.4	Criteria for splittability	342
19.5	Looped trunks	345
<b>20</b>	<b>Further Examples</b>	349
20.1	Fake singular fibers	349
20.2	Splitting families which give the same splitting	349
20.2.1	Example 1	351
20.2.2	Example 2	353
20.2.3	Three different complete propagations	357
20.3	Example of a practical computation of a compound barking	360
20.4	Wild cores	368
20.5	Replacement and grafting	370
20.6	Increasing multiplicities of simple crusts	377

---

## Part IV Singularities of Subordinate Fibers near Cores

---

<b>21</b>	<b>Singularities of Fibers around Cores</b>	383
21.1	Branched coverings and ramification points	385
21.2	Singularities of fibers	393
21.3	Zeros of the plot function	396
21.4	The number of subordinate fibers and singularities	400
21.5	Discriminant functions and tassels	404
21.6	Determination of the singularities	405
21.7	Seesaw phenomenon	413
21.8	Supplement: The case $m = ln$	417
<b>22</b>	<b>Arrangement Functions and Singularities, I</b>	421
22.1	Arrangement polynomials	422
22.2	Vanishing cycles	427
22.3	Discriminants of arrangement polynomials	430
22.4	The coefficients of arrangement polynomials take arbitrary values	432



<b>23 Arrangement Functions and Singularities, II</b> .....	439
23.1 Theta function .....	439
23.2 Genus 1: Arrangement functions .....	445
23.3 Riemann theta functions and Riemann factorization .....	449
23.4 Genus $\geq 2$ : Arrangement functions .....	455
<b>24 Supplement</b> .....	461
24.1 Riemann theta function and related topics .....	461

---

## Part V Classification of Atoms of Genus $\leq 5$

---

<b>25 Classification Theorem</b> .....	483
<b>26 List of Weighted Crustal Sets for Singular Fibers of Genus <math>\leq 5</math></b> .....	487
26.1 Genus 1 .....	492
26.1.1 Stellar singular fibers, $A = \mathbb{P}^1$ .....	492
26.1.2 $I_n^*$ .....	496
26.1.3 $mI_n$ .....	496
26.2 Genus 2 .....	497
26.2.1 Stellar singular fibers, $A = \mathbb{P}^1$ .....	497
26.2.2 Stellar singular fibers, $\text{genus}(A) = 1$ .....	502
26.2.3 Self-welding of stellar singular fibers of genus 1 .....	503
26.3 Genus 3 .....	503
26.3.1 Stellar singular fibers, $A = \mathbb{P}^1$ .....	503
26.3.2 Stellar singular fibers, $\text{genus}(A) = 1, 2$ .....	518
26.3.3 Self-welding of stellar singular fibers of genus 2 .....	519
26.3.4 Welding of stellar singular fibers of genus 2 and genus 1 .....	520
26.4 Genus 4 .....	521
26.4.1 Stellar singular fibers, $A = \mathbb{P}^1$ .....	521
26.4.2 Stellar singular fibers, $\text{genus}(A) = 1, 2$ .....	541
26.4.3 Self-welding and self-connecting of genus 3 or 2 .....	543
26.4.4 Welding of stellar singular fibers of genus 3 and genus 1 .....	546
26.4.5 Welding of stellar singular fibers of genus 2 and genus 2 .....	546
26.4.6 Welding of stellar singular fibers of genus 2, 1, and 1 ...	547
26.5 Genus 5 .....	547
26.5.1 Stellar singular fibers, $A = \mathbb{P}^1$ .....	547
26.5.2 Stellar singular fibers, $\text{genus}(A) = 1, 2, 3$ .....	567
26.5.3 Self-welding and self-connecting of genus 4 or 3 .....	570

26.5.4 Welding of stellar singular fibers of genus 4 and genus 1 .....	574
26.5.5 Welding of stellar singular fibers of genus 3 and genus 2 .....	575
<b>Bibliography</b> .....	581
<b>Index</b> .....	587

---

# Introduction

Wading through,  
And wading through,  
Yet green mountains still.  
(Santoka “Somokuto”<sup>1</sup>)

This is the third in our series of works on degenerations of complex curves. (We here use “complex curve” instead of “Riemann surface”.) The aim of the present volume is to develop a new deformation theory of degenerations of complex curves. This theory is very geometric and a particular class of subdivisors contained in singular fibers plays a prominent role in the construction of deformations. It also reveals the close relationship between the monodromy of a degeneration and existence of deformations of the degeneration. Moreover, via some diagrams, we may visually understand how a singular fiber is deformed. These deformations are called *barking deformations*, because in the process of deformation, some special subdivisor of the singular fiber looks like “barked” (peeled) off. We point out that barking deformations have a remarkable cross-disciplinary nature; they are related to algebraic geometry, low dimensional topology, and singularity theory.

We will further develop our theory: In [Ta,IV], we describe the vanishing cycles of the nodes of the singular fibers appearing in barking families; we then apply this result to give the Dehn twist decompositions of some automorphisms of Riemann surfaces. In [Ta,V], we develop the moduli theory of splitting deformations, which as a special case, includes the theory of barking deformations over several parameters (in the present volume, we mainly discuss the one-parameter deformation theory).

## Background

We will give a brief survey on history and recent development of degenerations of complex curves. Our review is not exhaustive but only covers related topics to our book.

---

<sup>1</sup> Translated by Hisashi Miura and James Green.

## Degenerations of complex curves

A degeneration of complex curves is a one-parameter family of smooth complex curves, which degenerates to a singular complex curve. More precisely, let  $\pi : M \rightarrow \Delta$  be a proper surjective holomorphic map from a smooth complex surface  $M$  to a small disk  $\Delta := \{s \in \mathbb{C} : |s| < \delta\}$  such that  $\pi^{-1}(0)$  is singular and  $\pi^{-1}(s)$  for  $s \neq 0$  is a smooth complex curve of genus  $g$  ( $g \geq 1$ ); so the origin  $0 \in \Delta$  is the critical value of  $\pi$ . (In what follows, unless otherwise mentioned, complex surfaces (curves) are always supposed to be smooth.) We say that  $\pi : M \rightarrow \Delta$  is a *degeneration* of complex curves of genus  $g$  with the *singular fiber*  $X := \pi^{-1}(0)$ . For simplicity, we sometimes say “a degeneration of genus  $g$ ”.

Let  $f : S \rightarrow C$  be a proper surjective holomorphic map from a compact complex surface  $S$  to a compact complex curve  $C$ , and then  $S$  is called a *fibered surface* (e.g. elliptic surface). We note that a degeneration appears as a local model of a fibered surface around a singular fiber: Let  $X$  be a singular fiber of  $f : S \rightarrow C$ , and then the restriction of  $f$  to a sufficiently small neighborhood (germ) of  $X$  in  $S$  is a degeneration. To classify fibered surfaces, it is important to understand their local structure — degeneration — around each singular fiber. It is also important to know when the signature  $\sigma(S)$  (or some other invariant) of the fibered surface concentrates on singular fibers. Namely, when does the equality  $\sigma(S) = \sum_i \sigma_{\text{loc}}(M_i)$  holds?, where  $M_i$  is a germ of a singular fiber  $X_i$  in  $S$ , and  $\sigma_{\text{loc}}(M_i)$  denotes the local signature of  $M_i$ , and the summation runs over all singular fibers (see a survey [AK]). These questions motivate us to study degenerations and their invariants.

Apart from the (local) signature, we have another basic invariant “monodromy” of a degeneration, which also plays an important role in studying degenerations. Given a degeneration  $\pi : M \rightarrow \Delta$  of complex curves of genus  $g$ , we may associate an element  $h$  of the symplectic group  $Sp(2g : \mathbb{Z})$  acting on the homology group  $H_1(\Sigma_g : \mathbb{Z})$ , where  $\Sigma_g$  is a smooth fiber of  $\pi : M \rightarrow \Delta$ . The element  $h$  is defined as follows. We take a circle  $S^1 := \{|s| = r\}$  contained in the disk  $\Delta$ , and then  $R := \pi^{-1}(S^1)$  is a real 3-manifold. The map  $\pi : R \rightarrow S^1$  is a fibration (all fibers are diffeomorphic); that is,  $R$  is a  $\Sigma_g$ -bundle over  $S^1$ , where  $\Sigma_g$  is a smooth fiber of  $\pi : M \rightarrow \Delta$ . Topologically,  $R$  is obtained from a product space  $\Sigma_g \times [0, 1]$  by the identification of the boundary  $\Sigma_g \times \{0\}$  and  $\Sigma_g \times \{1\}$  via a homeomorphism  $\gamma$  of  $\Sigma_g$ . We say that  $\gamma$  is the *topological monodromy* of the degeneration  $\pi : M \rightarrow \Delta$ . (It measures how the complex surface  $M$  is twisted around the singular fiber  $X$ .) Then  $\gamma$  induces an automorphism  $h := \gamma_*$  on  $H_1(\Sigma_g : \mathbb{Z})$ , which is called the *monodromy* of the degeneration. Note that  $h$  preserves the intersection form on  $H_1(\Sigma_g : \mathbb{Z})$ , and so  $h \in Sp(2g : \mathbb{Z})$ .

Monodromy already appeared in the early study of degenerations, notably the work of Kodaira [Kod1] on the classification of degenerations of elliptic curves (complex curves of genus 1). He showed that there are eight degenerations and determined their monodromies: The singular fibers of eight

degenerations are respectively denoted by  $I_n, I_n^*, II, III, IV, II^*, III^*, IV^*$ . (Apart from the three types  $II, III, IV$ , each corresponds to an extended Dynkin diagram.) Kodaira also gave explicit construction of these eight degenerations.

Subsequently, Namikawa and Ueno [NU] carried out the classification of degenerations of complex curves of genus 2: there are about 120 degenerations. Namikawa and Ueno encountered with new phenomena, which did not occur in the genus 1 case: (1) The topological type of a degeneration is not necessarily determined by its singular fiber: There are topologically different degenerations of complex curves of genus 2 with the same singular fiber. (2) The monodromy does not determine the topological type of a degeneration. In fact, if  $g \geq 2$ , there are a lot of topologically different degenerations with the trivial topological monodromy. The reason is as follows: The mapping class group  $MCG_g$  of a complex curve of genus  $g$  has a natural homomorphism  $MCG_g \rightarrow Sp(2g : \mathbb{Z})$  (homological representation), as  $\gamma \in MCG_g$  induces an automorphism  $\gamma_*$  of  $H_1(\Sigma_g : \mathbb{Z})$ . The kernel of this homomorphism is the *Torelli group*  $T_g$ . (Note: If  $g = 1$ , then  $T_g$  is trivial (i.e. the above homomorphism is injective), whereas if  $g \geq 2$ , then  $T_g$  is nontrivial.) In particular, if  $g \geq 2$ , and the topological monodromy  $\gamma$  of a degeneration belongs to  $T_g$ , then  $h := \gamma_*$  (monodromy) is the identity.

This fact indicates that monodromy is not powerful enough to classify degenerations. Moreover, as is suggested by Namikawa and Ueno's classification of 120 degenerations of genus 2, there seem a tremendous amount of degenerations of genus  $g$ , as  $g$  grows higher, and further classifications for genus 3, 4, ... got stuck. New development came from topology. Observe that in the converting process from a topological monodromy to a monodromy, some information may be lost, and hence it is natural to guess that a topological monodromy carries more information than a monodromy, and this is the starting point of the work of Matsumoto and Montesinos, which we shall explain. First of all, we note that the topological monodromy of a degeneration is a very special homeomorphism; it is either periodic or pseudo-periodic (see [Im], [ES], [ST]). Here, a homeomorphism  $\gamma$  of a complex curve  $C$  is *periodic* if for some positive integer  $m$ ,  $\gamma^m$  is isotopic to the identity, and *pseudo-periodic* if for some loops (simple closed curves)  $l_1, l_2, \dots, l_n$  on  $C$ , the restriction  $\gamma$  on  $C \setminus \{l_1, l_2, \dots, l_n\}$  is periodic. A Dehn twist  $\gamma$  along a loop  $l$  on  $C$  is an example of a pseudo-periodic homeomorphism, as the restriction of  $\gamma$  to  $C \setminus l$  is isotopic to the identity.

**Remark 1** There is a classical study of pseudo-periodic homeomorphisms due to Nielsen [Ni1] and [Ni2]; he referred to a pseudo-periodic homeomorphism as algebraically finite type.

For a pseudo-periodic homeomorphism  $\gamma$ , let  $m$  be the integer as above, i.e.  $\gamma^m$  on  $C \setminus \{l_1, l_2, \dots, l_n\}$  is isotopic to the identity. Then  $\gamma^m$  is generated by Dehn twists along  $l_1, l_2, \dots, l_n$ . According to the direction of the twist, a Dehn twist is called *right* or *left*. A pseudo-periodic homeomorphism  $\gamma$  is *right* or *left* provided that  $\gamma^m$  is generated only by right or left Dehn twists. The complex

**structure on a degeneration** poses a strong constraint on the property of its topological monodromy. Using the theory of Teichmüller spaces, Earle–Sipe [ES] and Shiga–Tanigawa [ST] demonstrated that any topological monodromy is a right pseudo-periodic homeomorphism — in [MM2], it is called a pseudo-periodic homeomorphism of *negative type*. For example, if the singular fiber is a Lefschetz fiber (a reduced curve with one node), then the topological monodromy is a right Dehn twist along a loop  $l$  on a smooth fiber  $C$ . Note that the singular fiber is obtained from  $C$  by pinching  $l$ ; in other words,  $l$  is the vanishing cycle.

## Matsumoto–Montesinos theory

Matsumoto and Montesinos established the converse of the result of Earle–Sipe and Shiga–Tanigawa. Namely, given a periodic or right pseudo-periodic homeomorphism  $\gamma$ , they constructed a degeneration with the topological monodromy  $\gamma$ . Their argument is quite topological, using “open book construction”. In [Ta,II], we gave algebro-geometric construction, clarifying the relationship between topological monodromies and quotient singularities.

We denote by  $\mathcal{P}_g$  the set of periodic and right pseudo-periodic homeomorphisms of a complex curve of genus  $g$ , and denote by  $\widehat{\mathcal{P}}_g$  the conjugacy classes of  $\mathcal{P}_g$ . Next, we denote by  $\mathcal{D}_g$  the set of degenerations of complex curves of genus  $g$ , and denote by  $\widehat{\mathcal{D}}_g$  its topologically equivalent classes. The main result of Matsumoto and Montesinos [MM2] is as follows:

**Theorem 2 (Matsumoto and Montesinos [MM2])** *The elements of  $\widehat{\mathcal{P}}_g$  are in one to one correspondence with the elements of  $\widehat{\mathcal{D}}_g$ .*

One important consequence of this theorem is that the topological classification of degenerations completely reduces to the classification of periodic and right pseudo-periodic homeomorphisms.

Matsumoto and Montesinos [MM2] also determined the shape (configuration) of the singular fiber of a degeneration in terms of the data of its topological monodromy — screw numbers and ramification data. Here, we must take care when using the word “shape”, because a shape depends on the choice of model of a degeneration, and it changes under blow up or down. Algebraic geometers usually work with the relatively minimal model of a degeneration — a degeneration is *relatively minimal* if any irreducible component of its singular fiber is not an exceptional curve (a projective line with the self-intersection number  $-1$ ). However, from the viewpoint of topological monodromies, the relatively minimal model is not so natural. The most natural one is the normally minimal model, because it reflects the topological monodromy very well [MM2]. We now review the definition. Express a singular fiber  $X$  as a divisor:  $X = \sum_i m_i \Theta_i$  where  $\Theta_i$  is an irreducible component and a positive integer  $m_i$  is its multiplicity. Then  $\pi : M \rightarrow \Delta$  is called *normally minimal* if  $X$  satisfies the following conditions:

- (1) the reduced curve  $X_{\text{red}} := \sum_i \Theta_i$  is normal crossing (i.e. any singularity of  $X_{\text{red}}$  is a node), and
- (2) if  $\Theta_i$  is an exceptional curve, then  $\Theta_i$  intersects other irreducible components at at least three points.

We point out that a relatively minimal degeneration, after successive blow up, becomes a normally minimal one, which is uniquely determined from the relatively minimal degeneration.

*In what follows, unless otherwise mentioned, we assume that a degeneration is normally minimal.* According to whether the topological monodromy is periodic or pseudo-periodic, the singular fiber is *stellar* (star-shaped) or *constellar* (constellation-shaped). Here, a singular fiber  $X$  is called stellar<sup>2</sup> if its dual graph is stellar (star-shaped):  $X$  has a central irreducible component (*core*), and several chains of projective lines emanating from the core (see Figure 4.2.1, p61). Such a chain of projective lines is called a *branch* of  $X$ . A constellar singular fiber is obtained by bonding branches of stellar fibers, and a resulting chain of projective lines after bonding is called a *trunk*; it is a bridge joining two stellar singular fibers.

The number of the singular fibers of genus  $g$  increases rapidly, as  $g$  grows higher; this is because a constellar singular fiber is constructed from stellar singular fibers in an inductive way with respect to the genus. For instance, a constellar singular fiber of genus 2 is bonding of two stellar singular fibers of genus 1. (Precisely speaking, there is also a constellar singular fiber of genus 2 obtained from one stellar singular fiber of genus 1 by bonding its two branches.) A constellar singular fiber of genus 3 is either bonding of three stellar singular fibers of genus 1, or bonding of two stellar singular fibers of genus 1 and 2. And as  $g$  grows, the partition of the integer  $g$  increases rapidly, and accordingly the number of constellar singular fibers increases rapidly.

Based on the work of Matsumoto and Montesinos, Ashikaga and Ishizaka [AI] proposed an algorithm to carry out the topological classification of degenerations of given genus. Although the practical computation becomes difficult as genus grows higher, their algorithm settled down the topological classification problem of degenerations at least theoretically. They applied their algorithm to achieve the topological classification for the genus 3 case (see [AI]): The number of degenerations is about 1600, and among them there are about 50 degenerations with stellar singular fibers. (For any genus, the number of stellar singular fibers is much less than that of constellar singular fibers.)

## Morsification

There are about 8, 120, and 1600 degenerations of genus 1, 2, and 3 respectively, and as the genus grows higher, the number of degenerations increases

---

<sup>2</sup> We have a similar notion in singularity theory, that is, a star-shaped singularity: A singularity  $V$  is *star-shaped* if the dual graph of the exceptional set in the resolution space of  $V$  is star-shaped, e.g. a singularity with  $\mathbb{C}^\times$ -action. See [OW], [Pn].

rapidly. This fact motivates us to consider another kind of classification — “classification of degenerations modulo deformations”. Before we explain it, we review related materials from Morse theory, which elucidates the relationship between the shapes of smooth manifolds and smooth functions on them. One of the key ingredients of Morse theory is the Morse Lemma, asserting that we may perturb a smooth function  $f : M \rightarrow \mathbb{R}$  in such a way that  $f_t : M \rightarrow \mathbb{R}$  has only non-degenerate critical points. A non-degenerate critical points is stable under arbitrary perturbation, and so the Morse lemma ensures that we may split critical points of  $f$  into stable ones under perturbation. Of course, the Morse lemma is a result in the smooth category, but its spirit is carried over to the complex category, for instance, *Morsification of singularities*: When does an isolated singularity  $V$  admits a deformation  $\{V_t\}$  such that  $V_t$  for  $t \neq 0$  possesses only  $A_1$ -singularities? (It is known that any hypersurface isolated singularity admits a Morsification, e.g. see Dimca [Di] p82)

We next explain *Morsification of singular fibers*, which was advocated by M. Reid [Re]. First of all, we review splitting deformations.

## Splitting deformations of degenerations

Let  $\Delta^\dagger := \{t \in \mathbb{C} : |t| < \varepsilon\}$  be a sufficiently small disk. Suppose that  $\mathcal{M}$  is a complex 3-manifold, and  $\Psi : \mathcal{M} \rightarrow \Delta \times \Delta^\dagger$  is a proper flat surjective holomorphic map. We set  $M_t := \Psi^{-1}(\Delta \times \{t\})$  and  $\pi_t := \Psi|_{M_t} : M_t \rightarrow \Delta \times \{t\}$ . (Hereafter, we denote  $\Delta \times \{t\}$  simply by  $\Delta$ , so that  $\pi_t : M_t \rightarrow \Delta$ .) We say that  $\Psi : \mathcal{M} \rightarrow \Delta \times \Delta^\dagger$  is a *deformation family* of  $\pi : M \rightarrow \Delta$  if  $\pi_0 : M_0 \rightarrow \Delta$  coincides with  $\pi : M \rightarrow \Delta$ . In this case,  $\pi_t : M_t \rightarrow \Delta$  is referred to as a *deformation* of  $\pi : M \rightarrow \Delta$ .

Suppose that  $\pi_t : M_t \rightarrow \Delta$  for  $t \neq 0$  has at least two singular fibers, say,  $X_1, X_2, \dots, X_n$  ( $n \geq 2$ ). Then we say that  $\Psi : \mathcal{M} \rightarrow \Delta \times \Delta^\dagger$  is a *splitting family* of the degeneration  $\pi : M \rightarrow \Delta$ , and that  $\pi_t : M_t \rightarrow \Delta$  is a *splitting deformation* of  $\pi : M \rightarrow \Delta$ . In this case, we say that the singular fiber  $X = \pi^{-1}(0)$  *splits into*  $X_1, X_2, \dots, X_n$ .

To the contrary, if a singular fiber  $X$  admits no splitting deformations at all, the degeneration  $\pi : M \rightarrow \Delta$  is called *atomic*. The singular fiber of the atomic degeneration is called an *atomic fiber*. (Caution: This terminology is not completely rigorous, because a singular fiber does not determine the topological type of a degeneration, so we must use it with care.) A Lefschetz fiber (i.e. a reduced curve with one node) and a multiple  $m\Theta$  of a smooth curve  $\Theta$ , where  $m \geq 2$  is an integer, are examples of atomic fibers (see [Ta, I]).

A *Morsification* of a degeneration  $\pi : M \rightarrow \Delta$  is a splitting family  $\Psi : \mathcal{M} \rightarrow \Delta \times \Delta^\dagger$  such that for  $t \neq 0$ , all singular fibers of  $\pi_t : M_t \rightarrow \Delta$  are atomic fibers. Unfortunately this notion is too restrictive, as many degenerations of high genus seem to admit no Morsifications. Instead, we work with a weaker notion “a finite-stage Morsification”, defined as follows. If  $\pi : M \rightarrow \Delta$  is not atomic, take a splitting family  $\Psi : \mathcal{M} \rightarrow \Delta \times \Delta^\dagger$ , say,  $X$  splits into  $X_1, X_2, \dots, X_n$  (the first-stage splitting). If all singular fibers  $X_1, X_2, \dots, X_n$



are atomic, the first-stage splitting is a Morsification. If some  $X_i$  is not atomic, then take a sufficiently small neighborhood  $M_i$  of  $X_i$  in  $M_t$ , and then consider the restriction of  $\pi_t$  to  $M_i$ , which is a degeneration  $\pi_i : M_i \rightarrow \Delta$  (called the *fiber germ* of  $X_i$  in  $\pi_t : M_t \rightarrow \Delta$ ). Next, take a splitting family  $\Psi_i : \mathcal{M}_i \rightarrow \Delta \times \Delta^\dagger$  of  $\pi_i : M_i \rightarrow \Delta$ , say,  $X_i$  splits into  $X_{i,1}, X_{i,2}, \dots, X_{i,m}$  (the second-stage splitting). Repeating this process, we finally reach to a set of atomic fibers, say,  $X'_1, X'_2, \dots, X'_l$ : Under the finite-stage Morsification,  $X$  splits into atomic fibers  $X'_1, X'_2, \dots, X'_l$ . In this case, we obtain a smooth 4-manifold  $M'$  together with a locally holomorphic map  $\pi' : M' \rightarrow \Delta$  such that (1)  $M'$  is diffeomorphic to  $M$  and (2) all singular fibers  $X'_1, X'_2, \dots, X'_l$  of  $\pi'$  are atomic. Here, “locally holomorphic map” means that  $M'$  has a complex structure around  $X'_i$ , and  $\pi'$  is holomorphic with respect to this complex structure. A finite-stage Morsification of a degeneration is useful for studying the topological types of fibered algebraic surfaces.

There is another motivation from algebraic geometry to study Morsification, inspired by the following question: How does an invariant of a degeneration (e.g. local signature, Horikawa index [AA1]) behave under splitting. Specifically, let  $\text{inv}(\pi)$  be some invariant of a degeneration  $\pi : M \rightarrow \Delta$ . Suppose that  $\pi_t : M_t \rightarrow \Delta$  is a splitting deformation, which splits the singular fiber  $X$  into singular fibers  $X_1, X_2, \dots, X_n$ . Then find a formula of the form

$$\text{inv}(\pi) = \sum_{i=1}^n \text{inv}(\pi_i) + c,$$

where  $\pi_i : M_i \rightarrow \Delta$  is a fiber germ of  $X_i$  in  $M_t$ , and  $c$  is a “correction term”. For these problems, we refer the reader to excellent surveys [AE], [AK], and also [Re].

A primary concern of the Morsification problem of degenerations is to classify all atomic degenerations. The number of atomic degenerations of genus  $g$  must be much less than that of all degenerations of genus  $g$ , and so this problem leads us to a reasonable classification — *classification of degenerations modulo deformations*.

When is a degeneration atomic? Before we discuss this problem, we explain several methods to construct splitting families.

## Double covering method for hyperelliptic degenerations

A hyperelliptic curve  $C$  is a complex curve which admits a double covering  $C \rightarrow \mathbb{P}^1$  branched over  $2g + 2$  points on  $\mathbb{P}^1$ , where  $g = \text{genus}(C)$ . (All complex curves of genus 1 and 2 are hyperelliptic.) A degeneration  $\pi : M \rightarrow \Delta$  is called *hyperelliptic* provided that any smooth fiber  $\pi^{-1}(s)$  is a hyperelliptic curve. In this case, the total space  $M$  is expressed as a double covering  $M \rightarrow \mathbb{P}^1 \times \Delta$  branched over a complex curve (*branch curve*)  $B$  in  $\mathbb{P}^1 \times \Delta$ , and conversely from this double covering, we may recover the hyperelliptic degeneration  $\pi : M \rightarrow \Delta$ . (Precisely speaking, instead of  $M$ , we need to take