Ideas of Space

Euclidean, Non-Euclidean, and Relativistic

Second Edition

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JEREMY GRAY

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Preface to the second edition

I have made a number of changes for the second edition, which make the book more historical. The most significant of these is the new chapter on Islamic investigations of the parallel postulate, which replaces some material on the Greek treatment of incommensurability. This not only does more justice to the historical story, but helps to explain how the postulate can be investigated at all. I would particularly like to draw the reader's attention to two recent books on the subject: K. Jaouiche, *La théorie des parallèles en pays d'Islam* (1986), which contains French translations of many original sources, and B. A. Rosenfeld, *A history of non-Euclidean geometry* (1989). I am most grateful to Abe Shenitzer for supplying me with a copy of the galley proofs of his English translation of Rosenfeld's book, which has a very detailed account of non-Euclidean geometry in Islam (among other topics). The work of Rosenfeld and Jaouiche means that for the first time we are in the position of having translations and thorough accounts of Islamic contributions in this area based on original sources, and I have happily made use of them.

I have taken the opportunity to correct a number of small mistakes, and to supply more references to original sources. The availability of primary sources in translation has also been increased somewhat with the publication of J. Fauvel and J. J. Gray (eds.), *The history of mathematics—a reader* (Macmillan, 1987). Finally, I would like to thank those who responded, in the spirit of my earlier invitation, with critical and helpful comments, and to repeat that invitation here.

Milton Keynes
1988

J.J.G.

Preface to the first edition

I hope in this book to say something about mathematics, what it is and how it has been done. I shall discuss Greek and modern geometry, in particular what came to be known as the problem of parallels, that 'blot on geometry' as Saville* called it in 1621. The problem is this: if a line meets a vertical line obliquely, must it necessarily meet any horizontal line as well? Stated as simply as that it may sound trivial, but the charm of the problem is that although it can be stated in classical terms it cannot be solved without a dramatic change in fundamental ideas. Its resolution is elusive, difficult, and surprising. I shall pursue the matter further and discuss Einstein's theories of relativity, both special and general, and modern ideas of the shape of the universe.

The approach I have taken is largely historical and chronological. I have not avoided discussing difficult problems—indeed to have done so would be to have sacrificed my objective—but I have assumed no specialist mathematical knowledge. A working familiarity with simple equations and the elements of trigonometry, such as students of science and engineering possess, is all that is needed. It is my hope that the study of past insights into a problem provides as valid a way into mathematics as the polished answers we now seem to regard as best. By using the history we can analyse problems, exposing and discussing difficulties and confusions as they arise, and thus learn about mathematics itself, and in part this book is an attempt to understand mathematics as a dynamic activity. We shall often encounter connections between mathematics, philosophy, and truth, which run as subsidiary themes throughout. However, this is not strictly a history book. I have not hesitated to abandon the history when the thread of mathematics runs thin or turns aside from the main subject. The reader should not feel it necessary to read every word of the book, but should select and skip as fancy suggests.

The book begins with early Greek mathematics, the Eastern legacy, and the transition to deductive and geometric thinking. Then we encounter parallels. The properties and problem of parallels were well formulated by the time of Euclid, and we start by looking at Greek and later Arab approaches. The second part of the book takes the story from Wallis, Saccheri, and Lambert to its resolution by Gauss, Lobachevskii, Bolyai, Riemann, and Beltrami. In contrast to most authors, who see the developments as primarily foundational, I see them as concerned more with the concepts and methods of geometry, and so I shall sketch the background of the nineteenth-century theory of surfaces and the relevant analysis. Chapter 14 revisits the earlier

* Saville, H. (1621). *Thirteen lectures on the elements of Euclid*. Oxford University Press Oxford.

material in the light of the later formulations; Chapter 15 summarizes the account and compares it with other versions. In the third part I give an account of Einstein's theories based on what has gone before, moving from a Newtonian–Euclidean picture to an Einsteinian–non-Euclidean one. This transition is often referred to in the literature, but rarely described. The book concludes with a brief modern account of gravitation, the nature of space, and black holes.

I believe that intelligible explanations of every subject can and should be made, giving their real flavour without descending to trivialities, and this has been my objective. I hope that this book will make mathematics accessible to some people who have been repelled by its technicalities, and that its historical approach will itself be of value to mathematicians.

It is with great pleasure that I thank the people who have helped me with this book: friends and colleagues who, by their interest and advice, have made it much better than it otherwise would have been. Among those who read it in whole or in part and commented valuably were Julia Annas, John Bell, David Charles, David Fowler, Luke Hodgkin, Clive Kilmister, Bill and Benita Parry, Colin Rourke, Graeme Segal, and Ian Stewart. I should also like to thank the reviewers whose comments so markedly helped to improve the book and of whom only Dana Scott is known to me by name, the British Society for the History of Mathematics for inviting me to address them on some of it and for the discussion afterwards, and Jennie Connell and the Open University typists whose excellent jobs of typing the manuscript helped to restore my confidence in it. Above all, I express my deepest thanks to my parents for their encouragement, comments, and advice.

Any helpful criticism and comments will be gratefully received. All mistakes that remain are mine.

Milton Keynes J.J.G.
1978

Contents

Part 1

Part 2

Part 3

Part 1

1

Early geometry

The civilizations of the Eastern Mediterranean and Middle East seem to have had an interest in mathematics from very early times. Egyptian and Babylonian scribes in about 1700 BC discussed not only matters of practical or commercial importance, but carried out abstract calculations as well. Estimates of areas and volumes are found alongside solutions to quite complicated numerical problems, and while the rules of mensuration are frequently wrong the skill with which the numerical problems were solved suggests very strongly that the Babylonians, at least, had a good grasp of elementary mathematics. The Babylonians, who generally surpassed the Egyptians, also developed an excellent positional astronomy which, it should be noted, had been preceded by over a thousand years of mathematics. However, the differences between Greek and Babylonian or Egyptian mathematics of around 300 BC are manifest. The Greeks were doing geometry, they were proving things, their methods were deductive, and there are signs of a lively interest in questions of rigour and logical validity. The Babylonians, on the other hand, had procedures but no proofs. Like the Greeks they possessed an impressive grasp of observational astronomy, but it did not rest on a theoretical or geometrical base. The Greeks, as is well known, gave mathematics a paramount position in their philosophical endeavours. Plato in numerous places directed his contemporaries towards mathematics. Aristotle drew many illustrations of argument from it, which are now collected in *Mathematics in Aristotle* by T. L. Heath (1949). At least one form of argument, that of *reductio ad absurdum*, was first used in mathematics before being used elsewhere. Naturally we look for the origins of this attitude to see how the transition to deductive mathematics might have been made.

Unfortunately the evidence for this period is scanty since Eudemus' *History* (*c*. 325 BC) is lost. Virtually the only nearly contemporary references to early Greek mathematics occur in Plato and Aristotle. Later writers, writing about work done three to eight hundred years before them, gave fuller accounts, but they brought to the task of a set of attitudes to mathematics which must have been different from those of their forerunners, and they may have credited the earlier mathematicians with a clarity and exactness of thought which they did not in fact possess. In some cases a later way of doing things has made it difficult to appreciate the problems originally raised. Furthermore, the transmission of the record may have been faulty.

Happily, there are now several good histories of early mathematics available. Foremost amongst the modern texts are the many works of T. L. Heath, chiefly his three-volume edition of the *Elements* (1956), his two-volume

History of Greek mathematics (1921), and the one-volume *Greek mathematics* (1930). Other texts are *Science awakening* by van der Waerden (1961), and *The exact sciences in antiquity* by O. Neugebauer (1969). The books in the bibliography by Fowler (1987), Knorr (1975) and (1986), Lloyd (1970) and (1973), Mueller (1981), and Szabo (1978) bear witness to the vitality of current studies in the history of Greek mathematics. Individual topics are treated in the *Oxford classical dictionary* and the *Dictionary of scientific biography*. These studies have shed much light on our questions concerning the origin and evolution of deductive mathematics.

The spread of learning

There is a particular problem involved in the transmission of mathematics across a region or between cultures which is not found in the transmission of other ideas or techniques. Mathematics is not simply a collection of facts or 'results'; it is also a set of procedures for isolating problems and for solving them, a set of assumptions and permissible deductions, a way of thinking about things. Isolated from these habits of mind the individual results can not only seem trivial, but they can lose their specifically mathematical character and become observational or 'inductive' instead. Conversely, if the procedures are transmitted they act as a check upon the body of transmitted facts, allowing them to be re-derived or excluded if no proof can be found. Yet the compelling character of mathematics is to interest cultures in similar problems and so to drive them after similar information, even if they cannot understand each other's activity, so we should not necessarily assume that information has passed when we find two cultures doing similar things. Indeed the evidence of direct cultural contact between Greece and Mesopotamia is slight, consisting of a few opinions, like that of Herodotus,[1] who gave the gnomon and the division of the day into twelve hours a Babylonian origin. It is salutary to remember that he was wrong about the twelve-hour day, for Neugebauer[2] has established that it has an Egyptian origin.

The characteristics of Babylonian mathematics were a good number system and a rhetorical formulation of mathematical problems, which has led to their formulation of mathematics being called 'rhetorical algebra' by many writers, but the limitations of rhetorical algebra made its transmission difficult. Essentially, rhetorical algebra is a set of procedures expressed in words and illustrated with numerical examples for solving certain problems: finding solution to equations, calculating areas and volumes. BM 13901,[3] a tablet containing 24 similar problems, starts as follows:

[1] Herodotus, Book II, 336 109. Loeb edition, transl. A. D. Godley. Heinemann, London.
[2] Neugebauer (1969, p. 81).
[3] A picture of the tablet appears in Unit N4 of *The history of mathematics* (Open University Course AM289), p. 30, Open University, Milton Keynes; the unit contains a discussion of Babylonian mathematics. A different translation appears in Fauvel and Gray (1987), p. 31.

I have added the area and the side of my square: 45. Take 1, divide it into two: 30, and multiply: 30 × 30 = 15. Add 15 and 45: 1, the square of 1. Subtract the 30 (which you had multiplied by itself) from the 1. You have 30, the side of the square.

Since all numbers have here been expressed as parts of 60, we should express the original equation as $x^2+x = \frac{3}{4}$. The coefficient of x is 1; halve that and square it $(\frac{1}{2})^2 = \frac{1}{4}$. Add $\frac{1}{4}$ and $\frac{3}{4}$ (and form $x^2+x+\frac{1}{4} = \frac{3}{4}+\frac{1}{4}$). Both sides are squares; take square roots ($(x+\frac{1}{2})^2 = 1^2$. Therefore $x+\frac{1}{2} = 1$. Subtract the half from both sides; $x = \frac{1}{2}$).

Now, a procedure expressed verbally is not a formula, it cannot be manipulated into equivalent forms or checked against another intended to solve the same problem. For these reasons rhetorical algebra is without proofs and can accommodate different and incompatible answers. It is tied to such operations with numbers as can be marshalled in words and therefore derived fairly directly from the elementary properties of number.

Teachers of it may have referred to a body of theory transmitted aurally which amplified the written remains we have, but it is most likely that the rhetorical techniques were taught as methods which check. If they were transmitted as such then they could well seem to anyone who encountered them a sterile body of facts without coherence or power to inform. In this form they probably did pass to the West, if only because of their use in commerce. We can trace the appearance of some rhetorical techniques in Greek mathematics, if not their passage there.[4]

There is only one way out of the profusion of contradictory and non-explanatory results in rhetorical algebra and that is to find a way of making coherent sense of its results—at least those which are right. I believe that it is in attempting to do that that the Greeks were led to geometry, not for its own sake but as a method of proof. The two go together and provide a deductive method for the treatment of mathematical problems. This point of view enables one to make sense of the otherwise confusing legends that have come down about the earliest Greek geometers, Thales and the school of Pythagoras.

Thales

According to Proclus[5] (AD 410–85) Thales (624?–548? BC) '... made many discoveries himself and taught the principles for many others to his successors, attacking some problems in a general way and others more empirically'. In particular he is supposed to have been the first to demonstrate that a circle is bisected by a diameter, that the base angles of an isosceles triangle are equal, that the angle in a semicircle is a right angle, and that two triangles are

[4] See p. 21.

[5] Proclus, *A commentary on the first book of Euclid's elements*, transl. G. R. Morrow, 1970, p. 52. I shall refer to this book as Proclus (Morrow edn) to distinguish it from the earlier English edition translated by Taylor.

congruent[6] if they have two pairs of corresponding angles equal and the sides between those angles equal. Proclus' commentary on the first of these results (Proclus (Morrow edn), p. 124) gives no clue as to Thales' proof, but Proclus did indicate how such a proof might go. Imagine that a diameter does not bisect the circle and then apply one part of the circle to the other by folding it over along the diameter. If the two parts are not to coincide one falls inside or outside the other somewhere, but then radial lines would not all be equal, which is absurd, and the result follows.

It is sometimes implied by some writers that mathematics is discovered in a way that reflects its logical order, so that propositions which appeal to others for their proof must likewise follow these others in their order of birth. Thus, to prove that the angle in a semicircle is a right angle[7] we nowadays use the result that the sum of the angles is two right angles. It is possible that Thales appealed to that result too, but we do not know. However, it is not possible to infer that, as a mathematician, he would automatically stop to prove the validity of any result he used. In the second heyday of foundational enquiry, the late nineteenth century, much of the work done was in providing theories whose conclusions were the basic assumptions of another man's work. When we are at the historical beginnings of deductive mathematics, therefore, we can well imagine that what was obvious to one man and not worthy of proof was an interesting puzzle to another. To extract basic assumptions would have taken time, as the deductive method was seen to apply to more and more of mathematics and to yield proofs of more already 'known' results. So it should not surprise us to see attributed to Thales results which seem to depend on others of which he is not known to have a proof and which he may have taken from the common store of factual knowledge. We cannot even be certain that he really knew what a proof was. Greek work on purely logical questions seems to have begun even later than their mathematical investigations.

Naïve geometry

We may reasonably imagine an initial, naïve formulation of mathematics in which numbers are represented by geometrical segments, say lines, squares, rectangles, or cubes. To represent a number[8] as a line one took a fixed, but arbitrary, unit length and repeated it as often as was necessary; representations of square numbers in terms of a unit square proceeded similarly. The method was traditional in Babylonian and Egyptian mathematics, and was referred to by Plato[9] as being common in Greek mathematics. The early, but not the late, work of the Pythagoreans was cast in such a form.

[6] Two figures are *congruent* if they can be made to coincide exactly with one another.

[7] This result is attributed to Thales by Pamphile in Diogenes Laertius (I, 24–5, p. 6, ed. Cobet), third century AD.

[8] Number meant positive integer throughout this period.

[9] Knorr (1975, p. 172) cites *Theaetetus*, 148A, amongst other passages.

Our knowledge of Pythagoras is little better than our knowledge of Thales, and has been conveniently summarized by Kurt von Fritz.[10] No theorem can be reliably attributed to the man rather than his school, which seems to have split into factions sometimes after the leader's death (*c.* 480 BC). In his day the school was primarily religious and philosophical in its preoccupations, and took as its programme the belief that all things are numbers. In mathematics their chief interest was in arithmetic, and they studied numbers geometrically through their representations as figured numbers. These are invariably described in histories of mathematics,[11] and will be briefly described here with a view to establishing a hypothetically Pythagorean proof of Pythagoras' Theorem due to Bretschneider.[12] The result, that in a right-angled triangle ABC with a right angle at C, $AB^2 = AC^2 + CB^2$, was known to the Babylonians by about 1700 BC. They used it over and over again in their problem solving, and one tablet, Plimpton 322, carries an impressive list of triples of numbers which reveals that they had a good grasp of how to construct integer triples a, b, c such that $c^2 = b^2 + a^2$ (see the Exercises for further details).

I have suggested that what is significant about this period is the move from procedures to proofs. One may therefore speak of theorems rather than results, a theorem being a result for which there is a proof. In this sense of 'theorem' the theorem which today bears the name of Pythagoras must surely, as Heath (1921, p. 145) suggested, have originated in the school, although we have no source allowing us to attribute it to Pythagoras directly and no proof attributable to the school has survived. Proclus, for instance, does not even attribute the result to Pythagoras.

I shall assume that the famous Theorem of Pythagoras originated in his school, which involves me in two further assertions:

(1) The result was known for *all* right-angled triangles and not just in various special cases (3, 4, 5; 5, 12, 13; . . .).
(2) A *proof* of the theorem was also obtained by them in more or less the sense in which we understand 'proof' today.

My reason for believing this is that as a theorem rather than a conjecture the result is non-trivial. However, if the Pythagoreans lacked a proof of the theorem, it is difficult to see why its most immediate corollary, the existence of 'irrational numbers', would so disturb them, and we do have evidence suggesting that the discovery of irrationals shook them profoundly. How much easier it would have been to reject the conjecture and with it the fateful corollary.

[10] *Dictionary of scientific biography* (1975), Vol. XI, pp. 219–25.
[11] In addition to Knorr's book there is e.g. Sambursky's *The physical world of the Greeks* which gives a fuller account of the Pythagorean attitude to number.
[12] Knorr (1975, Chap. VI) gives it particular emphasis. For Bretschneider, see Heath (1956), note after Prop I, p. 47.

Figured numbers and a proof of the Theorem of Pythagoras

A number may be represented by a row of uniformly spaced dots, and, since all numbers are built out of the unit by repetition, classical authors generally denied that 1 was a number, and numbers began at 2. The unit, 1, was, rather, the source of number. If the row can be broken into two equal rows the number is even, but if the dividing line hits a dot in the middle the number is odd. Figured numbers are obtained whenever the dots are arranged into shapes or figures (see Figs. 1.2 and 1.3). By no means every number can be figured in a pre-assigned way. The only numbers which can be represented as triangles are 1, 3, 6, 10, . . . ; the only squared numbers are, of course, 1, 4, 9, 16, ... The difference between two figured numbers or, more strictly, the number which when added to a figured number produces the next figured number of the same class is called a *gnomon*. Consecutive gnomons which generate the

Fig. 1.1. (a) 8, an even number, divides into 4+4; (b) 7, an odd number, cannot be divided into two equal halves.

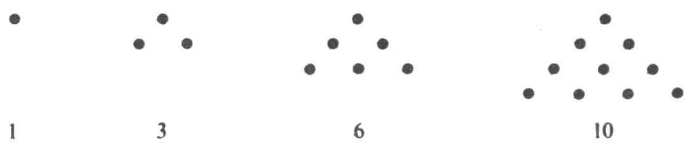

Fig. 1.2. Triangular numbers.

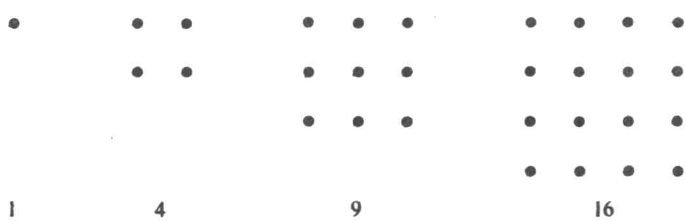

Fig. 1.3. Square numbers.

Fig. 1.4. A gnomon between two successive triangular numbers.